数理解析研究所講究録 第2093巻 2018年 65-75

THE GINZBURG-LANDAU MODEL WITH A VANISHING EXTERIOR MAGNETIC FIELD

BERNARD HELFFER AND AYMAN KACHMAR

ABSTRACT. This is a survey on various results on the Ginzburg-Landau functional with an exterior magnetic field changing sign along a smooth curve. This affects the distribution of bulk superconductivity in the large Ginzburg-Landau parameter regime. Furthermore, it features a concentration of the minimizing order parameter on the magnetic zero set.

1. THE GINZBURG-LANDAU MODEL

1.1. **Physical background.** Superconductors enjoy the property of loosing electrical resistance when cooled below a critical temperature. At the same time, when placed in an applied magnetic field (whose intensity is not too large), they repel the applied magnetic field. In 1950, Ginzburg and Landau proposed their phenomenological model on superconductors placed in an applied magnetic field. This theory allows for the classification of superconductors into two classes, Type I and Type II materials, based on their response to the applied magnetic field. Type I materials may have two states, the pure superconducting (Meissner state) or the pure normal state. Type II materials may have one more state, the mixed state where normal and superconducting regions coexist in the sample.

We will describe the Ginzburg-Landau model for a superconducting sample having the following typical geometry: A long cylinder with a vertical axis and a horizontal cross section $\Omega \subset \mathbb{R}^2$, placed in a vertical applied magnetic field (the magnetic field can be homogeneous or non-homogeneous). A characteristic scale of the sample is the Ginzburg-Landau parameter, $\kappa > 0$. For high temperature type II materials, this parameter is very large compared to the sample's size, so we will focus on the extreme asymptotic regime $\kappa \to +\infty$.

The applied magnetic field is of the form $H\,B_0\vec{e}$, where $\vec{e} = (0,0,1)$, $B_0 : \overline{\Omega} \to \mathbb{R}$ is a given function (the profile of the magnetic field) and $H > 0$ is a parameter measuring the intensity of the magnetic field.

We assume that the sample's cross section Ω is open, bounded, simply connected and with a smooth boundary. For a given (κ, H), the state of the material is described by the configuration $(\psi, \mathbf{A})_{\kappa,H}$, where $\psi : \Omega \to \mathbb{C}$ is the (complex-valued) order parameter and the vector field $\mathbf{A} : \Omega \to \mathbb{R}^2$ is the magnetic potential. At equilibrium, this configuration has to minimize the Ginzburg-Landau free energy, given in non-dimensional form as follows,

$$\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) = \int_\Omega e_{\kappa,H}(\psi, \mathbf{A})\,dx \tag{1.1}$$

where

$$e_{\kappa,H}(\psi, \mathbf{A}) := |(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + (\kappa H)^2|\operatorname{curl}\mathbf{A} - B_0|^2$$

is the Ginzburg-Landau *energy density*. We sometimes omit in the notation the dependence on (κ, H) for the energy and write $\mathcal{E}(\psi, \mathbf{A})$ for simplicity.

Date: September 7, 2017.

The configuration $(\psi, \mathbf{A})$ is a solution of the following Ginzburg-Landau equations (ν is the unit inward normal on the boundary)

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi &= \kappa^2(1-|\psi|^2)\psi & \text{in } \Omega\,, \\ -\nabla^\perp(\operatorname{curl}\mathbf{A} - B_0) &= (\kappa H)^{-1}\mathrm{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{in } \Omega\,, \\ \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi &= 0 & \text{on } \partial\Omega\,, \\ \operatorname{curl}\mathbf{A} &= B_0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

and it is interpreted as follows: $|\psi|^2$ measures the density of superconducting electrons; $\operatorname{curl}\mathbf{A}$ measures the induced magnetic field in the sample; and $\mathbf{j} = \mathrm{Im}\left(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi\right)$ measures the induced electric (super)-current. The order parameter ψ satisfies the celebrated universal bound (see [13, Prop. 10.3.1])

$$\|\psi\|_\infty \leq 1\,. \tag{1.3}$$

Then we can describe the various states of the sample as follows:

- The pure superconducting state: $|\psi| > 0$ everywhere.
- The pure normal state: $\psi \equiv 0$ and $\operatorname{curl}\mathbf{A} \equiv B_0$;
- The mixed state: $|\psi| \not\equiv 0$ and $\operatorname{curl}\mathbf{A} \not\equiv 0$.

1.2. **The constant magnetic field case.** In the case of a homogeneous applied magnetic field (which correponds simply to taking $B_0 \equiv 1$ in (1.1)) there exists a huge mathematical literature devoted to the analysis of the Ginzburg-Landau functional in (1.1). We refer to the two monographs [13] and [31] and the references therein. In the asymptotic regime $\kappa \to +\infty$, the existing results distinguish between three phase transitions related to three threshold values of the applied magnetic field (these are the critical fields H_{C_1}, H_{C_2} and H_{C_3}). These phase transitions can be described as follows (in terms of the variations of the parameter H measuring the intensity of the applied magnetic field) :

- For $H < H_{C_1}$, the sample is in the pure superconducting states;
- For $H_{C_1} < H < H_{C_2}$ the sample is in the mixed state;
- For $H_{C_2} < H < H_{C_3}$, the bulk of the sample is in the normal state, while the surface of the sample is in the superconducting state;
- For $H > H_{C_3}$, the sample is in the pure normal state (i.e. $\psi \equiv 0$ and $\operatorname{curl}\mathbf{A} \equiv 0$).

The foregoing discussion of the phase transitions can be seen through the variations of the following ground state energy with respect to the parameter H (as $\kappa \to +\infty$)

$$\mathrm{E_{gs}}(\kappa, H) = \inf\{\mathcal{E}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)\}\,, \tag{1.4}$$

where $\mathcal{E}(\psi, \mathbf{A})$ is the energy introduced in (1.1).

The regime $H_{C_1} < H < H_{C_2}$ is of particular interest because the mixed state is associated with a lattice structure in the following sense (this happens when H is very close to H_{C_2}). The function ψ measuring the density of superconductivity is evenly distributed in the sample with isolated zeros arranged along a lattice (these zeros are called *vortices*). That has been discovered by Abrikosov in [1]. Many mathematical contributions are devoted to the proof of this property [2, 3, 16, 23, 29], which is still a celebrated open problem in this field.

1.3. **The non-degenerately vanishing magnetic field case.** In this survey, we will focus on the situation where the applied magnetic field is allowed to vanish along a smooth curve. That is, we assume that the function B_0 (appearing in (1.1)) satisfies the following assumptions:

$$|B_0| + |\nabla B_0| > 0 \quad \text{in } \overline{\Omega}\,; \tag{1.5}$$

$$\Gamma = \{x \in \overline{\Omega} \ : \ B_0(x) = 0\} \text{ consists of a finite number of simple smooth curves}\,; \tag{1.6}$$

$$\Gamma \cap \partial\Omega \quad \text{is a finite set}\,; \tag{1.7}$$

$$\Gamma \text{ intersects } \partial\Omega \quad \text{transversally}\,. \tag{1.8}$$

Our assumptions in (1.5)-(1.8) force the function B_0 to change sign. In physical terms, the set Γ splits the domain Ω into two parts $\Omega_1 = \{B_0(x) > 0\}$ and $\Omega_2 = \{B_0(x) < 0\}$ such that the magnetic field applied on Ω_1 is along the opposite direction of the magnetic field applied on Ω_2.

The assumptions in (1.5)-(1.8) on the function B_0 are discussed in various contexts involving magnetic fields:

- The spectral analysis of the magnetic Schrödinger operator [11, 18, 22, 26];
- The time-dependent Ginzburg-Landau equations with an applied electric currents [4, 5];
- Superconducting surfaces placed in a constant magnetic field [9].

Many papers are devoted to the analysis of the Ginzburg-Landau functional in (1.1) with the magnetic field satisfying the assumptions in (1.5)-(1.8); see [6, 7, 8, 19, 20, 21, 25, 28]. We will review these results in the next sections. Compared with the existing results on the constant magnetic field case ($B_0 \equiv 1$), we record the following features:

- When the mixed state exists, the order parameter is no more evenly distributed in the domain Ω and (vortex) lattices do not show up.
- Superconductivity persists up to magnetic fields of order κ^2, much more than for samples placed in a uniform magnetic field.
- There exists a regime where the order parameter concentrates on the magnetic zero set Γ, which is the analogue of the surface superconductivity regime for samples placed in a uniform magnetic field.

2. The breakdown of superconductivity

In the constant magnetic field case (i.e. $B_0 \equiv 1$ in (1.1)), Giorgi-Phillips [12] prove that for sufficiently large values of H, the only critical points of (1.1) are the pure normal states, i.e. $\psi \equiv 0$ and $\operatorname{curl} \mathbf{A} \equiv B_0$. The same result holds in the case of a non-degenerately vanishing magnetic field (i.e. B_0 satisfying (1.5)-(1.8)). We will present this result here.

2.1. **Auxiliary operators.**

2.1.1. *The Montgomery operator.* We will introduce the constant $\lambda_0 > 0$ in (2.2) below. This is the ground state energy of the following self-adjoint operator in $L^2(\mathbb{R}^2)$,

$$P = -\left(\partial_{x_1} - i\frac{x_2^2}{2}\right)^2 - \partial_{x_2}^2\,; \tag{2.1}$$

that is

$$\lambda_0 = \inf \sigma(P)\,. \tag{2.2}$$

We can compute the constant λ_0 using the family of Montgomery one-dimensional operators as follows. For $\tau \in \mathbb{R}$, let $\lambda(\tau)$ be the ground state energy of the operator

$$P(\tau) = -\frac{d^2}{dx_2^2} + \left(\frac{x_2^2}{2} + \tau\right)^2 \quad \text{in } L^2(\mathbb{R})\,. \tag{2.3}$$

Then, by separation of variables, $\lambda_0 = \inf_{\tau\in\mathbb{R}} \lambda(\tau)$. By [17], there exists a unique $\tau_0 < 0$ such that

$$\lambda_0 = \lambda(\tau_0)\,, \tag{2.4}$$

and this minimum is non degenerate.

2.1.2. *The half-plane operator.* Here we will consider the analogue of the operator in (2.1) in the half-plane $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_2 > 0\}$. For $\theta \in [0, \pi/2]$, define the magnetic potential

$$\mathbf{A}_\theta(x) = -\frac{|x|^2}{2}(\cos\theta, \sin\theta) \quad (x = (x_1, x_2))\,,$$

which generates the following magnetic field

$$\operatorname{curl} \mathbf{A}_\theta(x) = x_2 \cos\theta - x_1 \sin\theta\,,$$

which vanishes along the line $x_2 \cos\theta = x_1 \sin\theta$. Note that θ is the angle between the line $\{\operatorname{curl} \mathbf{A}_\theta = 0\}$ and the boundary of $\mathbb{R}^2_+$.

We introduce the following positive constant

$$\zeta(\theta) = \inf \sigma\big(\mathcal{L}(\theta)\big)\,, \tag{2.5}$$

where $\mathcal{L}(\theta)$ is the self-adjoint operator in $L^2(\mathbb{R}^2_+)$,

$$\mathcal{L}(\theta) = -(\nabla - i\mathbf{A}_\theta)^2\,,$$

with Neumann boundary condition on $\partial\mathbb{R}^2_+ = \{x_2 = 0\}$.

2.2. **Extension of the Giorgi-Phillips theorem.** The Giorgi-Phillips theorem is essentially a consequence of bounds on the principal eigenvalue of the magnetic Schrödinger operator in the domain Ω (see [13, Sec. 10.4]).

Let $\mathbf{A}_0$ be the unique vector field in Ω such that $\operatorname{curl}\mathbf{A}_0 = B_0$, $\operatorname{div}\mathbf{A}_0 = 0$ and $\nu\cdot\mathbf{A}_0 = 0$, where ν is the unit inward normal vector of $\partial\Omega$. The function B_0 satisfies the assumptions in (1.5)-(1.8).

For $b > 0$, let $\lambda(b)$ be the ground state energy of the self-adjoint operator in $L^2(\Omega)$

$$-(\nabla - ib\mathbf{A}_0)^2$$

with the Neumann boundary condition on $\partial\Omega$.

Pan-Kwek [28] computed the leading order term of $\lambda(b)$ as $b \to +\infty$. The precise result is:

Theorem 2.1. *If B_0 satisfies (1.5)-(1.8), then*

$$\lim_{b\to+\infty} \frac{\lambda(b)}{b^{2/3}} = \min\left(\lambda_0^{3/2} \inf_{x\in\Gamma\cap\Omega} |\nabla B_0(x)|\,,\ \inf_{x\in\Gamma\cap\partial\Omega}\Big(\zeta(\theta(x))^{3/2}|\nabla B_0(x)|\Big)\right)\,,$$

where

- *λ_0 is the constant in (2.2);*
- *Γ is the magnetic zero set introduced in (1.6);*
- *$\theta(x)$ is the acute angle between $\partial\Omega$ and Γ at x.*

As a consequence of Theorem 2.1 and standard *a priori* estimates of solutions of (1.2), Attar [8, Thm. 8.5] obtains the following generalization of the Giorgi-Phillips theorem.

Theorem 2.2. *Assume that B_0 satisfies (1.5)-(1.8). There exists a constant $C > 0$ such that, for all $\kappa > 0$ and $H \geq C\max(\kappa^2, 1)$, every solution $(\psi, \mathbf{A})_{\kappa,H}$ of (1.2) satisfies*

$$\psi \equiv 0 \quad\text{and}\quad \operatorname{curl}\mathbf{A} \equiv B_0\,.$$

With Theorem 2.2 in hand, we can define the critical field

$$H_{C_3}(\kappa) = \sup\{H > 0 \ :\ \text{ if } (\psi, \mathbf{A})_{\kappa,H} \text{ is a solution of (1.2) then } \psi \equiv 0\,\&\, \operatorname{curl}\mathbf{A} \equiv B_0\}\,. \tag{2.6}$$

Theorem 2.1 then yields the following asymptotics for the critical field [28, 8]

$$\lim_{\kappa\to+\infty} \frac{H_{C_3}(\kappa)}{\kappa^2} = \max\left(\lambda_0^{-3/2} \sup_{x\in\Gamma\cap\Omega} |\nabla B_0(x)|^{-1}\,,\ \sup_{x\in\Gamma\cap\partial\Omega}\Big(\zeta(\theta(x))^{-3/2}|\nabla B_0(x)|^{-1}\Big)\right)\,. \tag{2.7}$$

Other reasonable definitions of the critical field $H_{C_3}(\kappa)$ are discussed in [13, Ch. 13] and [8], all of them satisfying the asymptotics in (2.7). Attar [8] discussed the equality between the various definitions of H_{C_3}, following the investigations by Fournais-Helffer on the same question in the constant magnetic field case [13, Ch. 13].

3. Reference energies

In the forthcoming Sections 4 and 5, we will present the asymptotics of the ground state energy $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$ in (1.4) for various values of H when $\kappa \to +\infty$. These asymptotics are displayed via simplified quantities, *the reference energies*, which we present now.

3.1. **The bulk energy.** We recall the construction of a celebrated function that describes the energy of the Ginzburg-Landau model with constant magnetic field [2, 15, 32].

For $b \in (0,\infty)$, $r > 0$, and $Q_r = (-r/2, r/2) \times (-r/2, r/2)$, define the functional,

$$F_{b,Q_r}(u) = \int_{Q_r} \left(b|(\nabla - i\mathbf{a}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx\,, \qquad \text{for } u \in H^1(Q_r)\,. \tag{3.1}$$

Here, $\mathbf{a}_0$ is the magnetic potential,

$$\mathbf{a}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \quad (x = (x_1, x_2) \in \mathbb{R}^2)\,, \tag{3.2}$$

which generates the constant unit magnetic field, $\operatorname{curl} \mathbf{a}_0 = 1$.

Define the Dirichlet and Neumann ground state energies as follows

$$e_D(b,r) = \inf\{F_{b,Q_r}(u) \ :\ u \in H_0^1(Q_r)\}\,, \tag{3.3}$$

$$e_N(b,r) = \inf\{F_{b,Q_r}(u) \ :\ u \in H^1(Q_r)\}\,. \tag{3.4}$$

It is known [7, 15, 32] that the following two limits exist and are equal

$$\lim_{r\to\infty} \frac{e_D(b,r)}{|Q_r|} = \lim_{r\to\infty} \frac{e_N(b,r)}{|Q_r|}\,,$$

where $|Q_r|$ denotes the area of the square Q_r ($|Q_r| = r^2$).

Thus, we define the function $g(\cdot)$ as follows (this is the bulk energy)

$$\forall\, b > 0\,, \quad g(b) = \lim_{r\to\infty} \frac{e_D(b,r)}{|Q_r|} = \lim_{r\to\infty} \frac{e_N(b,r)}{|Q_r|}\,. \tag{3.5}$$

The function $g(\cdot)$ has the following properties. First, there exists a constant C such that, for all $r \geq 1$ and $b > 0$,

$$g(b) - C\frac{\sqrt{b}}{r} \leq \frac{e_N(b,r)}{|Q_r|} \leq \frac{e_D(b,r)}{|Q_r|} \leq g(b) + C\frac{\sqrt{b}}{r}\,; \tag{3.6}$$

see [7, 15]. Furthermore, the function g is continuous, monotone increasing and satisfies [15, 32]

$$g(0) = -\frac{1}{2} \text{ and } g(b) = 0 \text{ when } b \geq 1\,. \tag{3.7}$$

The asymptotic behavior of the function $g(b)$ as $b \to 1_-$ was obtained in [2, 15], namely the following limit exists

$$\lim_{b\to 1_-} \frac{g(b)}{(b-1)^2} = E_{\mathrm{Ab}}\,, \tag{3.8}$$

where $E_{\mathrm{Ab}} \in [-\frac{1}{2}, 0)$. However, an estimate of the remainder term in (3.8) is not available yet. The constant E_{Ab} is related to the work of Abrikosov [1], hence called the Abrikosov constant [23]. It can be evaluated through a simplified energy defined on a lattice domain [2, 15].

Finally, we present the asymptotics of the function $g(b)$ as $b \to 0_+$,

$$g(b) = -\frac{1}{2} + \frac{1}{2} b \ln\frac{1}{b} + o\left(b \ln \frac{1}{b} \right)\,. \tag{3.9}$$

This formula was proved in [24]. It provides a link between the works [31] and [32].

3.2. **The magnetic zero line energy.**

3.2.1. *Definition of the energy.*

We now present a second reference energy, $E(\cdot)$, which involves a simplified Ginzburg-Landau functional with a magnetic field vanishing along a line. We introduce the following magnetic potential

$$\mathbf{F}(x) = \Big(-\frac{x_2^2}{2}, 0\Big), \quad \Big(x = (x_1, x_2) \in \mathbb{R}^2\Big). \tag{3.10}$$

This magnetic potential generates the magnetic field $\operatorname{curl} \mathbf{F} = x_2$ which vanishes along the line $x_2 = 0$. For $L > 0$ and $R > 0$, we introduce the domain $\mathcal{S}_R = (-R, R) \times \mathbb{R}$ and the following functional

$$\mathcal{E}_{L,R}(u) = \int_{\mathcal{S}_R} \left(|(\nabla - i\mathbf{F})u|^2 - L^{-2/3}|u|^2 + \frac{L^{-2/3}}{2}|u|^4 \right) dx\,, \tag{3.11}$$

with the ground state energy

$$\mathfrak{e}_{\rm gs}(L;R) = \inf\{\mathcal{E}_{L,R}(u) \ : \ (\nabla - i\mathbf{F})u \in L^2(\mathcal{S}_R)\,,\ u \in L^2(\mathcal{S}_R)\,,\ \text{and } u = 0 \text{ on } \partial\mathcal{S}_R\}\,. \tag{3.12}$$

The magnetic zero line energy $E(\cdot)$ is defined through the following theorem.

Theorem 3.1. ([19, Thm. 3.8])

(1) *For all $L > 0$, there exists $E(L) \leq 0$ such that*

$$\lim_{R\to\infty} \frac{\mathfrak{e}_{\rm gs}(L;R)}{2R} = E(L)\,.$$

(2) *The function $(0,\infty) \ni L \mapsto E(L) \in (-\infty, 0]$ is continuous, monotone increasing and*

$$E(L) = 0 \quad \textit{if and only if } L \geq \lambda_0^{-3/2}\,,$$

where λ_0 is the eigenvalue introduced in (2.2).

(3) *There exists a constant $C > 0$ such that*

$$\forall\, R \geq 2\,, \forall L > 0\,, \quad E(L) \leq \frac{\mathfrak{e}_{\rm gs}(L;R)}{2R} \leq E(L) + C\left(1 + L^{-2/3}\right) R^{-2/3}\,.$$

3.2.2. *Relation with the bulk energy.*
The asymptotic behavior of $E(L)$ as $L \to 0_+$ has been obtained in [20]; it is related to the bulk energy $g(\cdot)$.

Theorem 3.2. ([20, Thm. 1.2])

Let $E(\cdot)$ be as in Theorem 3.1. It holds

$$E(L) = 2L^{-4/3}\int_0^1 g(b)\,db + o(L^{-4/3}) \quad (L \to 0_+)\,,$$

where $g(\cdot)$ is the bulk energy introduced in (3.5).

3.2.3. *Relation with a* 1*D functional.*
The existing literature on the Ginzburg-Landau functional with a constant magnetic field [4, 10] indicates a simpler approach to define the reference energy $E(\cdot)$ in Theorem 3.1. Namely, it would be desirable to define it starting from a one-dimensional energy functional. The natural candidate is defined as follows. For $L > 0$ and $\alpha \in \mathbb{R}$, we introduce the energy functional (compare with the 2D functional in (3.11))

$$\mathcal{E}^{1D}_{\alpha,L}(f) = \int_{-\infty}^{\infty} \left(|f'(t)|^2 + \left(\frac{t^2}{2} + \alpha\right)^2 |f(t)|^2 - L^{-2/3}\,|f(t)|^2 + \frac{L^{-2/3}}{2}\,|f(t)|^4 \right) dt\,, \tag{3.13}$$

defined over configurations in the space

$$B^1(\mathbb{R}) = \{f \in H^1(\mathbb{R};\mathbb{R}) \ : \ t^2 f \in L^2(\mathbb{R})\}\,.$$

We introduce the ground state energy of the functional in (3.13)

$$\mathfrak{b}(\alpha, L) = \inf\{\mathcal{E}^{1D}_{\alpha,L}(f) \ : \ f \in B^1(\mathbb{R})\} . \tag{3.14}$$

We anticipate a relationship between the reference energy $E(\cdot)$ in Theorem 3.1 and the following *one-dimensional energy*

$$E^{1\mathrm{D}}(L) = \inf_{\alpha\in\mathbb{R}} \mathfrak{b}(\alpha, L) .$$

Conjecture 3.3. *For all* $L \in \left(0, \lambda_0^{-3/2}\right)$, $E(L) = E^{1\mathrm{D}}(L)$.

A similar statement to Conjecture 3.3 has been conjectured in [27], but for the following one-dimensional energy

$$\mathcal{F}^{1D}_{\alpha,b}(f) = \int_0^\infty \left(|f'(t)|^2 + (t+\alpha)^2\,|f(t)|^2 - b|f(t)|^2 + \frac{b}{2}|f(t)|^4\right) dt\,, \tag{3.15}$$

where an affirmative answer was given for particular regimes in the papers [4, 14], and eventually for the full regime in [10].

Recently, inspired by the contribution of Almog-Helffer [4], Conjecture 3.3 was proved in [25] when L is close to $\lambda_0^{-3/2}$. Essentially, that is a consequence of the following asymptotics of the energy $E(L)$ as $L \nearrow \lambda_0^{-3/2}$.

Theorem 3.4. ([25, Thm. 1.1])
Let $E(\cdot)$ *be the energy defined in* (3.1). *As* $L \nearrow \lambda_0^{-3/2}$, *it holds*

$$E(L) = -\frac{L^{2/3}}{2}\frac{(L^{-2/3} - \lambda_0)^2}{\|u_0\|_4^4}(1 + o(1))\,,$$

where u_0 *is the* L^2*-normalized ground state of the Montgomery operator* $P(\tau_0)$ *introduced in* (2.3) *and* τ_0 *is defined in* (2.4).

Note that Theorem 3.4 is complementary to Theorems 3.2 and 3.1.

4. The distribution of bulk superconductivity

In this section, we return to the analysis of the full Ginzburg-Landau functional in (1.1) and its ground state energy $\mathrm{E_{gs}}(\kappa, H)$ in (1.4). We will present the results obtained by Attar [6, 7] and the authors [21].

Recall that we work under the assumption that the function B_0 appearing in (1.1) satisfies (1.5)-(1.8). The presented results here are valid in the large κ regime and when the parameter H varies in the following manner. If $0 < \Lambda_1 < \Lambda_2$ are given constants and $H = H(\kappa)$ is a function of κ such that

$$\forall\, \kappa > 0\,, \quad \Lambda_1\kappa^{1/3} \le H \le \Lambda_2\kappa\,, \tag{4.1}$$

then we have the following result on the ground state energy $\mathrm{E_{gs}}(\kappa, H)$.

Theorem 4.1. ([6, Thm. 1.1] and [7, Thm. 1.1])
If (4.1) *holds, then the ground state energy in* (1.4) *satisfies*

$$\mathrm{E_{gs}}(\kappa, H) = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H\left(\left|\ln\frac{H}{\kappa}\right| + 1\right)\right),$$

where $g(\cdot)$ *is the bulk energy introduced in* (3.5).

Remark 4.2.

(1) Under the additional assumption $c\kappa \le H \le \Lambda_2\kappa$, Theorem 1.1 in [6] gives an explicit (uniform with respect to H) estimate of the remainder term $o\left(\kappa H\left(\left|\ln\frac{H}{\kappa}\right| + 1\right)\right)$, namely it is of order κ^τ with τ any constant in the interval $(1, 2)$. Even under the relaxed assumption in (4.1), the remainder term in Theorem 4.1 above can be controlled uniformly with respect to H.

(2) Under the restricted condition $\Lambda_1\kappa^{1/3} \le H \le o(\kappa)$, the asymptotics in Theorem 4.1 reads as follows (this is a consequence of the formula (3.6)):

$$\mathrm{E}_{\mathrm{gs}}(\kappa, H) = -\frac{\kappa^2}{2}|\Omega| + \frac{1}{2}\kappa H\left(\int_\Omega |B_0(x)| \ln \frac{\kappa}{|B_0(x)|H}\, dx\right)\big(1+o(1)\big)\,. \tag{4.2}$$

Furthermore, Attar proves that the minimizing order parameter ψ has isolated zeros filling up all the domain Ω except the magnetic zero set Γ (see [7, Thm. 1.6]). This is a generalization of the result by Sandier-Serfaty [30] devoted to the constant magnetic field case, $B_0 \equiv 1$. However, the result of Sandier-Serfaty holds under the relaxed assumption $\frac{\ln\kappa}{\kappa} \ll H \ll \kappa$. The restrictive assumption in the non-constant magnetic field case seems to be of technical nature (see [7, Rem. 1.2]).

(3) Since the function $g(b)$ vanishes for $b \ge 1$, Theorem 4.1 indicates that superconductivity is confined in the region $\{\frac{H}{\kappa}B_0(x) < 1\}$. We will measure this confinement more precisely in Theorem 4.3 below.

For all $b > 0$, let us introduce the following set

$$\omega(b) = \{x \in \Omega\ :\ |B_0(x)| > \frac{1}{b}\}\,.$$

Note that $\omega(b) \neq \emptyset$ if and only if $b \ge \beta_0^{-1}$, where

$$\beta_0 = \sup_{x\in\overline{\Omega}} |B_0(x)|\,.$$

Theorem 4.3. ([21, Thm 1.3])
Given $b > \beta_0^{-1}$ and an open set O such that $\overline{O} \subset \omega(b)$, there exist positive constants κ_0 and α such that, if $\kappa \ge \kappa_0$ and $(\psi, \mathbf{A})_{\kappa,H}$ is a solution of (1.2) for $H = b\kappa$, then the following inequality holds

$$\|\psi\|_{H^1(O)} \le C\, e^{-\alpha_0\kappa}\,. \tag{4.3}$$

Since $\omega(b)$ expands to $\Omega\setminus\Gamma$ as $b \to +\infty$, Theorem 4.3 states that superconductivity is confined near the magnetic zero set $\Gamma = \{B_0(x) = 0\}$ if the intensity of the applied magnetic field is sufficiently large.

Note that Theorems 4.1 and 4.3 do not describe what happens on the boundary of the domain Ω. The reason is that the bulk function $g(\cdot)$ is not adequate for capturing the boundary contributions. In order to understand the influence of the boundary, we need another *surface energy*, introduced first by Pan [27], then elaborated by Almog-Helffer [4], Fournais-Helffer-Persson [14] and Correggi-Rougerie [10]. All these contributions deal with the constant magnetic field case (i.e. $B_0 \equiv 1$ in (1.1)). In the non-constant magnetic field case, we refer to [21, Thms. 1.5 & 1.6] for results concerning surface superconductivity.

5. Concentration of superconductivity on the magnetic zero set

The results presented in Section 4, in particular Theorem 4.1, are valid when the parameter measuring the intensity of the applied magnetic field satisfies $H = \mathcal{O}(\kappa)$. However, in light of Theorem 2.2, one should understand what happens for the minimizers of the Ginzburg-Landau functional up to H of order $\mathcal{O}(\kappa^2)$. That has been the subject of the paper [19], whose main result is now presented.

Coming back to the full Ginzburg-Landau functional in (1.1), we assume that the function B_0 satisfies (1.5)-(1.8). We assume that the parameter H in (1.1) depends on κ in the following manner

$$H = b(\kappa)\kappa\,, \tag{5.1}$$

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying

$$\lim_{\kappa\to+\infty} b(\kappa) = +\infty \quad \text{and} \quad \limsup_{\kappa\to+\infty} \kappa^{-1}b(\kappa) < +\infty\,. \tag{5.2}$$

Note that, with the new assumption on H, we cover a regime complementary to the one in (4.1). In this regime, superconductivity is confined on the magnetic zero set Γ.

Theorem 5.1. ([19, Thm. 6.3]) *There exist constants $C > 0$, $m_0 > 0$ and $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$ and if (5.1) and (5.2) hold for H, then every solution $(\psi, \mathbf{A})_{\kappa,H}$ of the G-L equations (1.2) satisfies*

$$\int_\Omega \exp\left(2m_0\frac{H}{\kappa}\,t(x)\right)\left(\frac{1}{\kappa^2}|(\nabla - i\kappa H\mathbf{A})\psi|^2 + |\psi(x)|^2\right)dx \leq C\int_{\{t(x)\leq C\frac{\kappa}{H}\}}|\psi(x)|^2\,dx\,,$$

where $t(x) = \mathrm{dist}(x,\Gamma)$.

Remark 5.2. As a consequence of Theorem 5.1 and the uniform bound $\|\psi\|_\infty \leq 1$, we observe that for all $\delta \in (0,1)$, the order parameter ψ satisfies (for $H = b(\kappa)\kappa$ and κ sufficiently large)

$$\|\psi\|_{L^2(\Gamma_{\kappa,\delta})} \leq \exp\left(-m_0 b(\kappa)^{-\delta}\right),$$

where $\Gamma_{\kappa,\delta} = \{x \in \Omega \;:\; \mathrm{dist}(x,\Gamma) \geq b(\kappa)^{-1+\delta}\}$. This bound is the analogue of the one in Theorem 4.3 (in the limiting case $b = +\infty$).

In light of Theorem 5.1, it is natural to understand the behavior of the minimizers of the Ginzburg-Landau functional near the magnetic zero set Γ. We do this by estimating the ground state energy, $\mathrm{E_{gs}}(\kappa, H)$ in (1.4).

In a first regime, the leading order term in the expression of $\mathrm{E_{gs}}(\kappa,H)$ is the same as the one in Theorem 4.1 (it involves the bulk energy $g(\cdot)$ introduced in (3.5)).

Theorem 5.3. ([19, Thm. 1.1])
Assume that (5.1)-(5.2) hold with the additional condition $\limsup\limits_{\kappa\to+\infty}\kappa^{-1/2}b(\kappa) < +\infty$. Then, as $\kappa \to +\infty$, the ground state energy in (1.4) satisfies

$$\mathrm{E_{gs}}(\kappa,H) = \kappa^2\int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right)dx + o\left(\frac{\kappa^3}{H}\right). \tag{5.3}$$

Remark 5.4. Note that the leading order term in (5.3) is of order κ^3/H, thanks to the fact that $g\left(\frac{H}{\kappa}|B_0(x)|\right)$ vanishes for $\frac{H}{\kappa}|B_0(x)| \geq 1$ and the assumption on the zero set of B_0 in (1.5)-(1.8). Consequently, we notice a difference between Theorems 4.1 and 5.3; the remainder term in Theorem 4.1 is of order $o(\kappa H)$, which is not negligible compared to the principal term.

In the second regime (complementary to Theorem 5.3), the bulk energy $g(\cdot)$ is not enough to capture the leading order term in the expression of $\mathrm{E_{gs}}(\kappa,H)$, but we need in this case the reference energy $E(\cdot)$ introduced in Theorem 3.1.

Theorem 5.5. ([19, Thm. 1.1])
Assume that (5.1)-(5.2) hold with the additional condition $\lim\limits_{\kappa\to+\infty}\kappa^{-1/2}b(\kappa) = +\infty$. Then, as $\kappa \to +\infty$, the ground state energy in (1.4) satisfies

$$\mathrm{E_{gs}}(\kappa,H) = \kappa\left(\int_\Gamma\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)ds(x)\right) + o\left(\frac{\kappa^3}{H}\right), \tag{5.4}$$

where ds denotes the arc-length measure on Γ.

Remark 5.6. In light of Theorem 3.2, we can bridge the results in Theorems 5.3 and 5.5 and state them in the single statement [20, Thm. 1.4] as follows. If (5.1)-(5.2) hold, then as $\kappa \to +\infty$,

$$\mathrm{E_{gs}}(\kappa,H) = \kappa\left(\int_\Gamma\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)ds(x)\right) + o\left(\frac{\kappa^3}{H}\right).$$

Remark 5.7. Since $E(L)$ vanishes for $L \geq \lambda_0^{-3/2}$ (see Theorem 3.1 and (2.2)), we see that the principal term in (5.6) vanishes for $H \geq H_{C_2}(\kappa)$ where

$$H_{C_2}(\kappa) := \lambda_0^{-3/2} c_0^{-1} \kappa^2 \quad \text{with} \quad c_0 = \min_{x \in \Gamma} |\nabla B_0(x)| \,.$$

In [25, Thm. 1.5], the principal term of the ground state energy $\mathrm{E_{gs}}(\kappa, H)$ is computed as $H \nearrow H_{C_2}(\kappa)$ under a non-optimal assumption on the magnitude of the term $H_{C_2}(\kappa) - H$. Essentially, that was a consequence of Theorem 3.4.

Remark 5.8. We refer to [19, Thm. 1.6] for results concerning the convergence of the minimizing order parameter ψ.

Acknowledgements. The first author (B.H.) would like to thank Prof. Y. Tsutsumi and the RIMS Kyoto for their kind invitation at the occasion of the Tosio Kato Centennial Conference.

References

[1] A. Abrikosov. On the magnetic properties of superconductors of the second group. *J. Exp. Theor. Phys.* **5** 1174-82 (1957).

[2] A. Aftalion, S. Serfaty. Lowest Landau level approach for the Abrikosov lattice close to the second critical field. Selecta Math. **2** (13), 183-202 (2007).

[3] Y. Almog. Abrikosov lattices in finite domains. *Comm. Math. Phys.* **262** 677-702 (2006).

[4] Y. Almog, B. Helffer. Global stability of the normal state of superconductors in the presence of a strong electric current. *Comm. Math. Phys.* **330**, 1021–1094 (2014).

[5] Y. Almog, B. Helffer, X.B. Pan. Mixed normal-superconducting states in the presence of strong electric currents. *Arch. Rational Mech. Anal.* **223** (1) 419-462 (2017).

[6] K. Attar. The ground state energy of the two dimensional Ginzburg-Landau functional with variable magnetic field. *Annales de l'Institut Henri Poincaré - Analyse Non-Linéaire* **32**, 325-345 (2015).

[7] K. Attar. Energy and vorticity of the Ginzburg-Landau model with variable magnetic field. *Asymptotic Analysis* **93**, 75-114 (2015).

[8] K. Attar. Pinning with a variable magnetic field of the two dimensional Ginzburg-Landau model. *Non-linear Analysis: Theory Methods and Applications* **139**, 1-54 (2015).

[9] A. Contreras, X. Lamy. Persistence of superconductivity in thin shells beyond H_{c1}. *Commun. Contemp. Math.* **18** (4), 1550047 [32 pages] (2016).

[10] M. Correggi, N. Rougerie. On the Ginzburg-Landau functional in the surface superconductivity regime. *Comm. Math. Phys.* **332** (3) 1297-1343 (2014).

[11] N. Dombrowski, N. Raymond. Semi-classical analysis with vanishing magnetic fields. *Journal of Spectral Theory.* **3**, 423-464 (2013).

[12] T. Giorgi, D. Phillips. The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model. *SIAM J. Math. Anal.* **30** 341-359 (1999).

[13] S. Fournais, B. Helffer. *Spectral Methods in Surface Superconductivity.* Progress in Nonlinear Differential Equations and Their Applications. **77** Birkhäuser (2010).

[14] S. Fournais, B. Helffer, M. Persson. Superconductivity between HC2 and HC3. *Journal of Spectral Theory.* **1** (3) 273-298 (2011).

[15] S. Fournais, A. Kachmar. The ground state energy of the three dimensional Ginzburg-Landau functional. Part I. Bulk regime. *Communications in Partial Differential Equations* **38**, 339–383 (2013).

[16] S. Fournais, A. Kachmar. Nucleation of bulk superconductivity close to critical magnetic field. *Adv. Math.* **226**, 1213–1258 (2011).

[17] B. Helffer. The Montgomery operator revisited. *Colloquium Mathematicum,* Vol. 118, No 2, 391–400 (2011).

[18] B. Helffer, A. Mohamed. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.* **138** (1), 40-81 (1996).

[19] B. Helffer, A. Kachmar. The Ginzburg-Landau functional with a vanishing magnetic field. *Arch. Ration. Mech. Anal.* **218** (1), 55-122 (2015).

[20] B. Helffer, A. Kachmar. From constant to non-degenerately vanishing magnetic fields in superconductivity. *Ann. Institut Henri Poincaré (Section Analyse non-linéaire)* **34** (2) 423-438 (2017).

[21] B. Helffer, A. Kachmar. Decay of superconductivity away from the magnetic zero set. *Cal. Var. PDE.* **56** (5) art. no. 130 (2017).

[22] B. Helffer, Y.A. Kordyukov. Spectral gaps for periodic Schrödinger operators with hypersurface magnetic wells: Analysis near the bottom. *J. Funct. Anal.* **257** (10), 3043-3081 (2009).

[23] A. Kachmar. The Ginzburg-Landau order parameter near the second critical field. *SIAM J. Math. Anal.* **46** (1), 572-587 (2014).

[24] A. Kachmar. The ground state energy of the three-dimensional Ginzburg-Landau model in the mixed phase. *J. Funct. Anal.* **261**, 3328-3344 (2011).
[25] A. Kachmar, M. Nasrallah. On the Ginzburg–Landau energy with a magnetic field vanishing along a curve. *Asymptot. Anal.* **103** (3) 135-163 (2017).
[26] R. Montgomery. Hearing the zero locus of a magnetic field. *Commun. Math. Phys.* **168** (3), 651–675 (1995).
[27] X.B. Pan. Surface superconductivity in applied magnetic fields above H_{C_2}. *Commun. Math. Phys.* **228**, 228–370 (2002).
[28] X.B. Pan, K.H. Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. *Trans. Amer. Math. Soc.* **354** (10), 4201-4227 (2002).
[29] E. Sandier, S. Serfaty. From the Ginzburg-Landau model to vortex lattice problems. *Commun. Math. Phys.* **313** 635-743 (2012).
[30] E. Sandier, S. Serfaty. On the energy of type-II superconductors in the mixed phase. *Rev. Math. Phys.* **12** (9), 1219âĂŞ1257 (2000).
[31] E. Sandier, S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg-Landau model. *SIAM. J. Math. Anal.* **34** (4), 939–956 (2003).
[32] E. Sandier, S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg-Landau model. *SIAM J. Math. Anal.* **34** (4), 939-956 (2003).

(B. Helffer) Laboratoire Jean Leray, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes (France) and Laboratoire de Mathématiques, Univ. Paris-Sud.
E-mail address: `bernard.helffer@math.u-psud.fr`

(A. Kachmar) Department of Mathematics, Lebanese University, Section V, Nabatiyeh, Lebanon.
E-mail address: `ayman.kashmar@liu.edu.lb`