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The Schwarz inequality via operator-valued inner product and the geometric operator mean
一作用素値内積と作用素幾何平均を用いたシュワルツの不等式について一

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ABSTRACT. In this paper, by virtue of the Cauchy-Schwarz operator inequality due to J.I. Fujii, we show the covariance-variance operator inequality via the geometric operator mean which differs from Bhatia-Davis's one and estimate the upper bounds. By our formulation, we show a Robertson type inequality associated to the conditional expectation on a finite von Nuemann algebra.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. An operator A in $B(\mathcal{H})$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For selfadjoint operators A and B, the order relation $A \geq B$ means that $A - B$ is positive and we denote the absolute value of $A \in B(\mathcal{H})$ by $|A| = (A^*A)^{\frac{1}{2}}$. A map Φ on $B(\mathcal{H})$ is called 2-positive if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \qquad \text{implies} \qquad \begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{pmatrix} \geq 0.$$

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis. Regarding a sesquilinear map $\langle X, Y\rangle_\Phi = \Phi(Y^*X)$ for $X, Y \in B(\mathcal{H})$ as an operator-valued inner product with a positive linear map on $B(\mathcal{H})$, several operator versions for the Schwarz inequality are discussed by many researchers. In [3], Bhatia and Davis showed some new operator versions of the Schwarz inequality for a positive linear map: If Φ is a 2-positive linear map on $B(\mathcal{H})$, then $\langle Y, X\rangle_\Phi \langle Y, Y\rangle_\Phi^{-1} \langle X, Y\rangle_\Phi \in B(\mathcal{H})$ and

$$\langle Y, X\rangle_\Phi \langle Y, Y\rangle_\Phi^{-1} \langle X, Y\rangle_\Phi \leq \langle X, X\rangle_\Phi \tag{1.1}$$

for every $X, Y \in B(\mathcal{H})$. In fact, for every $X, Y \in B(\mathcal{H})$

$$\begin{pmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{pmatrix} = \begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \geq 0$$

and by 2-positivity of Φ

$$\begin{pmatrix} \Phi(X^*X) & \Phi(X^*Y) \\ \Phi(Y^*X) & \Phi(Y^*Y) \end{pmatrix} \geq 0.$$

Hence for any $\varepsilon > 0$ we have

$$\begin{pmatrix} \Phi(X^*X) & \Phi(X^*Y) \\ \Phi(Y^*X) & \Phi(Y^*Y) + \varepsilon I \end{pmatrix} \geq 0$$

and so

$$\Phi(X^*X)(\Phi(Y^*Y) + \varepsilon I)^{-1}\Phi(Y^*X) \leq \Phi(X^*X).$$

Since $\Phi(X^*X)(\Phi(Y^*Y)+\varepsilon I)^{-1}\Phi(Y^*X)$ are monotone increasing and bounded below for any $\varepsilon > 0$, there exists a strong-operator limit of $\Phi(X^*X)(\Phi(Y^*Y)+\varepsilon I)^{-1}\Phi(Y^*X)$ as $\varepsilon \to 0$ and we write

$$\Phi(X^*X)\Phi(Y^*Y)^{-1}\Phi(Y^*X) = \operatorname*{s-lim}_{\varepsilon\to 0} \Phi(X^*X)(\Phi(Y^*Y)+\varepsilon I)^{-1}\Phi(Y^*X) \in B(\mathcal{H})$$

and then we have the desired inequality (1.1).

In the framework of an operator-valued inner product, the formulation of the Schwarz operator inequality is very important, but the left-hand sides of the Schwarz inequalities (1.1) are expressed as the strong-operator limits unless $\langle Y, Y\rangle_\Phi$ is invertible. This fact is a cause of difficulty in application. Thus, we consider another version of the Schwarz operator inequality in terms of the geometric operator mean due to J.I. Fujii in [5]. For this, we recall the geometric operator mean, also see [7, Chap. 5]. Let A and B be two positive operators in $B(\mathcal{H})$. The geometric operator mean $A \sharp B$ of A and B is defined by

$$A \sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$$

if A is invertible. In [2], Ando showed the following characterizaion:

$$A \sharp B = \max\left\{X \geq 0 : \begin{pmatrix} A & X \\ X & B\end{pmatrix} \geq 0\right\}. \tag{1.2}$$

The geometric operator mean has the monotonicity:

$$0 \leq A \leq C \quad \text{and} \quad 0 \leq B \leq D \quad \text{implies} \quad A \sharp B \leq C \sharp D$$

and the subadditivity:

$$A \sharp B + C \sharp D \leq (A+C) \sharp (B+D).$$

By monotonicity, we can uniquely extend the definition of $A \sharp B$ for all positive operators A and B by setting

$$A \sharp B = \operatorname*{s-lim}_{\varepsilon\to 0}(A+\varepsilon I) \sharp (B+\varepsilon I).$$

In this case, the geometric operator mean $A \sharp B$ for positive operators A and B always exists in $B(\mathcal{H})$ and it has all the desirable properties as geometric mean such as monotonicity, continuity from above, transeformer inequality, subadditivity and self-duality so on.

In [5], Fujii showed the following Cauchy-Schwarz operator inequality in terms of the geometric operator mean:

Theorem A. *Let Φ be a 2-positive map on $B(\mathcal{H})$. Then*

$$|\langle X, Y\rangle_\Phi| \leq \langle X, X\rangle_\Phi \sharp U^*\langle Y, Y\rangle_\Phi U \tag{1.3}$$

for every $X, Y \in B(\mathcal{H})$, where U is a partial isometry in the polar decomposition of $\langle X, Y\rangle_\Phi = U|\langle X, Y\rangle_\Phi|$.

The purpose of this paper is to present applications of the operator Cauchy-Schwarz inequality (1.3) due to J.I. Fujii. We firstly show the covariance-variance operator inequality via the geometric operator mean which differs from Bhatia-Davis's one and estimate the upper bounds. By our formulation, we show a Robertson type inequality associated to the conditional expectation on a finite von Nuemann algebra.

2. Variance-Covariance inequality

We recall the notion of the covariance and the variance of operators defined by Fujii, Furuta, Nakamoto and Takahasi [6]. In 1954, the noncommutative probability theory is founded by H. Umegaki as an application of the theory of von Neumann algebra in [8]. An operator $A \in B(\mathcal{H})$ plays the role of a random variable, that is, for every unit vector $x \in \mathcal{H}$, the functional $\langle Ax, x\rangle$ on the operatyor algebra may be thought as an expectation at a state x (with $\|x\| = 1$). The covariance of operators A and B at a state x is introduced by

$$\operatorname{cov}_x(A, B) = \langle A^*Bx, x\rangle - \langle A^*x, x\rangle\langle Bx, x\rangle, \tag{2.1}$$

and the variance of A at a state x by

$$\operatorname{var}_x(A) = \langle A^*Ax, x\rangle - |\langle Ax, x\rangle|^2.$$

The following variance-covariance inequality is an application of the Cauchy-Schwarz inequality:

$$|\operatorname{cov}_x(A, B)| \leq \sqrt{\operatorname{var}_x(A)\operatorname{var}_x(B)}. \tag{2.2}$$

In [3], Bhatia and Davis studied a noncommutative analogue of variance and covariance in statistics, which is a generalization of the covariance (2.1) at a state: Let Φ be a unital completely positive linear map on $B(\mathcal{H})$. The convariance $\operatorname{cov}(A, B)$ between two operators A and B is defined by

$$\operatorname{cov}(A, B) = \Phi(A^*B) - \Phi(A)^*\Phi(B).$$

The variance of A is defined by

$$\operatorname{var}(A) = \operatorname{cov}(A, A) = \Phi(A^*A) - \Phi(A)^*\Phi(A).$$

Since Φ is completely positive, then the variance of A is positive, i.e., $\operatorname{var}(A) \geq 0$. Bhatia and Davis showed the following counterpart of the variance-covariance inequality in the context of noncommutative probability, which is a generalization of the variance-covariance inequality (2.2): For all $A, B \in B(\mathcal{H})$,

$$\operatorname{cov}(A, B)\operatorname{var}(B)^{-1}\operatorname{cov}(A, B)^* \in B(\mathcal{H})$$

and

$$\operatorname{cov}(A, B)\operatorname{var}(B)^{-1}\operatorname{cov}(A, B)^* \leq \operatorname{var}(A).$$

By virtue of the geometric operator mean, we show the following variance-covariance inequality:

Theorem 2.1. *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$ and A, B two operators in $B(\mathcal{H})$. Then*

$$|\operatorname{cov}(A, B)| \leq U^*\operatorname{var}(A)U \,\sharp\, \operatorname{var}(B), \tag{2.3}$$

where $\operatorname{cov}(A, B) = U|\operatorname{cov}(A, B)|$ *is the polar decomposition of* $\operatorname{cov}(A, B)$.

Proof. It follows from [3, Theorem 1] that the 2×2 operator matrix

$$\begin{pmatrix} \operatorname{var}(A) & \operatorname{cov}(A, B) \\ \operatorname{cov}(A, B)^* & \operatorname{var}(B) \end{pmatrix}$$

is positive. Then we have

$$0 \leq \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{var}(A) & \mathrm{cov}(A,B) \\ \mathrm{cov}(A,B)^* & \mathrm{var}(B) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} U^*\mathrm{var}(A)U & U^*U|\mathrm{cov}(A,B)| \\ |\mathrm{cov}(A,B)|U^*U & \mathrm{var}(B) \end{pmatrix} = \begin{pmatrix} U^*\mathrm{var}(A)U & |\mathrm{cov}(A,B)| \\ |\mathrm{cov}(A,B)| & \mathrm{var}(B) \end{pmatrix}$$

and so by (1.2) we have the desired inequality (2.3). □

If A is a selfadjoint operator with $mI \leq A \leq MI$ for some scalars $m \leq M$, then it follows from [6] that the variance of A at a state x is not greater than $(M-m)^2/4$: $\mathrm{var}_x(A) \leq \frac{1}{4}(M-m)^2$. To estimate the variance and the covariance of general operators, we need the notion of the accretivity. An operator $A \in B(\mathcal{H})$ is said to be accretive if $\mathrm{Re}\,\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The symbol $C_{a,b}(A)$ stands for $C_{a,b}(A) = (A-aI)^*(bI-A)$ for some $a, b \in \mathbb{C}$. We give the estimates of the variance and covariance by virtue of Theorem 2.1.

Theorem 2.2. *Let A be an operator in $B(\mathcal{H})$ and $a, b \in \mathbb{C}$. If the operator $C_{a,b}(A)$ is accretive, then*

$$\mathrm{var}(A) \leq \frac{1}{4}|a-b|^2 - \left|\Phi(A) - \frac{a+b}{2}\right|^2.$$

Theorem 2.3. *Let A and B be two operators in $B(\mathcal{H})$ and $a, b, c, d \in \mathbb{C}$ such that $C_{a,b}(A)$ and $C_{c,d}(B)$ are accretive. Then*

$$|\mathrm{cov}(A,B)| \leq \frac{1}{4}|a-b||c-d| - \left[U^*|\Phi(A) - \frac{a+b}{2}|^2 U\right] \sharp \left[|\Phi(B) - \frac{c+d}{2}|^2\right]$$
$$\left(\leq \frac{1}{4}|a-b||c-d|,\right)$$

where $\mathrm{cov}(A,B) = U|\mathrm{cov}(A,B)|$ *is the polar decomposition of* $\mathrm{cov}(A,B)$.

As an application of Theorem 2.3, we have the following noncommutative Kantorovich inequality:

Corollary 2.4. *Let A be a positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$. If Φ is a unital completely positive linear map on $B(\mathcal{H})$, then*

$$|I - \Phi(A)\Phi(A^{-1})| \leq \frac{(M-m)^2}{4Mm} I.$$

Remark 2.5. *If the range of Φ is abelian in Corollary 2.4, then $I \leq \Phi(A)\Phi(A^{-1})$ and*

$$\Phi(A)\Phi(A^{-1}) \leq \frac{M+m)^2}{4Mm} I.$$

3. Comutation relation and covariance

In this section, we discuss the near relation of the variance-covariance inequality with the Heisenberg uncertainty principle in quantum physics. In [4], Enomoto pointed out that the variance-covariance inequality (2.2) is exactly the generalized Schrödinger inequality: Let A and B be (not necessarily bounded) selfadjoint operators on a Hilbert space $\mathcal{H}$. Let

$D(AB)$ and $D(BA)$ be the domain of AB and BA, respectively. Let $\{A, B\}$ and $[A, B]$ be the Jordan product $AB + BA$ and the commutator $AB - BA$, respectively. Then

$$|\mathrm{cov}_x(A, B)|^2 = \left(\frac{1}{2}\langle\{A, B\}x, x\rangle - \langle Ax, x\rangle\langle Bx, x\rangle\right)^2 + \left(\frac{1}{2i}\langle[A, B]x, x\rangle\right)^2$$

for every unit vector $x \in D(AB) \cap D(BA)$. In particular, the following Robertson type inequality holds:

$$\sqrt{\mathrm{var}_x(A)\mathrm{var}_x(B)} \geq \frac{1}{2}|\langle[A, B]x, x\rangle|$$

and the following Schrödinger type inequality holds:

$$\mathrm{var}_x(A)\mathrm{var}_x(B) \geq \left|\frac{1}{2}\langle A, Bx, x\rangle - \langle Ax, x\rangle\langle Bx, x\rangle\right|^2 + \frac{1}{4}|\langle[A, B]x, x\rangle|^2.$$

We consider a Robertson type uncertainty relation associated to the conditional extectation on a finite von Nuemann algebra. Let $\mathcal{A}$ be a finite von Neumann algebra and $\mathcal{B} \subset \mathcal{A}$ a von Neumann subalgebra. Let $\Phi : \mathcal{A} \mapsto \mathcal{B}$ be a conditional expectation, that is, $\mathcal{B}$-linear projection and positive linear map. For $A, B \in \mathcal{A}$, we define the standard deviation of A and B by the formula

$$\Delta A = A - \Phi(A) \quad \text{and} \quad \Delta B = B - \Phi(B),$$

respectively. Then it follows from $\mathcal{B}$-linearlity of Φ that

$$\begin{aligned}
\langle \Delta B, \Delta A\rangle_\Phi &= \Phi((\Delta A)^*\Delta B) \\
&= \Phi((A^* - \Phi(A)^*)(B - \Phi(B)) \\
&= \Phi(A^*B - \Phi(A)^*B - A^*\Phi(B) + \Phi(A)^*\Phi(B)) \\
&= \Phi(A^*B) - \Phi(A)^*\Phi(B) - \Phi(A^*)\Phi(B) + \Phi(A)^*\Phi(B) \\
&= \Phi(A^*B) - \Phi(A)^*\Phi(B)
\end{aligned}$$

and thus we have

$$\mathrm{cov}(A, B) = \Phi(A^*B) - \Phi(A)^*\Phi(B) = \langle \Delta B, \Delta A\rangle_\Phi$$

and

$$\mathrm{var}(A) = \langle \Delta A, \Delta A\rangle_\Phi.$$

Hence we have the following variance-covariance inequality:

Theorem 3.1. *Let $\Phi : \mathcal{A} \mapsto \mathcal{B}$ be a conditional expectation. Then*

$$|\langle \Delta B, \Delta A\rangle_\Phi| \leq U^*\langle \Delta A, \Delta A\rangle_\Phi U \ \sharp\ \langle \Delta B, \Delta B\rangle_\Phi$$

for every $A, B \in \mathcal{A}$, where $\langle \Delta B, \Delta A\rangle_\Phi = U|\langle \Delta B, \Delta A\rangle_\Phi|$ is the polar decomposition of $\langle \Delta B, \Delta A\rangle_\Phi$.

In [1, Proposition 2.1], Akemann, Anderson and Pedersen showed that if $\mathcal{A}$ is finite and $x \in \mathcal{A}$ is selfadjoint, then there exists a unitary $v \in \mathcal{A}$ such that $v(\mathrm{Re}\, x)_+v^* \leq |x|$. By using the result, we show a Robertson type inequality associated to the conditional expectation on a finite von Nuemann alegebra:

Theorem 3.2. *Let $\Phi : \mathcal{A} \mapsto \mathcal{B}$ be a conditional expectation. Then for every selfadjoint $A, B \in \mathcal{A}$, there exists a unitary $v \in \mathcal{B}$ such that*

$$U^*\langle \Delta A, \Delta A\rangle_\Phi U \ \sharp\ \langle \Delta B, \Delta B\rangle_\Phi \geq v \left(\frac{\Phi([A,B]) - [\Phi(A), \Phi(B)]}{2i} \right)_+ v^*,$$

where $\langle \Delta B, \Delta A\rangle_\Phi = U|\langle \Delta B, \Delta A\rangle_\Phi|$ is the polar decomposition of $\langle \Delta B, \Delta A\rangle_\Phi$ and X_+ is the positive part of a selfadjoint element $X \in \mathcal{B}$.

Under the restricted condition, we have a Schrödinger type inequality associated to the conditional expectation on a finite von Nuemann algebra:

Corollary 3.3. *Let $\Phi : \mathcal{A} \mapsto \mathcal{B}$ be a conditional expectation and $A, B \in \mathcal{A}$ two selfadjoint elements. If $\Phi(AB) - \Phi(A)\Phi(B)$ is normal, then*

$$\begin{aligned} & U^*\langle \Delta A, \Delta A\rangle_\Phi U \ \sharp\ \langle \Delta B, \Delta B\rangle_\Phi \\ & \geq \left(\frac{1}{4} \left(\Phi(\{A,B\}) - \{\Phi(A), \Phi(B)\} \right)^2 + \left(\frac{\Phi([A,B]) - [\Phi(A), \Phi(B)]}{2i} \right)^2 \right)^{\frac{1}{2}} \\ & \geq \frac{1}{2} \left| \Phi([A,B]) - [\Phi(A), \Phi(B)] \right|, \end{aligned}$$

where $\Phi((\Delta A)^* \Delta B) = U|\Phi((\Delta A)^* \Delta B)|$ is the polar decomposition of $\Phi((\Delta A)^* \Delta B)$.

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