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Search for an Algebraic Structure of Not-Necessarily Algebraic Structures

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1. Introduction

This is a continuation of author's research presented and published earlier on the subject of symmetry and structural study of information. The present paper is directly related to the earlier paper in which the study of symmetry was generalized to arbitrary closure spaces.[1] Moreover, it continues the direction of research reported in another paper in which this generalization of symmetry to closure spaces together with the recognition of the fact that the set of closure spaces on a given set S is a family of closed subsets for the unique closure space on the power set of S justified the introduction and exploration of the concept of meta-symmetry.[2] Meta-symmetry, although it may be a subject of independent mathematical interest, was intended as an algebraic tool for the study of a wide range of structures across many mathematical theories from logic, algebra and geometry to topology and topological algebras, to study of information and information processing.

The original motivation for author's interest in symmetry was the question about the meaning of structural manifestation of information, when information is understood in a very general way which unifies multiple instances of the use of the term across a very wide range of contexts. With the question about structural manifestation of information comes the question about structure. In order to formulate a general theory of structural manifestations of information we need a general theory of structures. The present paper is still very far from giving definite form to a theory of structures. It is more an attempt to provide a background for this ultimate goal in which the question "What is a mathematical structure?" is the main point of reference.

The question is highly non-trivial, although frequently ignored by practicing mathematicians focused on own specialized field of expertise. Thus algebraists understand each other very well when they use the term "algebraic structure" or just "general algebra". Topologists do not question the concept of a topological space, although there are some ramifications regarding the direction of generalization. To some extent similar mutual understanding can be among those who use terms "geometric structure", "logical structure", etc., but this mutual understanding may be deceiving, as rarely the issue of general concepts of structures is discussed. More often the authors write a definition of what is the instance of a very specific mathematical object "By such and such structure we mean ..." and no questions are asked why the term "structure" is applied. We have an abundance of very different examples in mathematics how the term "structure" is used. However, this diversity is not only in properties of objects called structures, but also in the conceptual frameworks forming their contexts. Because of this conceptual variety making common generalization is very difficult.

The attempts to answer the question about the meaning of the concept of a mathematical structure were not successful. The opening paragraph in the Introduction to Leo Corry's book *Modern Algebra and the Rise of Mathematical Structures* is as accurate today as it was in 2004: "It is commonplace for mathematicians and non-mathematicians alike to refer to the structural character of twentieth century mathematics, or at least of considerable parts of it. Such references can be found in mathematical texts, as well as in historical accounts and in philosophical or semi-philosophical debates concerning the discipline. However, when one attempts to discern the meaning attached to the term 'mathematical structure' in the various places where it appears, one soon realizes that, while most writers using it take its meaning for granted and feel no need to add further clarifications, they often in fact ascribe diverging meanings to the term. In some cases, they ascribe to it no clear meaning at all."[3]

The opening, admittedly speculative hypothesis in the present paper is that the source of difficulties in defining and studying the general concept of a mathematical structure was in all earlier attempts a methodological hidden assumption that we should seek an elementary theory for this concept, i.e. that we should avoid the logical tools of higher order predicate calculus. The present author's experience in the studies of the concept of a structure, as well as that of symmetry is that both require engagement of at least sets, their power sets, and power sets of power sets. Whenever we try to restrict this multilevel hierarchy, we lose the opportunity to make transitions between different types of structures and on the other hand we lose the unifying power of the concept of a structure which can manifest itself in diverse ways (currently these diverse manifestations are called *cryptomorphic*).

In practice, these diverse manifestations of structures are left to the choice guided by an individual preference rather than a clear rule. Transition between different cryptomorphic formulations of the concepts understood as the same structure is more a translation of terminology than clearly defined general process. In each case translation is a subject of different procedure. Topological space can be defined by its family of open sets, closed sets, more than half a dozen of other families, by a closure operator, opening operator, derived set operator, or several other operators, and by many other means. We cannot say that the structures defined by this large variety of defining concepts are isomorphic as the idea of isomorphism requires conceptual homogeneity. However, we can reconstruct in one conceptual framework concepts of the other and then we can perform comparisons. The presence of equivalences or at least similarities between these constructions is the basis for establishing a cryptomorphic relationship.

In particular cases, e.g. in topology, as long as we are not drifting in generalization too far from the paradigmatic case of a metric space nothing prevents the use of a simple individual preference in the choice of the conceptual framework and of the definitions of particular instances of structures. But at some point ramifications generated by generalization block transitions between topological spaces defined in different conceptualizations. For instance more general topological structures considered by Eduard Čech in his influential monograph *Topological Spaces* are defined by pre-closure operators and cannot be determined by the family of open sets or family of closed sets. [4]

Thus the fundamental problems in the study of general structures are twofold, those related to the definition of a structure independent from the choice of specific concepts within mathematical theories and those related to the complementary issue of what constitutes the identity of a structure. Answers to the latter question are attempted in terms of concepts specific for particular mathematical theories in which appropriate definitions of an *isomorphism* (in algebra) or *homeomorphism* (in topology) preserving structure are possible. In each case when isomorphic or homeomorphic structures can be defined, they are considered essentially identical. However, as long as we do not have the concept of a general structure, we cannot select which bijective functions can be identified as structure-preserving. Therefore, the question about the meaning of the concept of a general structure is inseparable from the question about general isomorphism.

Although the general methodological hypothesis stated above that solving these two problems to provide a general definition of a structure and to provide criteria for the identity of structures with multiple cryptomorphic manifestations is very difficult, or maybe is impossible within an elementary theory of structures most likely cannot be verified, but only falsified (if such elementary theory is developed), the present paper provides several arguments for its justification. The further sections of the paper, after presentation of an additional historical background of the issues outlined above, complementary to already presented in an earlier paper direction of thought based on the study of symmetry,[1] present some introductory results pointing at the possible ways to formulate and to develop a future general theory of structures.

2. Attempts to Define Mathematical Structures

Corry's book *Modern Algebra and the Rise of Mathematical Structures* quoted above presents a very thorough, although insulated from other parts of mathematics historical account of the rise of mathematical structures within the main stream of algebra in the 19th and early 20th centuries.[3] In his interpretation the turning point was the publication in 1930 by Bartel Leendert van der Waerden his influential *Moderne Algebra*, [5] which ultimately changed the image of algebra as a theory of solving equations to a study of algebraic structures understood as sets equipped with one or more operations satisfying some axioms. Before publication of this book the expression algebraic structure was a generic name for a variety of objects of algebraic interest such as groups, rings, fields, etc., but not a clearly defined concept. Van der Waerden introduced our modern concept of a general algebraic structure as a set with one or more operations characterized by axioms describing the operations and their mutual relations.

Surprisingly Corry's presentation of the rise of algebraic structures did not refer directly to the concept of symmetry, although he refers to algebraic invariants and to some works of Emmy Noether, but not to her fundamental theorem (simply called Noether theorem without fear of confusion) associating symmetries for physical systems with conservation laws, theorem which gave the concept of symmetry its status of primary tool in physics.[6]

This is a manifestation of more general socio-scientific phenomenon in the study of structures and of its historical roots. There are two perspectives on the origins of the concept of structures and on the development of methodology of their studies. One derives the concept of a structure from the study of symmetry. In this approach associated with Felix Klein's Erlangen Program structures are invariants of some group of transformations.[7] The paradigm of this approach is Klein's idea to unify different forms of geometry as a study of the invariants of isometries, i.e. bijections which preserve metric (i.e. distance of points). Klein's program was focused on geometry, but it could be and actually was easily extended beyond mathematics, to physics and later even to philosophy, anthropology, etc.[1]

The other approach, exemplified by Corry's book on history of mathematical structures or by the book *Scientific Realism* edited by Alisa Bokulich and Peter Bokulich,[8] addressing ontological and epistemological status of structures in the context of scientific exploration of reality where Klein's program is marginalized or sometimes not even mentioned (as in the second of the books used here as illustration). This strange silence on symmetry as a way to study structures and as a source of structuralism as a dominating for several decades direction in philosophy is definitely myopic. Of course, not all studies of structures referred to symmetry. However, there is no doubt that importance of structures and structural analysis in mathematics, physics and science in general is a consequence of their association with symmetry.

This does not mean that the literature focused on the sources of interest in mathematical structures not related directly to symmetry is devoid of merit or value. Since the role of symmetry and its relation to the study of structures was already presented in the earlier paper of the present author, [1] in this paper and this section the emphasis is on the role of other ideas.

Corry in his book *Rise of Mathematical Structures* observes rightly that the structural way of thinking can be found in the works of Richard Dedekind, before Klein formulated his Erlangen Program. The construction of real numbers with what we now call "Dedekind cuts" was conceived by him at the beginning of his mathematical career at the end of the 1850's when he taught at the Polytechnic in Zürich (now ETH). This does not constitute evidence that that Dedekind in his thinking could not be influenced by the same ideas which motivated Klein. After all he already lectured on Galois Theory in Göttingen before moving to Zürich. It is more an example of an emergence of the great scale idea from collective work consisting of a network of mutually dependent "local" explorations and individual contributions.

Equally clear presence of structural thought can be identified in the work of William Hamilton on quaternions in the 1840's. This example is of special importance, since Hamilton constructed a new structure never considered before. What qualifies ways of thinking in the works of Dedekind and Hamilton as structural is the general idea of a collection of objects which is closed with respect to operations or constructions defined on them. The word "structure" was of humble provenience and in the mid-19th century did not mean much more than its Latin prototype corresponding to a heap of objects. Gradually it overshadowed and replaced the term "form" present in the intellectual intercourse for many centuries to become the most frequent, although vague term for anything which has components organized in some way.

The first comprehensive and programmatic attempt to develop a definition and theory of structures was initiated in the 1930's in the collaborative effort of the group of French mathematicians presenting themselves as Nicholas Bourbaki. At that time the word "structure" was always qualified by terms referring to particular set of axioms developed within a variety of mathematical disciplines (algebraic structure, topological structure, metric structure, order structure, etc.). Members of the Bourbaki group wanted to go beyond mathematics understood as a menagerie of creatures and to make it an organic whole. This was clear from the very beginning of the cooperation in the 1930's and appeared later in 1950 in a programmatic format as a Bourbaki manifesto *The Architecture of Mathematics* written by Jean Dieudonné (but of course signed Nicholas Bourbaki).[9] The architecture consisted of three fundamental pillars - three "mother structures" (or types of structures): algebraic, order and topological, and the edifice consisting of "multiple structures") constructed on this foundations by suitable hybridization.

It is a natural consequence of the programmatic interests of the members of Bourbaki group that they felt obligation to develop a general theory of structures. At first sight it may look that they achieved that which was declared above as the goal of present paper. In the volume devoted to set theory whose chapters appeared in the 1950's in French and 1968 in English,[10] but whose summary was already presented in 1939, [11] general structures were described in terms of a recursive process starting from the finite selection of fundamentals and proceeding to stepwise combinations of set theoretical operations of the direct product and the constructions of a power set and functions applied to the results of earlier steps.[3] The recursive character of the definition opened it to arbitrarily high hierarchy of potential levels. Bourbaki's definition was slightly simplified by Saunders MacLane in his 1996 article Structure in Mathematics in which he presents own view on the subject of mathematical structures:

"Their [Bourbaki] massive and widely used multivolume treatment of the 'Eléments de Mathématique', with a first part entitled 'Les structures fondamentales de l'analysé' began with volume 1, 'Theorie des ensembles, Fascicule de resultats'. In this volume, Bourbaki carefully describes what he means by a structure of some specific type T. We do not really need to use this description, but we will now present it, chiefly to show both that one can indeed define 'structure' and that the explicit definition does not really matter. It uses three familiar operations on sets: the product E x F of two sets E and F, consisting of all the ordered pairs (e, f) of their elements, the power set P(E), consisting of all the subsets S of E, and the function set E^F consisting of all the functions mapping F into E. For example, a topological structure on E is given by an element T of P(P(E)) satisfying suitable axioms—it is just the set consisting of all open sets, that is, the set T of all those sets U in P(E) which are open in the intended topology. Similarly, a group structure on a set G can be viewed as an element $M \in G^{GxG}$ with the usual properties of a group multiplication. With these examples in mind, one may arrive at Bourbaki's definition of a structure, say one built on three given sets E, F, and G. Adjoin to these sets any product set such as E x F, any function set such as E^G and any power set such as P(G). Continue to iterate this process to get the whole scale (échelle) of sets M successively so built up from E, F, and G. On one or more of the resulting sets M impose specified axioms on a relevant element (or elements) m. These axioms then define what Bourbaki calls a 'type' T of 'structure' on the given sets. This clearly includes algebraic structures like groups, topological structures, and combined cases such as topological groups (the definition also includes many bizarre examples of no known utility). Actually, I have here modified Bourbaki's account in an inconsequential way; he did not use function sets such as E^F. This modification does not matter; as best I can determine, he never really made actual use of his definition, and I will not make any use here of my variant. It is here only to show that it is indeed possible to define precisely 'type of structure' in a way that covers all the common examples."[12]

MacLane is right that neither Bourbaki, nor probably anyone else actually used this definition? In this assessment he is in agreement with very critical opinion of Corry.[3] Moreover, in spite of the declaration of desire for strict rigor the presentation of the definition was never made complete and some generalizations were left to the reader. [3] MacLane maintained that the source of the failure was in the fact that generalizations of this type make sense only, if they can be useful in mathematical practice of solving problems. The present author believes that the reason was much more fundamental.

Bourbaki's definition is an overgeneralization which does not permit study of structure's identity (second postulate for adequate studies of structures presented above). Practically unlimited and unspecified recursive procedure produces known structures as very few and undistinguished and the majority of the products are monsters devoid any meaning. The types of structures are either too broad, or too narrow. They do not help in understanding cryptomorphic manifestations of structures. Finally, it is interesting that the critics of this approach never pointed out inconsistency in Bourbaki's program. If this is a definition of a general mathematical structure, there is no reason to claim that mother structures (algebraic, order, and topological) have any special role to play. But, if not mother structures, what should be these distinctive, fundamental structures? The definition does not give answer, as the freedom of construction is too high. Instead of providing the answer to the question "What is a structure?" Bourbaki develops a conceptual framework (or language) for its study.

MacLane presented Bourbaki's approach to structures vis a vis category theory of his and Eilenberg's authorship. The confrontation was with a fair degree of objectivity, but not very convincing. He claimed possible equivalence between the two approaches, but only in hypothetical way. Of course, he did not hide the preference for category theory as simpler and of higher practical value. However, the present author believes that this is not an alternative for the study of structures, for the simple reason that category theory avoids entering the actual structural study. Objects (read structures) are black boxes which we study through the analysis of morphisms (here is the core of pragmatic value of category theory and the source of aesthetical simplicity). But as long as we do not know structures inside of black boxes, we cannot establish morphisms. Of course we get beautiful, clean results for properties which are not dependent on specific features of structures. We get a great tool to consider constructions combining structures without making our hands dirty with details. But, it is frequently this dirt which describes the essence of structures.

Both MacLane and Corry refer to the works of mathematicians studying structures independent from German or French mathematical schools. MacLane recalls from his youth own inclination to thinking about structures in algebraic way similar to his student time class mate from Göttingen Oystein Ore. Ore engaged in the study of lattices as tools for structural analysis in parallel to research interests of Garret Birkhoff. Actually Ore originally called lattices simply "structures" as he believed that lattices of substructures of given structure carry its entire structural information. [13,14] His hopes did not realize. Non-isomorphic algebraic structures can have isomorphic lattices of substructures, even for as regular structures as groups. It is a speculative thought, but it is possible that Ore's approach was not successful because he did not associate the study of structures with the study of symmetry. However, looking at the direction of his work it seems that he was aware where he could find tools for this study. After all he initiated the study of general closure spaces and of Galois connections.[15,16,17] Another source of potential obstacles could be in the attempt to avoid going beyond first order predicate calculus.

The direction of study of the present author is in some sense aligned with Ore's interests. However, with the collective experience of algebraic work in dozens of years of research in lattice theory and closure spaces it became clear that the lattice of substructures of given structure is only one of several tools for the study. Yet the question about the role of the complete lattice of substructures remains without full answer. However, it should be reformulated from the question "Why the lattice of substructures of a structure does not carry full structural information?" to "Why does it carry any structural information at all?"

3. Structures in Terms of Meta-Closure

The difficulty in the study of general structures consists in the temptation to formulate the question about structures as "What is the structure of a general structure?" Of course this question is faulty as it seeks the answer in terms of what we want to explain. However, this can suggest the approach to the study. The conceptual framework should have this feature that it can be applied to itself. To be sure, it is not a call for self-reference which leads to logical problems.

We have an example of mathematical concept with this characteristic, which was called in earlier papers of the author "autological".[2] In linguistics, the adjectives expressing properties which do not apply to themselves are called "heterological" (e.g. "long"), that which can (e.g. "short") are called "autological". In this case, mathematical concept is autological, if its individual instance can be used to describe all its instances. An example of autological concept is a closure space defined by its Moore family of closed subsets.

All possible closure spaces on a given set S (i.e. all possible closure operators on this common set S) can be associated with corresponding Moore families $\mathcal{M}$ of closed subsets. Then each of these Moore families of closed subsets $\mathcal{M}$ can be considered a closed subfamily of the power set $\wp = 2^S$ of S. For arbitrary family $\mathcal{B}$ of subsets of S we can find the least Moore family $\mathcal{M}$ of subsets including this family ($\mathcal{B} \subseteq \mathcal{M}$). Equivalently, for every family $\mathcal{B}$ of subsets of S we can consider a larger family $\mathcal{M}$ with all intersections of subsets belonging to the original family $\mathcal{B}$. Obviously this will be a Moore family $\mathcal{M}$ considered before, i.e. the least Moore family including the original one. We assume that the intersection of every empty family is entire power set of S.

Definition: *We can define this closure operator* $\boldsymbol{f}$ *on* $2^S = \wp(S)$ *by:*
$\forall\mathcal{B} \subseteq 2^S: f(\mathcal{B}) = \{B \subseteq S: \exists\mathfrak{C} \subseteq \mathcal{B} : B = \cap\, \mathfrak{C}\}$. *The power set equipped with this closure operator can be called a meta-closure space.* [1]

Thus, the set of all closure spaces defined on S, or all closure operators defined on S defines one specific closure operator $\boldsymbol{f}$ on the power set $\wp$. The closure space defined by this closure operator on the power set of S is called a meta-closure space. The closed subsets of a meta-closure space are directly and bijectively corresponding to closure operators on S. Some properties of this closure operator were presented elsewhere.[1]

Autological characteristic of closure spaces suggest that they are good candidates for the study of general structures. This can be amplified by discussed earlier special role of lattices in structural analysis of algebraic structures promoted by Ore. The missing link, often overlooked, between Ore's study and closure spaces is that when we consider the system of subalgebras (subgroups, subrings, etc.) we deal not just with lattices of subalgebras, but complete lattices of subalgebras. In the finite case there is no difference of course, but there is big difference in the infinite case. In complete lattice, all closed ideals are principal ideals. [18] Which means every complete lattice can be considered a Moore family of closed subsets for some closure space defined on the same set on which the lattice is defined. This shows that the role of the complete lattice of subalgebras is coming not from the special role of lattices, but more likely from the special role of closure spaces.

If there is one concept which is necessary in every mathematical theory and therefore in every study of mathematical structures, it is equivalence relation. Equivalence relation and its generalization – similarity or tolerance relation can be formalized within algebra of binary relations. A binary relation on set S is a subset of the direct product S×S. As the set $\mathcal{R}(S)$ of all binary relations on S is a set of subsets of S×S, and therefore a set, it can be ordered by inclusion. We have for relations R and T in $\mathcal{R}(S)$: $R \leq T$ *iff* $xRy \Rightarrow xTy$. We can consider Boolean operations on $\mathcal{R}(S)$ by importing set theoretical operations from S×S. The Boolean operations distinguish empty relation Ø and full relation S×S. We can also define complementary relation R^c in $\mathcal{R}(S)$ by: $\forall x,y\in S$: xR^cy *iff* not xRy, or in other words: $\forall x,y\in S$: xR^cy *iff* $(x,y)\notin R$.

The only nontrivial operation specific for relations giving $\mathcal{R}(S)$ its rich structure is the composition operation: $\forall x,y\in S$: xRTy *iff* $\exists z\in S$: xRz and zTy. The relation E of equality E={(x,y): x=y} is compatible with the order and gives $\mathcal{R}(S)$ the structure of ordered monoid. Another important unary operation on $\mathcal{R}(S)$ is converse R* defined by $\forall x,y\in S$: xR*y *iff* yRx. Now we can distinguish the classes of binary relations of special interest for us:

- R is *symmetric* if R = R*,

- R is *reflexive* if $E \leq R$,
- R is *transitive* if $R^2 = RR \leq R$,
- R is *antisymmetric* if $R \wedge R^* \leq E$,
- R is *weakly reflexive* if $\forall x \in S$: $(xR^c x \Rightarrow \forall y \in S: xR^c y)$.

With the basic concepts of relation algebra we can proceed to the concept of polarity, i.e. Galois connection generated by a binary relation. Let S be aset, and $R \subseteq S \times S$ be a binary relation in S. We define: $R^a(A) = \{y \in S: \forall x \in A: xRy\}$, $R^e(A) = \{y \in S: \exists y \in A: xRy\}$.

Since for one element sets the two sets coincide we can simplify our notation for single element subsets: $R(x) = R^a(\{x\}) = R^e(\{x\})$. Obviously: $R^a(A) = \cap\{R(x): x \in A\}$ and $R^e(A) = \cup\{R(x): x \in A\}$. If R is a binary relation in a set S, the compositions of the pair of functions $\varphi: 2^S \rightarrow 2^S$ and $\psi: 2^S \rightarrow 2^S$ between the power sets of S defined on subsets A, B of S by $\varphi: A \rightarrow R^a(A)$ and $\psi: B \rightarrow R^{*a}(B)$ are functions $f(A) = \varphi\psi(A) = R^a R^{*a}(A)$ and $g(B) = \varphi\psi(B) = R^{*a} R^a(B)$ which both are transitive closure operators. Also, the functions φ, ψ are dual isomorphisms between the complete lattices L_f and L_g of closed subsets for the closure operators f and g. Here we can see that every binary relation on a set S generates a closure operator f on this set.

Probably the best known example of the use of closure operators generated by polarity is Dedekind's "completion by cuts". In this case we have a reflexive, antisymmetric and transitive relation, i.e. a partially ordered set $<Q, \leq>$ in which: $\leq^a(A) = \{x \in S: \forall y \in A, y \leq x\}$, $\leq^{*a}(A) = \{x \in Q: \forall y \in A, y \geq x\}$ and the closure operator f defined by $f(A) = \leq^{*a}\leq^a(A)$. This gives us MacNeille's completion:

Let $[P, \leq]$ *be a poset and* φ *a function from P to the complete lattice* L_c *of the* f_c *-closed subsets of P defined by* $\varphi(x) = \leq^{*a}\leq^a(\{x\})$. *Then* φ *is an injective, isotone and inverse-isotone function preserving all suprema and infima that happen to exist in the poset* $[P, \leq]$. *MacNeille's completion is isomorphic to the original poset whenever it is already a complete lattice.*

In the very special case when Q is the set of rational numbers and the partial order is the usual order of rational numbers, the result of this completion by cuts is the set of real numbers with its order. This means, real numbers correspond to the closed subsets of rational numbers with respect to this Galois closure $f(A) = \leq^{*a}\leq^a(A)$.

All what was written about arbitrary relation R and generated by it Galois closure operators is of course valid for symmetric relations (defined by the additional condition $R = R^*$). Then Galois closures simplify to one with one function φ, and $f(A) = \varphi\varphi(A) = R^a R^a(A)$ with closed subsets distinguished by the condition: $A = f(A)$ *iff* $\exists B \subseteq S: A = R^a(B)$, which follows directly from the general rule: $\forall B \subseteq S: R^a(B) = R^a R^a R^a(B)$.

We can see that to every symmetric relation $R \in \mathscr{R}(S)$ corresponds a closure operator f, but it is possible that two different symmetric binary relations produce the same closure operator f. The correspondence becomes clear when we consider the complete lattice L_f of closed subsets with respect to the closure operator f. The function φ defines involution (involutive anti-isomorphism) which for symmetric, anti-reflexive or weakly anti-reflexive relations becomes an orthocomplementation.

Now we can focus our attention on relations which are subject of this study. We already have distinguished our equality relation E. Equivalence relations are defined as those which are reflexive, symmetric and transitive, conditions which combined can be written: $E \leq R^{*}=R=R^{2}$.

It is a very elementary fact that equivalence relations bijectively correspond to partitions of the set S on which they are defined. Subsets belonging to such partition are called classes of equivalence of corresponding equivalence. If we start from a partition, its corresponding equivalence relation is defined by the condition that the elements x and y are related, i.e. xRy if the belong to one of the subsets in the partition.

Equivalence relations are very simple, but of extremely high importance in mathematics, as they are involved in the process of abstraction understood as a transition from elements of a set S, to the elements of the partition associated with this equivalence.

Finally we can focus on similarity relations, or as they are called in mathematics on tolerance relations. Tolerance relations are reflexive and symmetric, but not necessarily transitive. Thus, every equivalence is a tolerance relation, but not the other way around. The name "tolerance relation" was introduced by the initiator of its study E.C. Zeeman [19].

In order to have perfect matching through complementarity tolerance relations were generalized to weak tolerance relations, which instead of being reflexive are weakly reflexive ($\forall x \in S$: $(xT^{c}x \Rightarrow \forall y \in S$: $xT^{c}y))$. Tolerance relations may not be transitive, but still they are associated (not in bijective way) with coverings of a set C, i.e. families of subsets of S whose union is entire set S. Transition to weak tolerances eliminates this condition. This means arbitrary family of subsets of S generates a weak tolerance. There is no space here for the presentation of the overview of weak tolerances which can be found elsewhere [20]

We can observe that the concept of a weak tolerance generates another example of a closure space on the power set of S of special interest.

Definition: *A binary relation T on a set S is a weak tolerance if it is symmetric and satisfies the condition: $\forall x \in S$: $[xT^{c}x \Rightarrow \forall y \in S{:}xT^{c}y]$. Every weak tolerance which is reflexive ($\forall x \in S$: xTx) is called a tolerance relation. Equivalence relations are transitive tolerance relations.*

Proposition: *There is a bijective correspondence between weak tolerance relations on set S which generalize equivalence relations extending them to a general concept of similarity and closed subsets of the closure operator on the power set of S, i.e. closure space $<2^{S}, f>$ defined by:*
$\forall \mathcal{B} \subseteq 2^{S}$: $f(\mathcal{B}) = \{B \subseteq S: \forall x,y \in B\ \exists A \in \mathcal{B}: \{x,y\} \subseteq A\}$.

We can see that yet another fundamental concept of mathematics equivalence or similarity on set S is closely associated with a closure space on the power set of S. This closure space is very different from the meta-closure space. Actually they do not have common elements in their Moore spaces, except for trivial cases. However, there are related by the fact that every binary symmetric relation defines a closure operator $f(A) = R^{a}R^{a}(A)$ described above. This shows that we can find connections between concepts on a set S defined in terms of its power set, when we consider closure space on this power set. This leads us to consideration of the three levels: sets, their power sets, and power sets of power sets.

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