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Kyoto University
On weakly \((\bar{\rho}, \bar{D})\)-separable polynomials in skew polynomial rings

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Abstract

Separable polynomials in skew polynomial rings were studied extensively by Y. Miyashita, T. Nagahara, S. Ikehata, and G. S eto. In particular, Ikehata gave the characteri ation of \((\bar{\rho}, \bar{D})\)-separable polynomials in skew polynomial rings. In this article, we shall introduce the notion of weakly \((\bar{\rho}, \bar{D})\)-separable polynomials in skew polynomial rings, and we shall give a characteri ation of the \((\bar{\rho}, \bar{D})\)-separability and that of the weak \((\bar{\rho}, \bar{D})\)-separability.

1 Introduction and Preliminaries

Throughout this paper, \(A/B\) will represent a ring extension with common identity 1. Let \(M\) be an \(A-A\)-bimodule, and \(x, y\) arbitrary elements in \(A\). An additive map \(\delta : A \rightarrow M\) is called a \(B\)-derivation of \(A\) to \(M\) if \(\delta(xy) = \delta(x)y + x\delta(y)\) and \(\delta(\alpha) = 0\) for any \(\alpha \in B\). Moreover, \(\delta\) is called inner if \(\delta(x) = mx - xm\) for some fixed element \(m \in M\). We say that a ring extension \(A/B\) is separable if the \(A-A\)-homomorphism of \(A \otimes_B A\) onto \(A\) defined by \(a \otimes b \mapsto ab\) splits. It is well known that \(A/B\) is separable if and only if for any \(A-A\)-bimodule \(M\), every \(B\)-derivation of \(A\) to \(M\) is inner (cf. [1, Satz 4.2]). A ring extension \(A/B\) is said to be weakly separable if every \(B\)-derivation of \(A\) to \(A\) is inner. The notion of a weakly separable extension was introduced by N. Hamaguchi and A. Nakajima (cf. [2]). Obviously, a separable extension is weakly separable.

Let \(B\) be a ring, \(\rho\) an automorphism of \(B\), \(D\) a \(\rho\)-derivation of \(B\). \(B[X; \rho, D]\) will mean the skew polynomial ring in which the multiplication is given by \(\alpha X = X\rho(\alpha) + D(\alpha)\) for any \(\alpha \in B\). We set \(B[X; \rho] := B[X; \rho, 0]\) and \(B[X; D] := B[X; 1_A, D]\). By \(B[X; \rho, D]_{(0)}\) we denote the set of all monic polynomials \(g\) in \(B[X; \rho, D]\) such that \(gB[X; \rho, D] = B[X; \rho, D]g\). For a polynomial \(f \in B[X; \rho, D]_{(0)}\), the residue ring \(B[X; \rho, D]/fB[X; \rho, D]\) is a free ring extension of \(B\). We say that a polynomial \(f \in B[X; \rho, D]_{(0)}\) is separable (resp. weakly separable) in \(B[X; \rho, D]\) if \(B[X; \rho, D]/fB[X; \rho, D]\) is separable (resp. weakly separable) over \(B\).
Throughout this article, we assume that $\rho D = D \rho$, and let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X]$ and

\[
f' := mX^{m-1} + (m-1)X^{m-2}a_{m-1} \cdots + Xa_2 + a_1 \quad \text{(the derivative of } f),
\]

\[
Y_0 := X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1,
\]

\[
\ldots
\]

\[
Y_j := X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1},
\]

\[
\ldots
\]

\[
Y_{m-2} := X + a_{m-1},
\]

\[
Y_{m-1} := 1.
\]

We shall use the following conventions:

\[
B^\rho := \{ \alpha \in B \mid \rho(\alpha) = \alpha \}
\]

\[
B^D := \{ \alpha \in B \mid D(\alpha) = 0 \}
\]

\[
B^{\rho, D} := B^\rho \cap B^D
\]

\[
C(B^{\rho, D}) := \{ \beta \in B^{\rho, D} \mid b\beta = \beta b \ (\forall b \in B^{\rho, D}) \} \quad \text{(the center of } B^{\rho, D})
\]

\[
A := B[X; \rho, D]/fB[X; \rho, D]
\]

\[
x := X + fB[X; \rho, D] \in A
\]

\[
f' := f' + fB[X; \rho, D] \in A
\]

\[
y_j := Y_j + fB[X; \rho, D] \in A \quad (0 \leq j \leq m-1)
\]

\[
\rho : \text{an automorphism of } A \text{ defined by } \rho \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j \rho(c_j) \ (c_j \in B)
\]

\[
D : \text{a } \rho-\text{derivation of } A \text{ defined by } D \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j D(c_j) \ (c_j \in B)
\]

For any subsets $T \subset B$ and $S \subset A$, we set

\[
J_{m-1}(T) := \{ z \in A \mid \rho^{m-1}(\alpha)z = z\alpha \ (\forall \alpha \in T) \},
\]

\[
V(T) := \{ z \in A \mid \alpha z = z\alpha \ (\forall \alpha \in T) \},
\]

\[
W(S) := \left\{ \sum_{j=0}^{m-1} y_j \omega \otimes x^j \mid \omega \in S \right\},
\]

\[
(A \otimes_B A)^S := \{ \varepsilon \in A \otimes_B A \mid \varepsilon w = w\varepsilon \ (\forall w \in S) \},
\]

\[
S^\rho := \{ z \in S \mid \rho(z) = z \},
\]

\[
S^D := \{ z \in S \mid D(z) = 0 \},
\]

\[
S^{\rho, D} := S^\rho \cap S^D.
\]

Note that $J_{m-1}(B') = V(B')$ for any subset $B'$ of $B^\rho$. 

We shall state some basic results which were already known.

**Lemma 1.1** ([7, Lemma 1.6]). $f$ is in $B[X; \rho, D]_{(0)}$ if and only if

1. $a_i \rho^m(\alpha) = \sum_{j=i}^{m} \binom{j}{i} \rho^j D^{j-i}(\alpha) a_j \quad (\alpha \in B, \ 0 \leq i \leq m - 1, \ a_m = 1)$

2. $D(a_i) = a_{i-1} - \rho(a_{i-1}) - a_i(\rho(a_{m-1}) - a_{m-1}) \quad (1 \leq i \leq m - 1)$

3. $D(a_0) = a_0(\rho(a_{m-1}) - a_{m-1})$

**Lemma 1.2** ([7, Corollary 1.7]). If $f$ is in $B[X; \rho, D]_{(0)} \cap B^\rho[X]$ then $f$ is in $C(B^\rho,D)[X]$. Moreover,

$$\alpha a_i = \sum_{j=i}^{m} (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \quad (\alpha \in B, \ 0 \leq j \leq m, \ a_m = 1).$$

**Lemma 1.3** ([6, Theorem 2.2]). Let $B$ be a commutative ring, and $f(X)$ a monic polynomial in $B[X]$. The following are equivalent.

1. $f(X)$ is weakly separable in $B[X]$.

2. $f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$, where $f'(X)$ is a derivative of $f(X)$.

3. $\delta(f(X))$ is a non-zero-divisor in $B$, where $\delta(f(X))$ is a discriminant of $f(X)$.

Now we consider the following $A$-$A$-homomorphisms:

- $\mu : \bigotimes \mathbb{A} A \otimes_B A \to \mathbb{A} A, \ \mu(z \otimes w) = zw$
- $\xi : \bigotimes \mathbb{A} A \otimes_B A \to \bigotimes \mathbb{A} \otimes_B A, \ \xi(z \otimes w) = D(z) \otimes \rho(w) + z \otimes D(w)$
- $\eta : \bigotimes \mathbb{A} A \otimes_B A \to \bigotimes \mathbb{A} \otimes_B A, \ \eta(z \otimes w) = \rho(z) \otimes \rho(w) - z \otimes w$

By making of the above mappings, S. Ikehata gave the following definition.

**Definition 1.4** ([4, pp.119]). $f$ is called $(\rho, D)$-separable in $B[X; \rho, D]$ if there exists an $A$-$A$-homomorphism $\nu : A \to A \otimes_B A$ such that

$$\mu \nu = 1_A, \ \xi \nu = \nu D, \ \eta \nu = \nu(\rho - 1_A).$$

Obviously, a $(\rho, D)$-separable polynomial in $B[X; \rho, D]$ is separable. In [4], S. Ikehata studied $(\rho, D)$-separable polynomials in $B[X; \rho, D]$ and he gave the following.
Lemma 1.5 ([4, Theorem 2.1]). The following are equivalent.

(1) $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$.

(2) There exists $h \in J_{m-1}(B)^{\bar{\rho}, \bar{D}}$ such that $f'h = hf' = 1$.

(3) $f$ is separable in $C(B^{\rho,D})[X]$.

Noting that Lemma 1.5 (3), we shall give the following definition as a generalization of $(\rho, D)$-separable polynomials in $B[X; \rho, D]$.

Definition 1.6. $f$ is called weakly $(\rho,D)$-separable in $B[X; \rho, D]$ if $f$ is weakly separable in $C(B^{\rho,D})[X]$.

The purpose of this article is to give characterizations of weakly $(\rho,D)$-separable in $B[X; \rho, D]$. Moreover, we shall characterize the difference between the $(\rho,D)$-separability and the weak $(\rho,D)$-separability in $B[X; \rho, D]$.

2 Main results

The conventions and notations employed in the preceding section will be used in this section. In particular, recall that $\rho D = D \rho$ and let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X]$. Note that $f$ is in $C(B^{\rho,D})[X]$ by Corollary 1.2. First we shall state the following.

Lemma 2.1. The following are equivalent.

(1) $f$ is weakly $(\rho, D)$-separable in $B[X; \rho, D]$.

(2) $f'$ is a non-zero-divisor in $C(B^{\rho,D})[X]/fC(B^{\rho,D})[X] \cong V(B^{\rho,D})^{\bar{\rho}, \bar{D}}$.

(3) $\delta(f)$ is a non-zero-divisor in $C(B^{\rho,D})$, where $\delta(f)$ is a discriminant of $f$.

Proof. It is obvious by Lemma 1.3. \qed

We recall that $A$-$A$-homomorphism $\mu : A \otimes_B A \to A$ defined by $z \otimes w \mapsto zw$. Noting that $\alpha f' = f'\rho^{m-1}(\alpha)$ for any $\alpha \in B$, we can see that $\mu \left( W(J_{m-1}(B)^{\bar{\rho}, \bar{D}}) \right) \subset V(B)^{\bar{\rho}, \bar{D}}$. In addition, it is easy to see that $\mu \left( W(V(B^{\rho,D})^{\bar{\rho}, \bar{D}}) \right) \subset V(B^{\rho,D})^{\bar{\rho}, \bar{D}}$. Then we shall state the following.

Theorem 2.2. (1) $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$ if and only if the following $A$-$A$-homomorphism is onto:

$$\mu|_{W(J_{m-1}(B)^{\bar{\rho}, \bar{D}})} : W(J_{m-1}(B)^{\bar{\rho}, \bar{D}}) \to V(B)^{\bar{\rho}, \bar{D}}$$
(2) $f$ is weakly $(\rho, D)$-separable in $B[X; \rho, D]$ if and only if the following $A$-$A$-homomorphism is one-to-one:

$$
\mu|_{W(V(B^\rho,D)\overline{\rho},\overline{D})} : W(V(B^\rho,D)\overline{\rho},\overline{D}) \longrightarrow V(B^\rho,D)\overline{\rho},\overline{D}
$$

Proof. Note that $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$ for any $h \in A^{\overline{\rho},\overline{D}}$.

(1) Assume that $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$. Then there exists $h \in J_{m-1}(B)^{\overline{\rho},\overline{D}}$ such that $f'h = hf' = 1$ by Lemma 1.5 (2). For any $g \in V(B)^{\overline{\rho},\overline{D}}$, we see that $hg = gh \in J_{m-1}(B)^{\overline{\rho},\overline{D}}$ and $\mu\left(\sum_{j=0}^{m-1} y_j hg \otimes x^j\right) = f'hg = g$. Thus $\mu|_{W(J_{m-1}(B)^{\overline{\rho},\overline{D}})}$ is onto.

Conversely, assume that $\mu|_{W(J_{m-1}(B)^{\overline{\rho},\overline{D}})}$ is onto. Since $1 \in V(B)^{\overline{\rho},\overline{D}}$, there exists $h \in J_{m-1}(B)^{\overline{\rho},\overline{D}}$ such that $1 = \mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$. Therefore $f$ is $(\rho, D)$-separable by Lemma 1.5 (2).

(2) Assume that $f$ is weakly $(\rho, D)$-separable in $B[X; \rho, D]$. Then $f'$ is a non-zero-divisor in $V(B^\rho,D)^{\overline{\rho},\overline{D}}$ by Lemma 2.1 (2). Let $\sum_{j=0}^{m-1} y_j h \otimes x^j$ be in $\text{Ker}\left(\mu|_{W(V(B^\rho,D)\overline{\rho},\overline{D})}\right)$ with $h \in V(B^\rho,D)^{\overline{\rho},\overline{D}}$. Then we have $0 = f'h = hf'$. Since $f'$ is a non-zero-divisor in $V(B^\rho,D)^{\overline{\rho},\overline{D}}$, we obtain $h = 0$ and hence $\text{Ker}\left(\mu|_{W(V(B^\rho,D)\overline{\rho},\overline{D})}\right) = \{0\}$. Thus $\mu|_{W(V(B^\rho,D)^{\overline{\rho},\overline{D}})}$ is one-to-one.

Conversely, assume that $\mu|_{W(V(B^\rho,D)^{\overline{\rho},\overline{D}})}$ is one-to-one. Let $hf' = 0$ for some $h \in V(B^\rho,D)^{\overline{\rho},\overline{D}}$. This implies that $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = 0$. Since $\mu|_{W(V(B^\rho,D)^{\overline{\rho},\overline{D}})}$ is one-to-one, we have $\sum_{j=0}^{m-1} y_j h \otimes x^j = 0$, namely, $h = 0$. Therefore $f'$ is a non-zero-divisor in $V(B^\rho,D)^{\overline{\rho},\overline{D}}$, and hence $f$ is weakly $(\rho, D)$-separable by Lemma 2.1 (2). \qed

Corollary 2.3. $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$ if and only if $W(J_{m-1}(B)^{\overline{\rho},\overline{D}}) \cong V(B)^{\overline{\rho},\overline{D}}$ as an $A$-$A$-bimodule.

Proof. Note that $W(J_{m-1}(B)^{\overline{\rho},\overline{D}}) \subset W(V(B^\rho,D)^{\overline{\rho},\overline{D}})$ and $V(B)^{\overline{\rho},\overline{D}} \subset V(B^\rho,D)^{\overline{\rho},\overline{D}}$. If $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$ then $f$ is also weakly $(\rho, D)$-separable, and so $\mu|_{W(J_{m-1}(B)^{\overline{\rho},\overline{D}})}$ is one-to-one. Therefore $\mu|_{W(J_{m-1}(B)^{\overline{\rho},\overline{D}})}$ is an isomorphism if and only if $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$. \qed

References


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