Von Neumann-Jordan constant of generalized Banaś-Frączek spaces [1]

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概 要

バナッハ空間の von Neumann-Jordan 定数に関する最近の結果について報告する．特に，Banaś-Frączek 空間を含む 2 次元ノルム空間を導入し，Banach-Mazur 距離の概念を用いてこの空間における von Neumann-Jordan 定数を計算する．

Definition 1 ([2]). The von Neumann-Jordan (NJ-) constant of a Banach space X, denoted by $C_{\mathrm{NJ}}(X)$, is the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leq C$$

holds for all $x, y \in X$ not both 0.

Definition 2 ([1, 10]). For $\lambda > 1$, the Banaś-Frączek space $\mathbb{R}^2_\lambda$ is the space $\mathbb{R}^2$ with the norm $|\cdot|_\lambda$ defined by

$$|(x,y)|_\lambda = \max\left\{\lambda|x|, \|(x,y)\|_2\right\},$$

where $\|\cdot\|_2$ is the ℓ_2-norm on $\mathbb{R}^2$.

Yang-Wang [13] は幾何学的定数 $\gamma_X(t)$ を導入し，これを用いて ℓ_2-ℓ_1 と ℓ_∞-ℓ_1 における NJ 定数を計算した．この方法と同様にして Yang [10] は Banaś-Frączek space の NJ 定数を計算した．

Theorem 3 ([10]) For $\lambda > 1$,

$$C_{\mathrm{NJ}}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}.$$

[1] *Keywords.* von Neumann-Jordan constant, Banach-Mazur distance
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本研究では，Banaś-Frączek space を含む次の 2 次元ノルム空間を導入し，Banach-Mazur distance を用いてこの空間の NJ 定数を計算する．

Definition 4. For $a \geq b \geq 1$ and $1 \leq p < \infty$, $\mathbb{R}^2_{a,b,p}$ is defined as $\mathbb{R}^2$ with the norm $\|\cdot\|$ on $\mathbb{R}^2$ by

$$\|(x,y)\| = \max\{a|x|, b|y|, \|(x,y)\|_p\},$$

where $\|\cdot\|_p$ is the ℓ_p-norm on $\mathbb{R}^2$.

Definition 5. For isomorphic Banach spaces X and Y, the Banach-Mazur distance between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y.

Lemma 6 ([4]). If X and Y are isomorphic Banach spaces, then

$$\frac{C_{\mathrm{NJ}}(X)}{d(X,Y)^2} \leq C_{NJ}(Y) \leq C_{\mathrm{NJ}}(X) d(X,Y)^2.$$

In particular, if X and Y are isometric, then $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(Y)$.

Lemma 7 ([4]). Let $X = (X, \|\cdot\|)$ be a Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$,

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|, \quad x \in X.$$

Then

$$\frac{\alpha^2}{\beta^2} C_{\mathrm{NJ}}(X) \leq C_{\mathrm{NJ}}(X_1) \leq \frac{\beta^2}{\alpha^2} C_{\mathrm{NJ}}(X).$$

Definition 8. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|,|y|)\| = \|(x,y)\|$ for any $x, y \in \mathbb{R}$.

Lemma 7 を用いて，absolute norm の NJ 定数に関する公式を得る．簡単のため $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|))$ を $C_{\mathrm{NJ}}(\|\cdot\|)$ とかく．

Theorem 9. Let $\|\cdot\|, \|\cdot\|_H$ be absolute norms on $\mathbb{R}^2$. Assume that the following hold:

(i) $(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space.

(ii) $\alpha\|(x,y)\|_H \leq \|(x,y)\| \leq \beta\|(x,y)\|_H$ for any $(x,y) \in \mathbb{R}^2$ (α, β are the best constants).
(iii) In (ii) it satisfies either $\alpha\|(1,0)\|_H = \|(1,0)\|$ and $\alpha\|(0,1)\|_H = \|(0,1)\|$, or $\beta\|(1,0)\|_H = \|(1,0)\|$ and $\beta\|(0,1)\|_H = \|(0,1)\|$.
Then

$$C_{\mathrm{NJ}}(\|\cdot\|) = \frac{\beta^2}{\alpha^2}.$$

上の公式を用いて，$\mathbb{R}^2_{a,b,p}$ の NJ 定数を計算する．$1/a^p + 1/b^p \leq 1$ のとき明らかに $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 2$．また $a = b = 1$ のとき $\|(x,y)\| = \|(x,y)\|_p$ であるから $a > 1, 1/a^p + 1/b^p > 1$ の場合のみ考えればよい．

Definition 10． For $a \geq b \geq 1$, $\|\cdot\|_H$ is defined by $\|(x,y)\|_H = \|(ax,by)\|_2$.

$p \geq 2$ とする．Definition 4 と Definition 10 で定義した $\|\cdot\|$, $\|\cdot\|_H$ は absolute であり，さらに Theorem 9 の (i)，(ii)，(iii) の条件をみたす．Theorem 9 を適用することで次が得られる．

Theorem 11 ([5]) Let $a > 1, a \geq b \geq 1$ and $p \geq 2$ with $\dfrac{1}{a^p} + \dfrac{1}{b^p} > 1$.
(i) If $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{\frac{2}{p}}.$$

(ii) If $b > a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = b^2\Big(1 + \Big(\frac{a}{b}\Big)^{\frac{2p}{p-2}}\Big)^{1-\frac{2}{p}}.$$

In particular, $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = d(\mathbb{R}^2_{a,b,p}, H)^2$, where H is a two-dimensional inner product space.

この結果は Theorem 3 を含む．また，$p < 2$ の場合も同様の方法で計算することができる．

Theorem 12 Let $a > 1, a \geq b \geq 1$ and $p < 2$ with $\frac{1}{a^p} + \frac{1}{b^p} > 1$ and $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$.
Then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{\frac{2}{p}}.$$

In particular, $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = d(\mathbb{R}^2_{a,b,p}, H)^2$, where H is a two-dimensional inner product space.

Corollary 13 Let $a > 1, a \geq b \geq 1$ with $\frac{1}{a} + \frac{1}{b} > 1$ and $\frac{1}{a^2} + \frac{1}{b^2} \leq 1$. Then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,1}) = 1 + b^2\Big(1 - \frac{1}{a}\Big)^2.$$

In particular,

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,a,1}) = a^2 - 2a + 2.$$

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