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Analyticity of solutions to the Navier-Stokes equtations with initial data in homogeneous Besov spaces

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1 Introduction.

Let us consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$;

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \pi = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} = a & \text{in } \mathbb{R}^n, \end{cases}$$

where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $\pi = \pi(x,t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and the time $t \in (0, \infty)$, respectively, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ is the given initial velocity vector.

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The first purpose of this article is to characterize the optimal space of the initial data a for existence of mild solution u of (N-S) in the Serrin class $L^s(0,\infty;L^p)$ with $2/s+n/p=1$ for $n<p<\infty$. In a bounded domain Ω, a similar investigation has been observed by Farwig-Sohr-Varnhorn [3] and Farwig-Sohr [2]. In the whole space $\mathbb{R}^n$, we shall establish a sharp estimate

$$\Big(\int_0^\infty \|e^{t\Delta}a\|_{\dot{B}^0_{p,1}}\,dt\Big)^{\frac{1}{s}} \le C\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,s}} \tag{1.1}$$

for all $a\in \dot{B}^{-1+\frac{n}{p}}_{p,s}(\mathbb{R}^n)$ provided $2/s+n/p=1$ with $n<p<\infty$. Since we are also successful to derive the continuous bilinear estimate of the Duhamel term $\displaystyle\int_0^t P\nabla\cdot e^{(t-\tau)\Delta}u\otimes v(\tau)d\tau$ for $u,v\in L^s(0,\infty;\dot{B}^0_{p,1})$, it follows from (1.1) that there exists a unique global mild solution $u\in L^s(0,\infty;\dot{B}^0_{p,1})$ provided a is sufficiently small in $\dot{B}^{-1+\frac{n}{p}}_{p,s}$. Conversely, if $a\in\mathcal{S}'$ satisfies $e^{t\Delta}a\in L^s(0,\infty;L^p)$ with $\mathcal{S}'$ denoting the class of temperated distribution, then it holds that $a\in \dot{B}^{-1+\frac{n}{p}}_{p,s}(\mathbb{R}^n)$ with the estimate

$$\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,s}(\mathbb{R}^n)} \le C\Big(\int_0^\infty \|e^{t\Delta}a\|^s_{L^p}\,dt\Big)^{\frac{1}{s}}. \tag{1.2}$$

Since the continuous bilinear estimate of the Duhamel term holds for $u,\ v\in L^s(0,\infty;L^p)$, by combining (1.1) with (1.2), we conclude that the mild solution u of (N-S) belongs to $L^s(0,\infty;L^p)$ if and only if $a\in \dot{B}^{-1+\frac{n}{p}}_{p,s}$ for $2/s+n/p=1$ with $n<p<\infty$.

The second purpose is to show analyticity of our mild solutions. In this direction, Giga-Sawada [5] proved that the strong solution $u\in C([0,T);L^n(\mathbb{R}^n)\cap C((0,T);L^p(\mathbb{R}^n))$ for $n<p\le\infty$ with the initial data $a\in L^n(\mathbb{R}^n)$ given by Giga-Miyakawa [4] and Kato [6] is analytic in the space variable. Later on, Miura-Sawada [11] showed that the mild solution u with $a\in vmo^{-1}$ given by Koch-Tataru [7] is also analytic in the space variable. At the end of the paper [11, Corollary 4.3], in spite of the special construction of Koch-Tataru's mild solution, they made it clear that uniqueness of mild solutions in the Serrin class necessarily implies analyticity in the space variable since any mild solution $u\in L^s(0,T;L^p(\mathbb{R}^n))$ with some T for $2/s+n/p=1$ with $n<p<\infty$ yields that $a\in vmo^{-1}$. Since our mild solution belongs also to such a Serrin class, our result is not altogether new. However, we should emphasize that analyticity of mild solutions is obtained even for the initial data $a\in \dot{B}^{-1+\frac{n}{p}}_{p,s}(\mathbb{R}^n)$. Moreover, our method of the proof of analyticity is different from that of [11] where they split the interval $(0,t)$ of integration of the Duhamel term and make use of the generalized Gronwall type inequality to obtain uniform estimate of derivatives $\partial_x^\alpha u(x,t)$ in $x\in\mathbb{R}^n$ for arbitrary $|\alpha|\in\mathbb{N}$. On the other hand, our method is based on the Hölder type estimate of $\|\partial_x^\alpha u(\cdot,t)\|_{L^p(\mathbb{R}^n)}$ with some $n<p<\infty$ in $t\in(0,\infty)$ for all $|\alpha|\in\mathbb{N}$.

By using the Stokes operator $-P\Delta$ on $P\dot{B}^s_{p,q}$, the original equations (N-S) can be rewritten to the abstract evolution equation:

$$\begin{cases} \dfrac{du}{dt} - \Delta u + P(u\cdot\nabla u) = 0 \quad \text{on } (0,T), \\ u(0)=a, \end{cases} \tag{1.3}$$

where we use a fact that $-P\Delta u = -\Delta P u = -\Delta u$ for u satisfying div $u=0$ in the whole space.

Definition 1 *Let $2 < s < \infty$, $n < p < \infty$ satisfy $2/s + n/p = 1$ and $a \in \mathcal{S}'$ with* $\operatorname{div} a = 0$. *A measurable function u on $\mathbb{R}^n \times (0, \infty)$ is called a mild solution of* (N-S) *if*

(i) $u \in L^s(0, \infty; PL^p)$;

(ii) *u satisfies*

$$u(t) = e^{t\Delta}a - \int_0^t P\nabla \cdot e^{(t-\tau)\Delta}(u \otimes u)(\tau)\, d\tau, \quad 0 < t < \infty. \tag{1.4}$$

We first state well-posedness of global solutions to (N-S) for small initial data a.

Theorem 1 *Let $n < p < \infty$ and $2 < s < \infty$ satisfy $2/s + n/p = 1$. There exists a constant $\delta = \delta(n, p, s) > 0$ such that if $a \in P\dot{B}^{-1+n/p}_{p,s}$ satisfies*

$$\|a\|_{\dot{B}^{-1+n/p}_{p,s}} \le \delta, \tag{1.5}$$

then there exists a unique mild solution u of (N-S) with the following properties

$$u \in BC([0, \infty); \dot{B}^{-1+n/p}_{p,s}) \cap L^s(0, \infty; \dot{B}^0_{p,1}), \tag{1.6}$$

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}u(\cdot) \in BC([0, \infty); \dot{B}^0_{p,1}), \tag{1.7}$$

$$\lim_{t\to+0} \|u(t) - a\|_{\dot{B}^{-1+n/p}_{p,s}} = 0, \tag{1.8}$$

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}} = 0 \tag{1.9}$$

$$\lim_{t\to\infty} \|u(t)\|_{\dot{B}^{-1+n/p}_{p,s}} = 0. \tag{1.10}$$

Remark 1 (1) Since $\dot{B}^0_{p,1} \subset L^p$, our class (1.6) shows that the solution u given by Theorem 1 belongs to the Serrin class $L^s(0, \infty; L^p)$, and so uniqueness holds.

(2) The decay (1.10) of u in the same space $\dot{B}^{-1+\frac{n}{p}}_{p,s}$ as the initial data a is the corresponding result to that which is stated at the end of Kato [6, Note] in such a way that the solution of (N-S) behaves like $\lim_{t\to\infty} \|u(t)\|_{L^n} = 0$ for initial data $a \in L^n$.

The next theorem shows the class of initial data when the mild solution u belongs to the Serrin class globally, i.e., $L^s(0, \infty; PL^p)$.

Theorem 2 *Let $a \in \mathcal{S}'$ and* $\operatorname{div} a = 0$ *in the distribution sense. Suppose that u is a mild solution of (N-S) in $L^s(0, \infty; PL^p)$ with $2/s + n/q = 1$ for $n < p < \infty$. Then it holds necessarily that $a \in P\dot{B}^{-1+n/p}_{p,s}$.*

The third result on analyticity of mild solutions now reads:

Theorem 3 *Let $2 < s < \infty$ and $n < p < \infty$ satisfy $2/s + n/p = 1$. Suppose that $a \in P\dot{B}^{-1+\frac{n}{p}}_{p,s}$ satisfies (1.5). The mild solution u of (N-S) given by Theorem 1 is smooth in the space variable as $D^\alpha u(\cdot, t) \in L^\infty$, $0 < t \le \infty$ for all multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n_0$ with the estimate*

$$\sup_{0<t<\infty} t^{\frac{1}{2}+\frac{|\alpha|}{2}}\|D^\alpha u(t)\|_{L^\infty} \le CK^{|\alpha|}|\alpha|^{|\alpha|}, \tag{1.11}$$

with an absolute constant K, where $C = C(n,p)$. In particular, such a mild solution $u(x,t)$ is uniformly analytic in $x \in \mathbb{R}^n$, namely

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0,t)}{k!}(x-x_0)^\alpha, \quad 0 < t < \infty \tag{1.12}$$

for all $x_0, x \in \mathbb{R}^n$ with $|x - x_0| < \frac{\sqrt{t}}{eK}$.

Remark 2 By (1.11) it holds that

$$\frac{\|D^\alpha u(t)\|_{L^\infty}}{k!} \leq \frac{CK^k k^k t^{-\frac{1}{2}-\frac{k}{2}}}{k!}, \quad \forall \alpha \in \mathbb{N}_0^n,$$

By the Stirling formula, we have that $\lim_{k\to\infty} \left(\frac{CK^k k^k t^{-\frac{1}{2}-\frac{k}{2}}}{k!} \right)^{\frac{1}{k}} = \frac{eK}{\sqrt{t}}$, from which it follows that the convergence radius in such a Taylor expansion as in (1.12) may be taken as $\frac{\sqrt{t}}{eK}$ uniformly at any point $x_0 \in \mathbb{R}^n$.

Remark 3 Based on Koch-Tataru's argument, Miura-Sawada [11, Theorem 1.1] constructed a mild solution for the initial data $a \in vmo^{-1}$ which is analytic in $\mathbb{R}^n$. They also showed that every solution in the Serrin class is analytic in $\mathbb{R}^n$. However, for existence of solutions in the Serrin class, they [11, Proposition 4.2] impose on $a \in bmo^{-1}$ the condition that $\int_0^\infty \|e^{t\Delta}a\|_{L^p}^s \, dt$ is sufficiently small. On the other hand, we make it clear that the solutions in the Serrin class exists if and only if the initial data a belongs to $\dot{B}_{p,s}^{-1+\frac{n}{p}}$. It should be noted that even for $a \in L^{n,\infty}$, $e^{t\Delta}a \notin L^s(0,\infty; L^p)$ for only s and p such that $2/s + n/p = 1$.

In this article, we only sketch the proof. The detail of the proof will be appeared in [9].

2 Outline of the proof of Theorems 1 and 2

We construct solutions by use of the implicit function theorem for Banach spaces (see, [10]). For using implicit function theorem, It needs controlling the Stokes flow in the Serrin class, and bilinear estimate of the Duhamel term. Therefore we prepare two following lemmata. The following first lemma plays a key role for the proof of Theorem 2.

Lemma 2.1 *Let $2 < s < \infty$ and $n < p < \infty$ satisfy $2/s + n/p = 1$.*
(1) For $a \in \dot{B}_{p,s}^{-1+n/p}$, it holds that $e^{t\Delta}a \in L^s(0,\infty; \dot{B}_{p,1}^0)$

$$\left(\int_0^\infty \|e^{t\Delta}a\|_{\dot{B}_{p,1}^0} \, dt \right)^{\frac{1}{s}} \leq C\|a\|_{\dot{B}_{p,s}^{-1+n/p}}, \tag{2.13}$$

where $C = C(n,p,s)$ is independent of a.
(2) Assume that $a \in \mathcal{S}'$ satisfies

$$e^{t\Delta}a \in L^s(0,\infty; L^p).$$

Then it holds that $a \in \dot{B}_{p,s}^{-1+\frac{n}{p}}$ *with the estimate*

$$\|a\|_{\dot{B}_{p,s}^{-1+\frac{n}{p}}} \le C\Big(\int_0^\infty \|e^{t\Delta}a\|_{\dot{B}^0_{p,1}}\,dt\Big)^{\frac{1}{s}}, \tag{2.14}$$

where $C = C(n,p,s)$ *is independent of* a.

Proof. (1) We take $n < p_0 < p < p_1$ and $0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Since $-1 + n/p_i < 0$ for $i = 0,1$, by using estimates of the heat semigroup $\{e^{t\Delta}\}_{t>0}$ in homogeneous Besov spaces (see, [8], [10]), we have

$$\|e^{t\Delta}a\|_{\dot{B}^0_{p_i,1}} \le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{p_i})}\|a\|_{\dot{B}_{p_i,\infty}^{-1+\frac{n}{p}_i}}, \quad i = 0,1. \tag{2.15}$$

We see that the mapping

$$\dot{B}_{p_i,\infty}^{-1+\frac{n}{p}_i} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{p_i,1}} \in L^{\alpha_i,\infty}(0,\infty), \quad i = 0,1,$$

is a bounded sub-additive operator for $\frac{1}{\alpha_i} = \frac{n}{2}(\frac{1}{n} - \frac{1}{p_i})$, $i = 0,1$. Here $L^{p,q}$ denotes the Lorentz space (see, e.g., Bergh-Löfström [1, Chapter 5]). Then it follows from the real interpolation theorem that

$$(\dot{B}_{p_0,\infty}^{-1+\frac{n}{p}_0}, \dot{B}_{p_1,\infty}^{-1+\frac{n}{p}_1})_{\theta,s} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{p,1}} \in (L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,s}. \tag{2.16}$$

Since $2/s + n/p = 1$, it holds that $1/s = (1-\theta)/\alpha_0 + \theta/\alpha_1$, which yields that

$$(L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,s} = L^{\alpha,s}(0,\infty) \subset L^{\alpha,\alpha}(0,\infty) = L^{\alpha}(0,\infty).$$

Since $(\dot{B}_{p_0,\infty}^{-1+\frac{n}{p_0}}, \dot{B}_{p_1,\infty}^{-1+\frac{n}{p_1}})_{\theta,s} = \dot{B}_{p,s}^{-1+\frac{n}{p}}$, we conclude from (2.16) that the mapping

$$\dot{B}_{p,s}^{-1+\frac{n}{p}} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{p,1}} \in L^s(0,\infty)$$

is a bounded sub-additive operator, which yields the desired estimate. This proves (1).

(2) We make use of the following characterization of the equivalent norm of the homogeneous Besov space $\dot{B}_{p',s'}^{1-\frac{n}{p}}$ due to Triebel [12]:

$$\|\varphi\|_{\dot{B}_{p',s'}^{1-\frac{n}{p}}} \backsimeq \Big\{\int_0^\infty (t^{1-\frac{1}{2}(1-\frac{n}{2p})}\|-\Delta e^{t\Delta}\varphi\|_{L^{p'}})^{s'}\frac{dt}{t}\Big\}^{\frac{1}{s'}} = \Big(\int_0^\infty \|-\Delta e^{t\Delta}\varphi\|_{L^{p'}}^{s'}\,dt\Big)^{\frac{1}{s'}},$$

where we have used the relation $2/s + n/p = 1$ with $1 - n/p = 2/s > 0$.

For $a \in \mathcal{S}'$, we take a dual coupling with $\varphi \in \mathcal{S}$.Since φ is expressed by $\varphi = e^{t\Delta}\varphi + \int_0^t(-\Delta)e^{\tau\Delta}\varphi\,d\tau$. We consider the coupling

$$|\langle a,\varphi\rangle| \le |\langle a, e^{t\Delta}\varphi\rangle| + \int_0^t |\langle a, (-\Delta)e^{\tau\Delta}\varphi\rangle|d\tau =: I_1(t) + I_2(t). \tag{2.17}$$

The second term of r.h.s. is estimated as

$$\begin{aligned}\int_0^t |\langle a, (-\Delta)e^{\tau\Delta}\varphi\rangle| d\tau &\le \int_0^t |\langle e^{\frac{\tau}{2}\Delta}a, (-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\rangle| d\tau \\ &\le (\int_0^t \|e^{\frac{\tau}{2}\Delta}a\|_{L^p}^s\, d\tau)^{\frac{1}{s}}(\int_0^t \|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{p'}}^{s'} d\tau)^{\frac{1}{s'}} \\ &\le 2(\int_0^t \|e^{\tau'\Delta}a\|_{L^p}^s\, d\tau')^{\frac{1}{s}}(\int_0^t \|(-\Delta)e^{\tau'\Delta}\varphi\|_{L^{p'}}^{s'} d\tau')^{\frac{1}{s'}}\end{aligned}$$

Since $a \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, it is easy to see that $I_1(t) \to 0$ as $t \to \infty$. Letting $t \to \infty$ in both side of (2.17), we obtain

$$|\langle a, \varphi\rangle| \le 2(\int_0^\infty \|e^{\tau'\Delta}a\|_{L^p}^s\, d\tau')^{\frac{1}{s}}\, \|\varphi\|_{\dot{B}^{1-\frac{n}{p}}_{p',s'}} \quad \text{for all } \varphi \in \mathcal{S}.$$

Since $\mathcal{S}$ is dense in $\dot{B}^{1-\frac{n}{p}}_{p',s'}$, it follows from the above estimate

$$\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,s}} = \sup_{\varphi\in\mathcal{S},\ \|\varphi\|_{\dot{B}^{1-\frac{n}{p}}_{p',s'}}=1} |\langle a, \varphi\rangle| \le 2\|e^{t\Delta}a\|_{L^s(0,\infty;L^p)},$$

which implies (2.14). This completes the proof of Lemma 2.1. ∎

We define the nonlinear term

$$N(u, v) = \int_0^t e^{(t-\tau\Delta}P(u\cdot\nabla v)(\tau)\, d\tau = \int_0^t P\nabla\cdot e^{(t-\tau)\Delta}(u\otimes v)(\tau)\, d\tau.$$

Next lemma shows bilinear estimates which will be used to control the nonlinear term $N(u, u)$.

Lemma 2.2 *Let $2 < s < \infty$ and $n < p < \infty$ satisfy $2/s + n/p = 1$.*

(1) It holds that

$$\|N(u, v)\|_{L^s(0,T;\dot{B}^0_{p,1})} \le C\|u\|_{L^s(0,T;\dot{B}^0_{p,1})}\|v\|_{L^s(0,T;\dot{B}^0_{p,1})} \tag{2.18}$$

for u, $v \in L^s(0, T; \dot{B}^0_{p,1})$ and for all $0 < T \le \infty$, where $C = C(n, p, s)$ is independent of T.

(2) We assume that $\sup_{0<t<\infty} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{\dot{B}^0_{p,1}} < \infty$ and $\sup_{0<t<\infty} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(t)\|_{\dot{B}^0_{p,1}} < \infty$. It holds that

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|N(u, v)(t)\|_{\dot{B}^0_{p,1}} \le C(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(\tau)\|_{\dot{B}^0_{p,1}})(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(\tau)\|_{\dot{B}^0_{p,1}}) \tag{2.19}$$

$$\|N(u, v)(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,s}} \le C(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(\tau)\|_{\dot{B}^0_{p,1}})(\sup_{0<\tau<t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|v(\tau)\|_{\dot{B}^0_{p,1}}) \tag{2.20}$$

for all $0 < t \le \infty$, where $C = C(n, p)$ is independent of T.

Proof. Using estimates of the heat semigroup $\{e^{t\Delta}\}_{t>0}$ in homogeneous Besov spaces (see, [8], [10]) and Hardy-Littlewood-Sobolev inequality.

3 Outline of the proof of Theorem 3

3.1 Hölder estimates for higher order derivatives

We assume the following assumption.

Assumption 1 *For* $\max(p, 2n) < q < \infty$*, there exist* $C = C(n,p,q)$ *such that for every* $k \in \mathbb{N}$*, u satisfies the estimate*

$$\|D^\alpha u(t)\|_{L^q} \le K_1 K_2^j j^j t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{j}{2}}, \quad t>0$$

for all multi-index $\alpha \in \mathbb{N}_0^n$ *with* $0 \le |\alpha| = j \le k-1$*, where* $K_1 = K_1(n,p,q)$ *and* $K_2 = K_2(n,p,q)$*.*

Proposition 3.1 *Let* $\max(p,2n) < q < \infty$ *and let* $a \in P\dot{B}_{p,s}^{-1+\frac{n}{p}}$ *for* $2/s + n/p = 1$ *with* $n < p < \infty$*. Suppose that u is a mild solution of (N-S) satisfying Assumption 1. Then for every* $k \in \mathbb{N}$*, u fulfills the estimate*

$$\|D^\alpha u(t+h) - D^\alpha u(t)\|_{L^q} \le C(h^{\frac{1}{2}-\frac{n}{2q}} t^{-\frac{1}{2}+\frac{n}{2q}} + h^{\frac{1}{4}} t^{-\frac{1}{4}}) K^j j^{j+\frac{1}{2}} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{j}{2}}, \quad t>0 \qquad (3.1)$$

for all $\alpha \in \mathbb{N}_0^n$ *with* $0 \le |\alpha| = j \le k-1$*, where* $C = C(n,p,q)$ *and* $K = K_2(n,p,q)$ *same as in Assumption 1.*

In order to prove Proposition 3.1, we make use of the following representation formula:

$$\begin{aligned} u(t+h) - u(t) &= (e^{h\Delta} - I)e^{t\Delta}a - \int_t^{t+h} e^{(t+h-\tau)\Delta} P\nabla\cdot(u\otimes u)(\tau)\,d\tau \\ &\quad - \int_0^{(1-\epsilon_j)t} (e^{h\Delta}-I)e^{(t-\tau)\Delta} P\nabla\cdot(u\otimes u)(\tau)\,d\tau \\ &\quad - \int_{(1-\epsilon_j)t}^{t} (e^{h\Delta}-I)e^{(t-\tau)\Delta} P\nabla\cdot(u\otimes u)(\tau)\,d\tau \\ &= I_1^h(t) + I_2^h(t) + I_3^h(t) + I_4^h(t), \end{aligned}$$

where we set $\epsilon_j = \frac{1}{2}$ for $j = 0$, 1 and $\epsilon_j = \frac{1}{j}$ for $j \ge 2$.

We prepare the following lemmata. Using the following lemmata and Proposition, we can estimate the higher order derivatives of $I_1^h(t)$ to $I_4^h(t)$ and obtain Proposition 3.1.

Lemma 3.1 *Let* $G_t = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ *be the Gauss kernel and* $\alpha \in \mathbb{N}_0^n$ *be a multi-index with* $|\alpha| = k$*. Then it holds that*

$$\|D^\alpha G_t\|_{L^1} \le \pi^{-\frac{k}{2}} k^{\frac{k}{2}} t^{-\frac{k}{2}}, \quad t>0.$$

Lemma 3.2 *For every* $k \in \mathbb{N}$ *it holds that*

$$\sum_{\ell=0}^{k} \binom{k}{\ell} \ell^\ell (k-\ell)^{k-\ell} \le C k^{k+\frac{1}{2}},$$

with an absolute constant $C > 0$*.*

Here $\binom{k}{\ell} = \dfrac{k!}{\ell!(k-\ell)!}$.

3.2 Analyticity of the solution

In order to prove Theorem 3, it is enough to prove for $2n < q < \infty$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ there exist $K_1 = K_1(n,p,q)$ and $K_2 = K_2(n,p,q)$ such that

$$\|D^\alpha u(t)\|_{L^q} \le K_1 K_2^k k^k t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} \tag{3.1}$$

by use of the Gagliardo-Nirenberg inequality. In order to prove (3.1), we make use of the following representation formula:

$$\begin{aligned} u(t) &= e^{t\Delta}a - \int_{(1-\frac{1}{k})}^{t} e^{(t-\tau)\Delta}P\nabla\cdot(u\otimes u)(t)\,d\tau \\ &- \int_0^{(1-\frac{1}{k})} e^{(t-\tau)\Delta}P\nabla\cdot(u\otimes u)(\tau)\,d\tau - \int_{(1-\frac{1}{k})}^{t} e^{(t-\tau)\Delta}P\nabla\cdot\{(u\otimes u)(\tau)-(u\otimes u)(t)\}\,d\tau \\ &= J_1(t)+J_{2,k}(t)+J_{3,k}(t)+J_{4,k}(t), \end{aligned}$$

which is defined for $k \ge 2$.

There are difficult to deal with singularity at $\tau = t$. Therefore, we estimate $J_{4,k}(t)$ by use of Propsition 3.1.

Lemma 3.3 *Let $n < p < \infty$, and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$. For $\max(p, 2n) < q < \infty$, there exists a constant $C_{J_4} = C_{J_4}(n,p,q)$ satisfying*

$$\|D^\alpha J_{4,k}(t)\|_{L^q} \le C_{J_4} K_1^3 K_2^{k-1} k^k t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \quad t > 0. \tag{3.2}$$

Proof. By Lemmata 3.1, 3.2, and Proposition 3.1, for β, $\gamma \in \mathbb{N}_0^n$ with $|\beta| = k-1$ and $|\gamma| = \ell$ we have

$$\begin{aligned} &\|D^\alpha J_{4,k}(t)\|_{L^q} \\ \le& \int_{(1-\frac{t}{k})t}^{t} \|D^\alpha e^{(t-\tau)\Delta}P\nabla\cdot((u\otimes u)(\tau)-(u\otimes u)(t))\|_{L^q}\,d\tau \\ \le& C\int_{(1-\frac{t}{k})t}^{t} (t-\tau)^{-1-\frac{n}{2q}}\|D^\beta((u\otimes u)(\tau)-(u\otimes u)(t))\|_{L^{\frac{q}{2}}}\,d\tau \\ \le& C\int_{(1-\frac{t}{k})t}^{t} (t-\tau)^{-1-\frac{n}{2q}}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\{\|D^{\beta-\gamma}(u(\tau)-u(t))\|_{L^p}\|D^\gamma u(\tau)\|_{L^q} \\ &\qquad + \|D^{\beta-\gamma}u(t)\|_{L^q}\|D^\gamma(u(\tau)-u(t))\|_{L^q}\}\,d\tau \\ \le& CK_1^3K_2^{k-1}(k-1)^k t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \end{aligned}$$

where we used $\frac{n}{q} < \frac{1}{2}$ and $\frac{n}{2q} < \frac{1}{4}$ by the assumption $2n < q$. This completes the proof of the lemma. ∎

Therefore, we can take sufficiently large K_1, K_2 such that

$$\|D^\alpha u(t)\|_{L^q} \le K_1 K_2^k k^k t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}.$$

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