数理解析研究所講究録 第2093巻 2018年 57-64

LOCAL WELL-POSEDNESS OF SEMILINEAR HIGHER ORDER DISPERSIVE EQUATIONS ON THE TORUS

中央大学理工学部数学科 津川 光太郎
KOTARO TSUGAWA
DEPARTMENT OF MATHEMATICS, CHUO UNIVERSITY,
1-13-27 KASUGA, BUNKYO-KU, TOKYO, 112-8551, JAPAN

1. Introduction

The purpose of this paper is to explain the idea of the author's recent papers [2], [3] and his further research plan intuitively. We consider the following $(2j)$-th order dispersive equaions:

$$(\partial_t + i\partial_x^{2j})u = F(\partial_x^{2j-1}u, \partial_x^{2j-1}\overline{u}, \dots, u, \overline{u}), \quad (t,x) \in [-T,T] \times \mathbb{T}, \tag{1.1}$$

and $(2j+1)$-st order dispersive equations:

$$(\partial_t + \partial_x^{2j+1})u = F(\partial_x^{2j-1}u, \partial_x^{2j-2}u, \dots, u), \quad (t,x) \in [-T,T] \times \mathbb{T}, \tag{1.2}$$

for $j = 1, 2, 3, \dots$. We give the initial condition:

$$u(0,x) = \varphi \in H^s(\mathbb{T}). \tag{1.3}$$

The unknown function u and the initial data φ are complex (resp. real) valued for (1.1) (resp. (1.2)) with (1.3). For simplicity, we assume F is a polynomial of $\partial_x^{2j-1}u, \partial_x^{2j-1}\overline{u}, \dots, u, \overline{u}$ (resp. $\partial_x^{2j-1}u, \partial_x^{2j-2}u, \dots, u$) for (1.1) (resp. (1.2)) without any constants and linear terms, though this assumption is not essential and we consider more general functions F in [3]. The final goal of the author's research plan is to find some algebraic conditions on F to ensure the local well-posedness of (1.1) or (1.2) with (1.3) for sufficiently large s. The difficulty of this problem comes from so called "derivative losss". The smoothing effect of the linear part of (1.1) (resp. (1.2)) on the torus is so weak that we can not control the terms including some derivatives of u in F. In fact, if F does not include any derivatives of u, that is $F = F(u, \overline{u})$ or $F = F(u)$, we can easily show the local well-posedness of (1.1) or (1.2) with (1.3) for sufficiently large s by the standard fixed point argument. On the real line case $x \in \mathbb{R}$, the smoothing effect of the linear parts of (1.1) (resp. (1.2)) is strong enough to obtain the local well-posedness of (1.1) or (1.2) with (1.3) in a weighted Sobolev space without any additional assumption on F.

Remark that we can deal with "one derivative loss" when u is real valued by the argument below. As an example, we consider the case $F = u^{p_1}(\partial_x u)^{p_2}$ with $p_1, p_2 \in \mathbb{N}$. By the Leibniz rule,

$$\langle F, u\rangle_{H^s} = \sum_{j=0}^{s} \langle \partial_x^j F, \partial_x^j u\rangle_{L^2} = p_2\langle u^{p_1}(\partial_x u)^{p_2-1}\partial_x^{s+1}u, \partial_x^s u\rangle_{L^2} + \cdots . \tag{1.4}$$

The terms omitted in (1.4) do not have "derivative loss", that is, they are bounded by $C\|u\|_{H^s}^{p+2}$ if $s > 3/2$. Since u is real valued, by the integration by parts, we have

$$\begin{aligned} \langle u^{p_1}(\partial_x u)^{p_2-1}\partial_x^{s+1}u, \partial_x^s u\rangle_{L^2} &= \left\langle u^{p_1}(\partial_x u)^{p_2-1}, \frac{\partial_x}{2}(\partial_x^s u)^2\right\rangle_{L^2} \\ &= \frac{-1}{2}\langle \partial_x(u^{p_1}(\partial_x u)^{p_2-1}), (\partial_x^s u)^2\rangle_{L^2} \end{aligned} \tag{1.5}$$

for $s \in \mathbb{N}$. Therefore, by (1.4), (1.5) and the Sobolev embedding, we obtain $|\langle F, u\rangle_{H^s}| \leq C\|u\|_{H^s}^{p+2}$ when $s > 5/2$. Though we assumed $s \in \mathbb{N}$ for simpleness, we can generalize this argument for fractional s. This argument also works for any polynomial $F(\partial_x u, u)$ when $s > 5/2$, that is to say, we obtain $|\langle F(\partial_x u, u), u\rangle_{H^s}| \leq C(1 + \|u\|_{H^s})^m$ for sufficiently large m and $s > 5/2$. This means that we have the local well-posedness of (1.2) with (1.3) for any polynomial $F(\partial_x u, u)$ when $s > 5/2$ by the energy method. Therefore, (1.2) with $j = 1$ is easy problem and we are interested in (1.2) only for $j \geq 2$.

On the other hand, we can not deal with even "one derivative loss" when u is complex valued by the standard energy method and the integration by parts. Precisely, for general $F(\partial_x u, \partial_x \overline{u}, u, \overline{u})$, it does not hold that $|\langle F(\partial_x u, \partial_x \overline{u}, u, \overline{u}), u\rangle_{H^s}| \leq C(1 + \|u\|_{H^s})^m$ even for large m and s. Therefore, we are interested in (1.1) for $j \geq 1$.

Here we mention some model equations in mathematical physics. The KdV hierarchy:

$$\begin{aligned} &\partial_t u + \partial_x^3 u = \partial_x(u^2), \\ &\partial_t u + \partial_x^5 u = 5\partial_x(\partial_x u)^2 - 10\partial_x(u\partial_x^2 u) - 10\partial_x(u^3), \\ &\qquad\vdots \end{aligned}$$

and the mKdV hierarchy:

$$\partial_t u + \partial_x^3 u = \partial_x(u^3),$$
$$\partial_t u + \partial_x^5 u = 5\partial_x(u\partial_x^2(u^2)) - 6\partial_x(u^5),$$
$$\vdots$$

are known as complete integrable systems. Each member of this hierarchy satisfies (1.2). The derivative nonlinear Schrödinger equation:

$$\partial_t u + i\partial_x^2 u = \partial_x(|u|^2 u)$$

satisfies (1.1) with $j = 1$.

2. MAIN RESULTS

In this section we mention the main results in [2], [3]. First, we mention the main results in [2]. In this paper, we consider (1.2), (1.3) with $j = 2$, that is

$$(\partial_t + \partial_x^5)u = F(\partial_x^3 u, \partial_x^2 u, \partial_x u, u), \quad (t, x) \in [-T, T] \times \mathbb{T}, \tag{2.1}$$
$$u(0, x) = \varphi. \tag{2.2}$$

For sufficiently smooth function f on $\mathbb{T}$, we put

$$J_F^{(5,2)}(f) := \frac{1}{2\pi} \int_{\mathbb{T}} \partial_{\omega_2} F(\omega_3, \omega_2, \omega_1, \omega_0)\Big|_{(\omega_3,\omega_2,\omega_1,\omega_0)=(\partial_x^3 f(x), \partial_x^2 f(x), \partial_x f(x), f(x))} dx.$$

We say $J_F^{(5,2)} \equiv 0$ if $J_F^{(5,2)}(f) = 0$ holds for any $f \in C^\infty(\mathbb{T})$.

Theorem 2.1 (L.W.P. of dispersive type). *Let $J_F^{(5,2)} \equiv 0$, $s \in \mathbb{N}$ and $s \geq 13$. Then, we have the followings.*
(Existence) Let $\varphi \in H^s(\mathbb{T})$. Then, there exist $T = T(\|\varphi\|_{H^{12}}) > 0$ and a solution to (2.1)-(2.2) on $[-T, T]$ satisfying $u \in C([-T, T]; H^s(\mathbb{T}))$.
(Uniqueness) Let $T > 0$, $u_1, u_2 \in L^\infty([-T, T]; H^{12}(\mathbb{T}))$ be solutions to (2.1)-(2.2) on $[-T, T]$. Then, $u_1(t) = u_2(t)$ on $[-T, T]$.
(Continuous dependence) Assume that $\{\varphi^j\}_{j\in\mathbb{N}} \subset H^s(\mathbb{T}), \varphi \in H^s(\mathbb{T})$ satisfy $\|\varphi^j - \varphi\|_{H^s} \to 0$ as $j \to \infty$. Let u^j (resp. u) be the solution obtained above with initial data φ^j (resp. φ) and $T = T(\|\varphi\|_{H^{12}})$. Then $\sup_{t\in[-T,T]} \|u^j(t) - u(t)\|_{H^s} \to 0$ as $j \to \infty$.

Theorem 2.2 (L.W.P. of parabolic type). *Let $J_F^{(5,2)} \not\equiv 0$, $s \in \mathbb{N}$ and $s \geq 13$. Then, we have the followings.*
(Existence) Let $\varphi \in H^s(\mathbb{T})$ and $J_F^{(5,2)}(\varphi) > 0$ (resp. $J_F^{(5,2)}(\varphi) < 0$). Then, there exist $T = T(J_F^{(5,2)}(\varphi), \|\varphi\|_{H^{12}}) > 0$ and a solution to (2.1)–(2.2) on $[0,T]$ (resp. $[-T,0]$) satisfying $u \in C([0,T]; H^s(\mathbb{T})) \cap C^\infty((0,T] \times \mathbb{T})$ and $J_F^{(5,2)}(u(t)) \geq P_N(\varphi)/2$ on $[0,T]$ (resp. $u \in C([-T,0]; H^s(\mathbb{T})) \cap C^\infty([-T,0) \times \mathbb{T})$ and $J_F^{(5,2)}(u(t)) \leq J_F^{(5,2)}(\varphi)/2$ on $[-T,0]$).
(Uniqueness) Assume that $T > 0$, $u_1, u_2 \in L^\infty([0,T]; H^{12}(\mathbb{T}))$ (resp. $u_1, u_2 \in L^\infty([-T,0]; H^{12}(\mathbb{T}))$) be solutions to (2.1)–(2.2) and $P_N(u_1(t)) \geq 0$ on $[0,T]$ (resp. $P_N(u_1(t)) \leq 0$ on $[-T,0]$). Then, $u_1(t) = u_2(t)$ on $[0,T]$ (resp. $[-T,0]$).
(Continuous dependence) Assume that $\{\varphi^j\}_{j\in\mathbb{N}} \subset H^s(\mathbb{T}), \varphi \in H^s(\mathbb{T})$ satisfy $P_N(\varphi) > 0$ (resp. $P_N(\varphi) < 0$)) and $\|\varphi^j - \varphi\|_{H^s} \to 0$ as $j \to \infty$. Let u^j (resp. u) be the solution obtained above with initial data φ^j (resp. φ) and $T = T(P_N(\varphi), \|\varphi\|_{H^{12}})$. Then $\sup_{t\in[0,T]} \|u^j(t) - u(t)\|_{H^s(\mathbb{T})} \to 0$ (resp. $\sup_{t\in[-T,0]} \|u^j(t) - u(t)\|_{H^s(\mathbb{T})} \to 0$) as $j \to \infty$.

Theorem 2.3 (non existence of parabolic type). *Let $\varphi \notin C^\infty(\mathbb{T})$ and $J_F^{(5,2)}(\varphi) < 0$ (resp. $J_F^{(5,2)}(\varphi) > 0$). Then, for any small $T > 0$, there does not exist any solution to (2.1)–(2.2) on $[0,T]$ (resp. $[-T,0]$) satisfying $u \in C([0,T]; H^{13}(\mathbb{T}))$ (resp. $u \in C([-T,0]; H^{13}(\mathbb{T}))$).*

Theorem 2.1 is a typical result for dispersive equations in the following sense: they can be solved on both positive and negative time intervals and the regularity of the solution is same as that of initial data. Theorems 2.2 and 2.3 are typical results for parabolic equations in the following sense: they can be solved on either positive or negative time interval with strong smoothing effect and they are ill-posed on the other time interval. Since (2.1) are semilinear dispersive equations, Theorem 2.1 seems to be a natural result. On the other hand, Theorems 2.2 and 2.3 are somewhat surprising. These theorems mean that when $J_F^{(5,2)} \not\equiv 0$, the nonlinear term cannot be treated as a perturbation of the linear part and the effect by the second derivative in the nonlinear part is dominant.

In [3], we consider (1.1), (1.3) with $j = 1$, that is

$$(\partial_t + i\partial_x^2)u = F(\partial_x u, \partial_x \overline{u}, u, \overline{u}), \quad (t,x) \in [-T,T] \times \mathbb{T}, \tag{2.3}$$

$$u(0,x) = \varphi \in H^s(\mathbb{T}). \tag{2.4}$$

For sufficiently smooth function f on $\mathbb{T}$, we put

$$J_F^{(2,1)}(f) := \frac{1}{2\pi}\int_{\mathbb{T}} \partial_{\omega_1} F(\omega_1,\overline{\omega_1},\omega_0,\overline{\omega_0})\Big|_{(\omega_1,\omega_0)=(\partial_x f(x),f(x))} dx.$$

We say $\operatorname{Im} J_F^{(2,1)} \equiv 0$ if $\operatorname{Im} J_F^{(2,1)}(f) = 0$ holds for any $f \in C^\infty(\mathbb{T})$.

Theorem 2.4 (L.W.P. of dispersive type). *Assume that* $\operatorname{Im} J_F^{(2,1)} \equiv 0$. *For any* $s \geq s_0 > 5/2$ *and* $K > 0$, *there exists* $T = T(K, s_0) > 0$ *which satisfies the followings:*
(Existence) For any complex valued function $\varphi \in H^s(\mathbb{T})$ *satisfying* $\|\varphi\|_{H^{s_0}} \leq K$, *there exists a solution* $u \in C([-T,T]; H^s(\mathbb{T}))$ *to* (2.3)–(2.4) *on* $[-T,T]$.
(Uniqueness) If $u_1, u_2 \in C([-\delta_1,\delta_2]; H^{s_0}(\mathbb{T}))$ *satisfy* (2.3)–(2.4) *on* $[-\delta_1,\delta_2]$ *for* $\delta_1,\delta_2 \geq 0$, *then* $u_1(t) = u_2(t)$ *on* $[-\delta_1,\delta_2]$.
(Continuous dependence) Assume that $\varphi^{(j)} \to \varphi^{(\infty)}$ *in* $H^s(\mathbb{T})$ *as* $j \to \infty$ *and* $\|\varphi^{(j)}\|_{H^{s_0}}, \|\varphi^{(\infty)}\|_{H^{s_0}} \leq K$. *Let* $u^{(j)}$ *(resp.* $u^{(\infty)}$*)* $\in C([-T,T]; H^s(\mathbb{T}))$ *be the solution to* (2.3)–(2.4) *on* $[-T,T]$ *with* $\varphi = \varphi^{(j)}$ *(resp.* $\varphi = \varphi^{(\infty)}$*). Then,* $u^{(j)} \to u^{(\infty)}$ *in* $C([-T,T]; H^s(\mathbb{T}))$.

Theorem 2.5 (non-existence). *Assume that* $\operatorname{Im} J_F^{(2,1)} \not\equiv 0$. *For any* $s > 5/2$, $T > 0$, *there exists* $\varphi \in H^s(\mathbb{T})$ *such that no solution* u *of* (2.3)–(2.4) *on* $[0,T]$ *exist in* $C([0,T]; H^s(\mathbb{T}))$ *and no solution* u *of* (2.3)–(2.4) *on* $[-T,0]$ *exist in* $C([-T,0]; H^s(\mathbb{T}))$.

Theorem 2.4 is a typical result for dispersive equations. These theorems mean that when $\operatorname{Im} J_F^{(2,1)} \not\equiv 0$, the nonlinear term cannot be treated as a perturbation of the linear part and the effect by the first derivative in the nonlinear part is dominant.

3. IDEA AND CONJECTURES

In this section, we explain the idea of Theorems 2.1–2.5 intuitively. Note that it is not rigorous proof. We use the energy method to prove Theorems 2.1–2.5. First, we explain the proof of Theorems 2.1–2.3. We consider the parabolic regularized equation of (2.1):

$$(\partial_t + \varepsilon\partial_x^4 + \partial_x^5)u_\varepsilon = F(\partial_x^3 u_\varepsilon, \partial_x^2 u_\varepsilon, \partial_x u_\varepsilon, u_\varepsilon), \quad (t,x) \in [-T,T] \times \mathbb{T}, \tag{3.1}$$

$$u_\varepsilon(0,x) = \varphi, \tag{3.2}$$

where $\varepsilon \in (0,1]$. By the presence of $+\varepsilon\partial_x^4$, the linear part has strong smoothing effect and we can easily prove the local well-poedness of (3.1)–(2.2) on $[0, T_\varepsilon)$, where T_ε is the maximal of the time interval of the existence. Taking $\varepsilon \to 0$, we construct a solution of (2.1)–(2.4) as the limit of the solution of (3.1)–(3.2). In this process, it is important to ensure that T_ε does not shrink to 0. T_ε has the property such that $T_\varepsilon = \infty$ or $\liminf_{t\to T_\varepsilon} \|u_\varepsilon(t)\|_{H^s} \to \infty$. Therefore, we obtain $T_\varepsilon \geq T > 0$ if we have $\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s} \leq C$. For that purpose we need to prove a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ To prove the continuous dependence, a priori estimate is not enough and we also need an estimate for the difference of two solutions with different initial condition, which is much complicated. For simplicity, we omit it and explain only how to obtain a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$. Differentiating (3.1) $j \in \mathbb{N}$ times, by the chain rule, we obtain

$$\begin{aligned}(\partial_t + \varepsilon\partial_x^4 + \partial_x^5)\partial_x^j u_\varepsilon =& \partial_{\omega_3} F \partial_x^{j+3} u_\varepsilon + (j\partial_x\partial_{\omega_3}F + \partial_{\omega_2}F)\partial_x^{j+2}u_\varepsilon \\ & \left(\frac{j(j-1)}{2}\partial_x^2\partial_{\omega_3}F + j\partial_x\partial_{\omega_2}F + \partial_{\omega_1}F\right)\partial_x^{j+1}u_\varepsilon + \cdots,\end{aligned} \tag{3.3}$$

where the terms which include only $\partial_x^k u$ ($k = 0, \ldots, j$) are omitted since the difficulty comes from "derivative loss".

Here, we consider the following linear equation of constant coefficients:

$$(\partial_t + \varepsilon\partial_x^4 + \partial_x^5)\partial_x^j u_\varepsilon = a\partial_x^{j+3}u_\varepsilon + b\partial_x^{j+2}u_\varepsilon + c\partial_x^{j+1}u_\varepsilon + d, \tag{3.4}$$

where $a, b, c, d, \in \mathbb{R}$. We compute the inner products as below:

$$\begin{aligned}&\langle(\partial_t + \varepsilon\partial_x^4 + \partial_x^5)\partial_x^j u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = \frac{1}{2}\frac{d}{dt}\|\partial_x^j u_\varepsilon\|_{L^2}^2 + \varepsilon\|\partial_x^{j+2}u_\varepsilon\|_{L^2}^2, \\ &\langle a\partial_x^{j+3}u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = \langle c\partial_x^{j+1}u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = 0, \\ &\langle b\partial_x^{j+2}u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = -b\|\partial_x^{j+1}u_\varepsilon\|_{L^2}^2, \quad \langle d, \partial_x^j u_\varepsilon\rangle_{L^2} \leq C(d)\|u_\varepsilon\|_{H^j}^2.\end{aligned}$$

Here, we used the integration by parts. Therefore, we obtain the following energy inequality for the solution of (3.4):

$$\frac{d}{dt}\|\partial_x^j u_\varepsilon\|_{L^2}^2 + 2b\|\partial_x^{j+1}u_\varepsilon\|_{L^2}^2 \leq C\|u\|_{H^j}^2.$$

Thus, we have a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ if $b \geq 0$. If $b > 0$, we also have a priori estimate of $\|\partial_x^{s+1}u_\varepsilon\|_{L^2}$. By this, we obtain the parabolic smoothing effect. Of course, the nonlinear equation (3.3) is much more difficult to treat than (3.4). But roughly speaking, b in (3.4) corresponds to

$$j\partial_x\partial_{\omega_3}F + \partial_{\omega_2}F = \left(j\partial_x\partial_{\omega_3}F + \partial_{\omega_2}F - \frac{1}{2\pi}\int_{\mathbb{T}} j\partial_x\partial_{\omega_3}F + \partial_{\omega_2}F\,dx\right) + J_F^{(5,2)}(u)$$

in (3.3). The first term does not have 0 mode in the Fourier space with x. By the effect of the oscillation, we can deal with this term and we have $b \sim J_F^{(5,2)}(u) \sim$

$J_F^{(5,2)}(\varphi)$ on sufficiently small interval $[0,T]$. Therefore, we obtain a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ for (3.1)–(3.2) when $J_F^{(5,2)}(\varphi) \geq 0$. We also obtain the smoothing effect when $J_F^{(5,2)}(\varphi) > 0$. Therefore, we have Theorems 2.1, 2.2. The smoothing effect on positive time direction $t : t_1 \to t_2$ means the solution lose smoothness on negative time direction $t : t_2 \to t_1$ and we obtain Theorem 2.3.

Next, we explain the idea of Theorems 2.4, 2.5 intuitively. We consider the parabolic regularized equation of (2.3):

$$(\partial_t - \varepsilon\partial_x^2 + i\partial_x^2)u_\varepsilon = F(\partial_x u_\varepsilon, \partial_x \overline{u}_\varepsilon, u_\varepsilon, \overline{u}_\varepsilon), \quad (t,x) \in [-T,T] \times \mathbb{T}, \tag{3.5}$$

$$u_\varepsilon(0,x) = \varphi, \tag{3.6}$$

In the same manner as (3.3), we obtain

$$(\partial_t - \varepsilon\partial_x^2 + i\partial_x^2)\partial_x^j u_\varepsilon = \partial_{\omega_1} F \partial_x^{j+1} u_\varepsilon + \partial_{\overline{\omega}_1} F \partial_x^{j+1} \overline{u}_\varepsilon + \cdots, \tag{3.7}$$

Thus, we consider the following linear equation of constant coefficients:

$$(\partial_t - \varepsilon\partial_x^2 + i\partial_x^2)\partial_x^j u_\varepsilon = a\partial_x^{j+1} u_\varepsilon + b\partial_x^{j+1}\overline{u}_\varepsilon + c, \tag{3.8}$$

where $a, b, c \in \mathbb{C}$. We compute the inner products as below:

$$\begin{aligned}
&\operatorname{Re}\langle(\partial_t - \varepsilon\partial_x^2 + i\partial_x^2)\partial_x^j u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = \frac{1}{2}\frac{d}{dt}\|\partial_x^j u_\varepsilon\|_{L^2}^2 + \varepsilon\|\partial_x^{j+1} u_\varepsilon\|_{L^2}^2, \\
&\operatorname{Re}\langle a\partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} = (\operatorname{Re} a)\operatorname{Re}\langle\partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} - (\operatorname{Im} a)\operatorname{Im}\langle\partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} \\
&\qquad = -(\operatorname{Im} a)\operatorname{Im}\langle\partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} \\
&|\langle b\partial_x^{j+1}\overline{u}_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2}| = \frac{b}{2}\int_{\mathbb{T}} \partial_x(\partial_x^j \overline{u}_\varepsilon)^2\, dx, = 0 \\
&|\langle c, \partial_x^j u_\varepsilon\rangle_{L^2}| \leq C(c)\|u\|_{H^j}^2.
\end{aligned}$$

Thus, we obtain the following energy inequality for the solution of (3.8):

$$\frac{d}{dt}\|\partial_x^j u_\varepsilon\|_{L^2}^2 + 2(\operatorname{Im} a)\operatorname{Im}\langle\partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2} \leq C\|u_\varepsilon\|_{H^j}^2.$$

Thus, we have a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ if $\operatorname{Im} a = 0$. Of course, the nonlinear equation (3.7) is much more difficult to treat than (3.8). But roughly speaking, $\operatorname{Im} a$ in (3.8) corresponds to

$$\operatorname{Im}\partial_{\omega_1} F = \operatorname{Im}\left(\partial_{\omega_1} F - \frac{1}{2\pi}\int_{\mathbb{T}} \partial_{\omega_1} F\, dx\right)\right) + \operatorname{Im} J_F^{(2,1)}(u)$$

in (3.7). The first term does not have 0 mode in the Fourier space with x. By the effect of the oscillation, we can deal with this term. Therefore, we obtain a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ for (3.5)–(3.6) when $\operatorname{Im} J_F^{(2,1)} \equiv 0$ and have Theorem 2.4. Since $\operatorname{Im}\langle \partial_x^{j+1} u_\varepsilon, \partial_x^j u_\varepsilon\rangle_{L^2}$ is not positive definite, we can not have a priori estimate of $\|u_\varepsilon(t)\|_{H^s}$ when $\operatorname{Im} J_F^{(2,1)} \not\equiv 0$. To avoid this difficulty, we introduce the restriction operator in the Fourier space, $P_+ = \mathcal{F}^{-1} 1_{\xi>0} \mathcal{F}$, $P_- = \mathcal{F}^{-1} 1_{\xi<0} \mathcal{F}$ and we compute $\|P_\pm \partial_x^j u_\varepsilon\|_{L^2}$ instead of $\|\partial_x^j u_\varepsilon\|_{L^2}$. Then, we can obtain the energy inequality for one of $\|P_+ \partial_x^j u_\varepsilon\|_{L^2}$ or $\|P_- \partial_x^j u_\varepsilon\|_{L^2}$. By this, we obtain Theorem 2.5.

Finally, we mention some conjectures. We consider (1.2) with $j \geq 3$. By the analogy of the argument above, The author guess the following: if $J_F^{(2j+1,l)} \equiv 0$ holds for all $l = 2, 4, \ldots, 2j-2$, then L.W.P. of dispersive type similar to Theorem 2.1 holds, and if $J_F^{(2j+1,l)} \not\equiv 0$ holds at least for one of $l = 2, 4, \ldots, 2j-2$, then L.W.P. of parabolic type similar to Theorem 2.2 and non existence of parabolic type similar to Theorem 2.3 hold. We believe the method used in the proof of Theorems 2.1–2.3 also works for this conjecture. But, it is difficult to find the functional $J_F^{(2j+1,l)}$. We believe $J_F^{(2j+1,2j-2)}$ should be defined by

$$J_F^{(2j+1,2j-2)}(f) := \frac{1}{2\pi} \int_{\mathbb{T}} \partial_{\omega_{2j-2}} F(\omega_{2j-1}, \ldots, \omega_0)\Big|_{(\omega_{2j-2},\ldots,\omega_0)=(\partial_x^{2j-2} f(x),\ldots,f(x))} dx.$$

However, $J_F^{(2j+1,l)}$ for small l must be much complicated.

The problem for (1.1) with $j \geq 2$ is more complicated than (1.2) because u is comlex valued. The author guess that there exist functionals and we obtain theorems similar to Theorems 2.1–2.5, but the functionals and the conditions should be much complicated. Mizuhara, studied the well-posedness of the fourth order linear equations of variable coefficients in [1]. By his results, we know that the functionals and the conditions are complicated even for the linear case.

References

[1] R. Mizuhara, *The initial value problem for third and fourth order dispersive equations in one space dimension*, Funkcial. Ekvac. **49** (2006), no. 1, 1–38.

[2] K. Tsugawa, *Parabolic smoothing effect and local well-posedness of fifth order semilinear dispersive equations on the torus*, `arXiv:1707.09550 [math.AP]`

[3] K. Tsugawa, *Local well-posedness of derivative Schrödinger equations on the torus*, in preparation.

E-mail address, K. Tsugawa: `tsugawa@math.chuo-u.ac.jp`