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On weakly $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings

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Abstract

Separable polynomials in skew polynomial rings were studied extensively by Y. Miyashita, T. Nagahara, S. Ikehata, and G. S eto. In particular, Ikehata gave the characteri ation of $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings. In this article, we shall introduce the notion of weakly $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings, and we shall give a characteri ation of the $(\tilde{\rho}, \tilde{D})$-separability and that of the weak $(\tilde{\rho}, \tilde{D})$-separability.

1 Introduction and Preliminaries

Throughout this paper, A/B will represent a ring extension with common identity 1. Let M be an A-A-bimodule, and x, y arbitrary elements in A. An additive map $\delta : A \to M$ is called a B-*derivation* of A to M if $\delta(xy) = \delta(x)y + x\delta(y)$ and $\delta(\alpha) = 0$ for any $\alpha \in B$. Moreover, δ is called *inner* if $\delta(x) = mx - xm$ for some fixed element $m \in M$. We say that a ring extension A/B is *separable* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits. It is well known that A/B is separable if and only if for any A-A-bimodule M, every B-derivation of A to M is inner (cf. [1, Satz 4.2]). A ring extension A/B is said to be *weakly separable* if every B-derivation of A to A is inner. The notion of a weakly separable extension was introduced by N. Hamaguchi and A. Nakajima (cf. [2]) . Obviously, a separable extension is weakly separable.

Let B be a ring, ρ an automorphism of B, D a ρ-derivation of B. $B[X;\rho,D]$ will mean the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We set $B[X;\rho] := B[X;\rho,0]$ and $B[X;D] := B[X;1_A,D]$. By $B[X;\rho,D]_{(0)}$ we denote the set of all monic polynomials g in $B[X;\rho,D]$ such that $gB[X;\rho,D] = B[X;\rho,D]g$. For a polynomial $f \in B[X;\rho,D]_{(0)}$, the residue ring $B[X;\rho,D]/fB[X;\rho,D]$ is a free ring extension of B. We say that a polynomial $f \in B[X;\rho,D]_{(0)}$ is *separable* (resp. *weakly separable*) in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable) over B.

Throughout this article, we assume that $\rho D = D\rho$, and let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X;\rho,D]_{(0)} \cap B^\rho[X]$ and

$$\begin{aligned}
f' &:= mX^{m-1} + (m-1)X^{m-2}a_{m-1}\cdots + Xa_2 + a_1 \text{ (the derivative of } f\text{)},\\
Y_0 &:= X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1,\\
&\cdots\cdots\\
Y_j &:= X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1},\\
&\cdots\cdots\\
Y_{m-2} &:= X + a_{m-1},\\
Y_{m-1} &:= 1.
\end{aligned}$$

We shall use the following conventions:

$$\begin{aligned}
B^\rho &:= \{\alpha \in B \mid \rho(\alpha) = \alpha\}\\
B^D &:= \{\alpha \in B \mid D(\alpha) = 0\}\\
B^{\rho,D} &:= B^\rho \cap B^D\\
C(B^{\rho,D}) &:= \{\beta \in B^{\rho,D} \mid b\beta = \beta b \ (\forall\, b \in B^{\rho,D})\} \text{ (the center of } B^{\rho,D}\text{)}\\
A &:= B[X;\rho,D]/fB[X;\rho,D]\\
x &:= X + fB[X;\rho,D] \in A\\
f' &:= f' + fB[X;\rho,D] \in A\\
y_j &:= Y_j + fB[X;\rho,D] \in A \quad (0 \leq j \leq m-1)
\end{aligned}$$

ρ : an automorphism of A defined by $\rho\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j \rho(c_j)$ $(c_j \in B)$

D : a $\rho-$derivation of A defined by $D\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j D(c_j)$ $(c_j \in B)$

For any subsets $T \subset B$ and $S \subset A$, we set

$$\begin{aligned}
J_{m-1}(T) &:= \{z \in A \mid \rho^{m-1}(\alpha)z = z\alpha \ (\forall\, \alpha \in T)\},\\
V(T) &:= \{z \in A \mid \alpha z = z\alpha \ (\forall\, \alpha \in T)\},\\
W(S) &:= \left\{\sum_{j=0}^{m-1} y_j\omega \otimes x^j \,\middle|\, \omega \in S\right\},\\
(A \otimes_B A)^S &:= \{\varepsilon \in A \otimes_B A \mid \varepsilon w = w\varepsilon \ (\forall\, w \in S)\},\\
S^{\tilde{\rho}} &:= \{z \in S \mid \rho(z) = z\},\\
S^{\tilde{D}} &:= \{z \in S \mid D(z) = 0\},\\
S^{\tilde{\rho},\tilde{D}} &:= S^{\tilde{\rho}} \cap S^{\tilde{D}}.
\end{aligned}$$

Note that $J_{m-1}(B') = V(B')$ for any subset B' of B^ρ.

We shall state some basic results which were already known.

Lemma 1.1 ([7, Lemma 1.6]). *f is in $B[X;\rho,D]_{(0)}$ if and only if*

(1) $a_i\rho^m(\alpha) = \sum_{j=i}^{m} \binom{j}{i} \rho^j D^{j-i}(\alpha) a_j \quad (\alpha \in B,\ 0 \le i \le m-1,\ a_m = 1)$

(2) $D(a_i) = a_{i-1} - \rho(a_{i-1}) - a_i\big(\rho(a_{m-1}) - a_{m-1}\big) \quad (1 \le i \le m-1)$

(3) $D(a_0) = a_0\big(\rho(a_{m-1}) - a_{m-1}\big)$

Lemma 1.2 ([7, Corollary 1.7]). *If f is in $B[X;\rho,D]_{(0)} \cap B^\rho[X]$ then f is in $C(B^{\rho,D})[X]$. Moreover,*

$$\alpha a_i = \sum_{j=i}^{m} (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \quad (\alpha \in B,\ 0 \le j \le m,\ a_m = 1).$$

Lemma 1.3 ([6, Theorem 2.2]). *Let B be a commutative ring, and $f(X)$ a monic polynomial in $B[X]$. The following are equivalent.*

(1) *$f(X)$ is weakly separable in $B[X]$.*

(2) *$f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$, where $f'(X)$ is a derivative of $f(X)$.*

(3) *$\delta(f(X))$ is a non-zero-divisor in B, where $\delta(f(X))$ is a discriminant of $f(X)$.*

Now we consider the following A-A-homomorphisms:

$$\mu : {}_A A \otimes_B A_A \to {}_A A_A, \quad \mu(z \otimes w) = zw$$
$$\xi : {}_A A \otimes_B A_A \to {}_A A \otimes_B A_A, \quad \xi(z \otimes w) = D(z) \otimes \rho(w) + z \otimes D(w)$$
$$\eta : {}_A A \otimes_B A_A \to {}_A A \otimes_B A_A, \quad \eta(z \otimes w) = \rho(z) \otimes \rho(w) - z \otimes w$$

By making of the above mappings, S. Ikehata gave the following definition.

Definiton 1.4 ([4, pp.119]). f is called *(ρ, D)-separable* in $B[X;\rho,D]$ if there exists an A-A-homomorphism $\nu : A \to A \otimes_B A$ such that

$$\mu\nu = 1_A, \quad \xi\nu = \nu D, \quad \eta\nu = \nu(\rho - 1_A).$$

Obviously, a (ρ, D)-separable polynomial in $B[X;\rho,D]$ is separable. In [4], S. Ikehata studied $(\rho,\ D)$-separable polynomials in $B[X;\rho,D]$ and he gave the following.

Lemma 1.5 ([4, Theorem 2.1]). *The following are equivalent.*

(1) *f is (ρ, D)-separable in $B[X;\rho,D]$.*

(2) *There exists $h \in J_{m-1}(B)^{\tilde{\rho},\tilde{D}}$ such that $f'h = hf' = 1$.*

(3) *f is separable in $C(B^{\rho,D})[X]$.*

Noting that Lemma 1.5 (3), we shall give the following definition as a generalization of (ρ, D)-separable polynomials in $B[X;\rho,D]$.

Definiton 1.6. f is called *weakly (ρ,D)-separable* in $B[X;\rho,D]$ if f is weakly separable in $C(B^{\rho,D})[X]$.

The purpose of this article is to give characterizations of weakly (ρ,D)-separable in $B[X;\rho,D]$. Moreover, we shall characterize the difference between the (ρ,D)-separability and the weak (ρ,D)-separability in $B[X;\rho,D]$.

2 Main results

The conventions and notations employed in the preceding section will be used in this section. In particular, recall that $\rho D = D\rho$ and let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X;\rho,D]_{(0)} \cap B^{\rho}[X]$. Note that f is in $C(B^{\rho,D})[X]$ by Corollary 1.2. First we shall state the following.

Lemma 2.1. *The following are equivalent.*

(1) *f is weakly (ρ, D)-separable in $B[X;\rho,D]$.*

(2) *f' is a non-zero-divisor in $C(B^{\rho,D})[X]/fC(B^{\rho,D})[X]$ ($\cong V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$).*

(3) *$\delta(f)$ is a non-zero-divisor in $C(B^{\rho,D})$, where $\delta(f)$ is a discriminant of f.*

Proof. It is obvious by Lemma 1.3. □

We recall that A-A-homomorphism $\mu : A \otimes_B A \to A$ defined by $z \otimes w \mapsto zw$. Noting that $\alpha f' = f'\rho^{m-1}(\alpha)$ for any $\alpha \in B$, we can see that $\mu\left(W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})\right) \subset V(B)^{\tilde{\rho},\tilde{D}}$. In addition, it is easy to see that $\mu\left(W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})\right) \subset V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$. Then we shall state the following.

Theorem 2.2. (1) *f is (ρ, D)-separable in $B[X;\rho,D]$ if and only if the following A-A-homomorphism is onto:*

$$\mu\,|_{W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})} : W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}}) \longrightarrow V(B)^{\tilde{\rho},\tilde{D}}$$

(2) *f is weakly (ρ, D)-separable in $B[X;\rho,D]$ if and only if the following A-A-homomorphism is one-to-one:*

$$\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})} : W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}) \longrightarrow V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$$

Proof. Note that $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$ for any $h \in A^{\tilde{\rho},\tilde{D}}$.

(1) Assume that f is (ρ, D)-separable in $B[X;\rho,D]$. Then there exists $h \in J_{m-1}(B)^{\tilde{\rho},\tilde{D}}$ such that $f'h = hf' = 1$ by Lemma 1.5 (2). For any $g \in V(B)^{\tilde{\rho},\tilde{D}}$, we see that $hg = gh \in J_{m-1}(B)^{\tilde{\rho},\tilde{D}}$ and $\mu\left(\sum_{j=0}^{m-1} y_j hg \otimes x^j\right) = f'hg = g$. Thus $\mu\,|_{W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})}$ is onto.

Conversely, assume that $\mu\,|_{W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})}$ is onto. Since $1 \in V(B)^{\tilde{\rho},\tilde{D}}$, there exists $h \in J_{m-1}(B)^{\tilde{\rho},\tilde{D}}$ such that $1 = \mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$. Therefore f is (ρ, D)-separable by Lemma 1.5 (2).

(2) Assume that f is weakly (ρ, D)-separable in $B[X;\rho,D]$. Then f' is a non-zero-divisor in $V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$ by Lemma 2.1 (2). Let $\sum_{j=0}^{m-1} y_j h \otimes x^j$ be in Ker $\left(\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})}\right)$ with $h \in V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$. Then we have $0 = f'h = hf'$. Since f' is a non-zero-divisor in $V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$, we obtain $h = 0$ and hence Ker $\left(\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})}\right) = \{0\}$. Thus $\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})}$ is one-to-one.

Conversely, assume that $\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})}$ is one-to-one. Let $hf' = 0$ for some $h \in V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$. This implies that $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = 0$. Since $\mu\,|_{W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})}$ is one-to-one, we have $\sum_{j=0}^{m-1} y_j h \otimes x^j = 0$, namley, $h = 0$. Therefore f' is a non-zero-divisor in $V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$, and hence f is weakly (ρ, D)-separable by Lemma 2.1 (2). □

Corollary 2.3. *f is (ρ, D)-separable in $B[X;\rho,D]$ if and only if $W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}}) \cong V(B)^{\tilde{\rho},\tilde{D}}$ as an A-A-bimodule.*

Proof. Note that $W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}}) \subset W(V(B^{\rho,D})^{\tilde{\rho},\tilde{D}})$ and $V(B)^{\tilde{\rho},\tilde{D}} \subset V(B^{\rho,D})^{\tilde{\rho},\tilde{D}}$. If f is (ρ, D)-separable in $B[X;\rho,D]$ then f is also weakly (ρ, D)-separable, and so $\mu\,|_{W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})}$ is one-to-one. Therefore $\mu\,|_{W(J_{m-1}(B)^{\tilde{\rho},\tilde{D}})}$ is an isomorphism if and only if f is (ρ,D)-separable in $B[X;\rho,D]$. □

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