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Mixed Morrey spaces

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Abstract

We introduce mixed Morrey spaces and show some basic properties. These properties extend the classical ones. We investigate the boundedness in these spaces of the iterated maximal operator. Furthermore, as a corollary, we obtain the boundedness of the iterated maximal operator in classical Morrey spaces.

1 Mixed Morrey spaces

In this section, we define the Mixed Morrey spaces $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. To do this, we prepare some definitions. Throughout the paper, we use the following notation. The letters $\vec{p}, \vec{q}, \vec{r}, \ldots$ will denote n-tuples of the numbers in $[0, \infty]$ $(n \geq 1)$, $\vec{p} = (p_1, \ldots, p_n), \vec{q} = (q_1, \ldots, q_n), \vec{r} = (r_1, \ldots, r_n)$. By definiton, the inequality, for example, $0 < \vec{p} < \infty$ means that $0 < p_i < \infty$ for each i. Furthermore, for $\vec{p} = (p_1, \ldots, p_n)$ and $r \in \mathbb{R}$, let

$$\frac{1}{\vec{p}} = \left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right), \quad \frac{\vec{p}}{r} = \left(\frac{p_1}{r}, \ldots, \frac{p_n}{r}\right), \quad \vec{p'} = (p'_1, \ldots, p'_n),$$

where $p'_j = \frac{p_j}{p_j - 1}$ is a conjugate exponent of p_j. Let $Q = Q(x, r)$ be a cube having center x and radius r, whose sides parallel to the cordinate axes. $|Q|$ denotes the volume of the cube Q. By $A \lesssim B$, we denote that $A \leq CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

In [4], Benedek and Panzone introduced mixed Lebesgue spaces.

Definition 1.1 (*Mixed Lebesgue spaces)*. [4] Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then define the *mixed Lebesgue norm* $\|\cdot\|_{\vec{p}}$ or $\|\cdot\|_{(p_1, p_2, \ldots, p_n)}$ by

$$\begin{aligned} \|f\|_{\vec{p}} &= \|f\|_{(p_1, p_2, \ldots, p_n)} \\ &\equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} \mathrm{d}x_1\right)^{\frac{p_2}{p_1}} \mathrm{d}x_2\right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n\right)^{\frac{1}{p_n}}, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$, then we have to make appropriate modifications. We define the *mixed Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ or $L^{(p_1,p_2,\ldots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{\vec{p}} < \infty$, where $L^0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$.

Remark. Let $\vec{p} \in (0,\infty]^n$.

(i) If for each $p_i = p$, then

$$\|f\|_{\vec{p}} = \|f\|_{(p_1,p_2,\ldots,p_n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} = \|f\|_p$$

and

$$L^{\vec{p}}(\mathbb{R}^n) = L^p(\mathbb{R}^n). \tag{1}$$

(ii) Let f be a measureable function on $\mathbb{R}^n$. For any $(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$,

$$\|f\|_{(p_1)}(x_2,\ldots,x_n) \equiv \left(\int_{\mathbb{R}} |f(x_1,\ldots,x_n)|^{p_1} \mathrm{d}x_1\right)^{\frac{1}{p_1}}$$

is a measureable function and defined on $\mathbb{R}^{n-1}$. Moreover, we define

$$\|f\|_{\vec{q}} = \|f\|_{(p_1,p_2,\ldots,p_j)} \equiv \|[\|f\|_{(p_1,p_2,\ldots,p_{j-1})}]\|_{(p_j)},$$

where $\|f\|_{(p_1,p_2,\ldots,p_{j-1})}$ denotes $|f|$, if $j = 1$ and $\vec{q} = (p_1,\ldots,p_j), j \le n$. Note that $\|f\|_{\vec{q}}$ is a measureable function of $(x_{j+1},\ldots,x_n)$ for $j < n$.

First, we define Morrey spaces. Let $1 \le q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} : \ Q \text{ is a cube in } \mathbb{R}^n\right\} \tag{2}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite.

Next, we define mixed Morrey spaces.

Definition 1.2 (*Mixed Morrey spaces*)**.** Let $\vec{q} = (q_1,\ldots,q_n) \in (0,\infty]^n$ and $p \in (0,\infty]$ satisfy $\sum_{j=1}^n \frac{1}{q_j} \ge \frac{n}{p}$.

Then define the *mixed Morrey norm* $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$ by

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}} : \ Q \text{ is a cube in } \mathbb{R}^n\right\}.$$

We define the *mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to be the set of all $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} < \infty$.

Remark. Let $\vec{q} \in (0,\infty]^n$.

(i) If for each $q_i = q$, then by (1),

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q}\right)}\|f\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_q.$$

Thus, taking supremum over the all cubes in $\mathbb{R}^n$, we obtain

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)},$$

and

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n),$$

with coincidence of norms.

(ii) In particular, let

$$p = \frac{n}{1/q_1 + \cdots + 1/q_n}.$$

Then, since

$$\begin{aligned}\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}} : \ Q \text{ is a cube in } \mathbb{R}^n\right\} \\ &= \sup\{\|f\chi_Q\|_{\vec{q}} : \ Q \text{ is a cube in } \mathbb{R}^n\} = \|f\|_{\vec{q}},\end{aligned}$$

we obtain

$$L^{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n),$$

with coincidence of norms.

2 The boundedness of the iterated maximal operator

In this section, we show the boundedness of the iterated mximal operator in the mixed spaces.

First, we recall the maximal operator. For all measureable functions f, we define the Hardy-Littlewood maximal operator M by

$$Mf(x) = \sup_{Q\in\mathcal{Q}} \frac{\chi_Q(x)}{|Q|}\int_Q |f(y)|\mathrm{d}y,$$

where $\mathcal{Q}$ denotes the set of all cubes in $\mathbb{R}^n$. Let $1 \le k \le n$. Then, we define the maximal operator M_k for x_k as follows:

$$M_k f(x) \equiv \sup_{x_k\in I} \frac{1}{|I|}\int_I |f(x_1,\ldots,y_k,\ldots,x_n)|\mathrm{d}y_k,$$

where I is an interval. Furthermore, for all measurable functions f, define the iterated maximal operator $\mathcal{M}_t$ by

$$\mathcal{M}_t f(x) \equiv \left(M_n \cdots M_1 \left[|f|^t\right](x)\right)^{\frac{1}{t}}$$

for every $t > 0$ and $x \in \mathbb{R}^n$.

Remark. Let $\mathcal{R}$ be a set of all rectangles in $\mathbb{R}^n$. By M_R, denote the strong maximal operator generated by a rectangle R:

$$M_R f(x) = \sup_{R \in \mathcal{R}} \frac{\chi_R(x)}{|R|} \int_R |f(y)| \mathrm{d}y.$$

Then, the followings follow [10]:

$$M_R f(x) \leq M_n \cdots M_1 f(x) = \mathcal{M}_1 f(x),$$

and

$$M_R f(x) \leq M_1 \cdots M_n f(x),$$

and so on. But, the relation between $M_1 \cdots M_n$ and $M_n \cdots M_1$ seems unknown.

We describe the boundedness of the iterated maximal operator in the mixed spaces. First, we consider the boundedness in mixed lebesgue spaces.

Theorem 2.1. *Let $0 < \vec{p} < \infty$. If $0 < t < \min(p_1, \ldots, p_n)$, then*

$$\|\mathcal{M}_t f\|_{\vec{p}} \leq C \|f\|_{\vec{p}},$$

for $f \in L^{\vec{p}}(\mathbb{R}^n)$.

In 1935, Jessen, Marcinkiewicz and Zygmund showed the boundedness of the strong maximal operator in the classical L^p spaces [10]. To show the boundedness of the strong maximal operator in mixed Lebesgue spaces, we use the followimg lemma, which is showed by Bagby in 1975 [3].

Lemma 2.2. *Let $1 < q_i < \infty (i = 1, \ldots, m)$ and $1 < p < \infty$. Let (Ω_i, μ_i) be σ-finite measure spaces, and let $t = (t_1, \ldots, t_m) \in \Omega_1 \times \cdots \times \Omega_m = \Omega$. For $f(x,t) \in L^0(\mathbb{R}^n \times \Omega)$,*

$$\int_{\mathbb{R}^n} \|Mf(x, \cdot)\|^p_{(q_1, \ldots, q_m)} \, \mathrm{d}x \lesssim \int_{\mathbb{R}^n} \|f(x, \cdot)\|^p_{(q_1, \ldots, q_m)} \, \mathrm{d}x,$$

Using this lemma, we prove Theorem 2.1.

Proof. Since

$$\|\mathcal{M}_t f\|_{\vec{p}} = \left\| \left(M_n \cdots M_1 \left[|f|^t\right]\right)^{\frac{1}{t}} \right\|_{\vec{p}} = \left\|M_n \cdots M_1 \left[|f|^t\right]\right\|_{\left(\frac{p_1}{t},\ldots,\frac{p_n}{t}\right)}^{\frac{1}{t}},$$

we have only to check the result for $t = 1$ and $1 < \vec{p} < \infty$.
Let $t = 1$. Then the result can be written as

$$\|\mathcal{M}_1 f\|_{\vec{p}} = \|M_n \cdots M_1 f\|_{\vec{p}} \leq C\|f\|_{\vec{p}}.$$

We use induction on n. Let $n = 1$. Then, the result follows by the classical case of the boundedness of the Hardy-Littlewood maximal operator.
Suppose that the result holds for $n-1$, that is, for $h \in L^0(\mathbb{R}^{n-1})$ and $1 < (q_1, \ldots, q_{n-1}) < \infty$,

$$\|M_{n-1} \cdots M_1 h\|_{(q_1,\ldots,q_{n-1})} \leq C\|h\|_{(q_1,\ldots,q_{n-1})}.$$

By Lemma 2.2,

$$\|M_n f\|_{\vec{p}} = \left\|\left[\|M_n f\|_{(p_1,\ldots,p_{n-1})}\right]\right\|_{(p_n)} \lesssim \left\|\left[\|f\|_{(p_1,\ldots,p_{n-1})}\right]\right\|_{(p_n)} = \|f\|_{\vec{p}}.$$

Thus, by induction assumption, we obtain

$$\begin{aligned}
\|M_n M_{n-1} \cdots M_1 f\|_{\vec{p}} &= \|M_n[M_{n-1} \cdots M_1 f]\|_{\vec{p}} \\
&\lesssim \|M_{n-1} \cdots M_1 f\|_{\vec{p}} \\
&= \left\|\|M_{n-1} \cdots M_1 f\|_{(p_1,\ldots,p_{n-1})}\right\|_{p_n} \\
&\lesssim \left\|\|f\|_{(p_1,\ldots,p_{n-1})}\right\|_{p_n} = \|f\|_{\vec{p}}.
\end{aligned}$$

□

Next, we consider the boundedness in mixed Morrey spaces.

Theorem 2.3. *Let $0 < \vec{q} \leq \infty$ and $0 < p < \infty$ satisfy*

$$\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n-1}{n} p < \max(q_1, \ldots, q_n).$$

If $0 < t < \min(q_1, \ldots, q_n, p)$, then

$$\|\mathcal{M}_t f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

for all $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

As a corollary, we obtain the result in classical Morrey spaces.

Corollary 2.4. *Let*

$$0 < \frac{n-1}{n}p < q \leq p < \infty.$$

If $0 < t < q$, *then*

$$\|\mathcal{M}_t f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)},$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

To show the boundedness of the strong maximal operator in mixed Morrey spaces, we use the following proposition.

Proposition 2.5. *Let* $1 < \vec{q} < \infty$ *and* $w_k \in A_{q_k}(\mathbb{R})$ *for* $k = 1, \ldots, n$. *Then,*

$$\left\| \mathcal{M}_1 f \cdot \bigotimes_{k=1}^n w_k^{\frac{1}{q_k}} \right\|_{\vec{q}} \lesssim \left\| f \cdot \bigotimes_{k=1}^n w_k^{\frac{1}{q_k}} \right\|_{\vec{q}}.$$

Note that a locally integrable weight w is said to be an A_{q_k}*-weight*, if $0 < w < \infty$ almost everywhere, and

$$[w]_{A_{q_k}} \equiv \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(y) \mathrm{d}y \right) \left(\frac{1}{|Q|} \int_Q w(y)^{-\frac{1}{q_k - 1}} \mathrm{d}y \right)^{q_k - 1} < \infty.$$

Proposition 2.6. *Let* $0 < p < \infty$, $0 < \vec{q} \leq \infty$ *and* $\eta \in \mathbb{R}$ *satisfy*

$$0 < \sum_{j=1}^n \frac{1}{q_j} - \frac{n}{p} < \eta < 1.$$

Then

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \|f(\mathcal{M}_1 \chi_Q)^\eta\|_{\vec{q}}.$$

Let us prove Theorem 2.3.

Proof. We have only to check for $t = 1, 1 < p < \infty$ and $1 < \vec{q} < \infty$. For $\eta \in \mathbb{R}$ satisfying

$$0 < \sum_{j=1}^n \frac{1}{q_j} - \frac{n}{p} < \eta < 1, \tag{3}$$

once we show

$$\|\mathcal{M}_1 f(\mathcal{M}_1 \chi_Q)^\eta\|_{\vec{q}} \lesssim \|f(\mathcal{M}_1 \chi_Q)^\eta\|_{\vec{q}}, \tag{4}$$

we get

$$|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \|\mathcal{M}_1 f(\mathcal{M}_1 \chi_Q)^\eta\|_{\vec{q}} \lesssim |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \|f(\mathcal{M}_1 \chi_Q)^\eta\|_{\vec{q}}.$$

Taking supremum for all cubes and using above proposition, we conclude the result.

We shall show (4). Let $Q = I_1 \times I_2 \times \cdots \times I_n$. Then,

$$(\mathcal{M}_1\chi_Q)^\eta = \left(\bigotimes_{j=1}^{n} M_j\chi_{I_j}\right)^\eta = \bigotimes_{j=1}^{n} \left(M_j\chi_{I_j}\right)^\eta.$$

Here, $(M_j\chi_{I_j})^\eta$ is A_1-weight if and only if

$$0 \le \eta q_j < 1, \tag{5}$$

and so $(M_j\chi_{I_j})^\eta \in A_1 \subset A_{q_j}$ for all q_j. Thus,

$$\begin{aligned}\|(\mathcal{M}_1 f)(\mathcal{M}_1\chi_Q)^\eta\|_{\vec{q}} &= \left\|(\mathcal{M}_1 f)\bigotimes_{j=1}^{n}\left(M_j\chi_{I_j}\right)^\eta\right\|_{\vec{q}} \\ &\lesssim \left\|f\bigotimes_{j=1}^{n}\left(M_j\chi_{I_j}\right)^\eta\right\|_{\vec{q}} \\ &= \|f(\mathcal{M}_1\chi_Q)^\eta\|_{\vec{q}}.\end{aligned}$$

Thus, (4) holds. Moreover, by (3) and (5), we get the condition

$$\frac{n-1}{n}p < \max(q_1, \ldots, q_n).$$

□

Note that Corollary 2.4 is a special case of Theorem 2.3. Letting $q_1 = \ldots = q_n$, we conclude the result.

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