数理解析研究所講究録 1355 巻 2004 年 117-127

Global solutions for the nonlinear Dirac equation and endpoint Strichartz estimates

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Abstract. Global wellposedness of the nonlinear Dirac equation is shown for small data in the energy class with some regularity assumption for the angular variable. Main tool for the proof, endpoint Strichartz estimates for Klein-Gordon and wave equations on the polar coordinates in three spatial dimension are studied.

1. Introduction

We consider the Cauchy problem for the nonlinear Dirac equation:

$$\begin{gathered}\sum_{\alpha=0}^{3} i\gamma^{\alpha}\partial_{\alpha}u - mu = \lambda(\gamma^0 u, u)u,\\ u(0,x) = \varphi(x),\end{gathered} \tag{1.1}$$

where $\varphi(x) : \mathbb{R}^3 \to \mathbb{C}^4$ is the given, $u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is the unknown function, $m \geq 0$ and $\lambda \in \mathbb{C}$ are given constants, $(\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla_x)$ is the space-time derivative, $(\cdot,\cdot)$ denotes the inner product on $\mathbb{C}^4$, and $\gamma^{\alpha} \in GL(\mathbb{C}, 4)$ $(\alpha = 0, 1, 2, 3)$ denote the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \tag{1.2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

We study the global existence of solutions of (1.1) with small data. We have already shown the existence of global solution in H^s with small data $\varphi \in H^s$ for $s > 1$ in [7]. Local existence was proved by Escobedo and Vega in $H^s, s > 1$ [3]. Here the value $s = 1$ is a scaling critical exponent which is given by the homogeneity of the Cauchy problem (1.1) with $m = 0$, see the introduction in [3]. In this note we concentrate on the critical case, that is, on searching for the H^1 solution of (1.1).

Before trying to the critical problem, we review the situation in [7] of subcritical case. The main tool there is Strichartz estimates for Klein–Gordon equations. However, we are faced the $L_t^2 L_x^{\infty}$ norm of u when we estimate the nonlinear term, and it is known that the estimates $H^1 \to L_t^2 L_x^{\infty}$ (initial data $\to$ free solutions) so-called endpoint Strichartz estimates does not hold [4], see also unsuccessful estimates $H^1 \to L_t^2 BMO_x$ [8]. Therefore we provided the more regular initial data and used the embedding theorems to obtain the solutions. To tell the truth, we could derive the H^1 global solvability easily, if the endpoint estimates held. How do we overcome the lack of endpoint Strichartz estimates? We show the one of answers for this difficulty in this note. We give the Strichartz estimate which deal with the variables of radius and angular independently. For the special solution sufficiently regular for rotation, this estimate corresponds to the endpoint estimates. By virtue of this estimates, we prove the global existence of solutions for small H^1 data with additional regularity for rotation.

Our original motivation to considering this type estimate is the following. It is well known that the endpoint Strichartz estimate for wave equations holds for a radial function [4]. Therefore there are the results of global solvability for small radial data of some nonlinear wave equations which preserve the spherical symmetry [11]. Here we give the one unhappy remark. Dirac equation, even if free Dirac equation, does not preserve spherical symmetry. So we could not use the available endpoint estimates of wave equations directly for Dirac equation, even under the assumption data is radial. Then we take notice of the fact that general functions turn to radial functions after averaging in L^p_θ over angular variable. We study the Strichartz estimates for wave and Klein–Gordon equations on the norms $L^q_r L^p_\theta$ that firstly take L^p_θ for angular variable and secondly take L^q_r for radius variable. A similar estimates for the Schrödinger equation in two spatial dimension were studied in [15].

Now we are in the position to state our results.

Theorem 1.1. *Let $m \geq 0$, $\lambda \in \mathbb{C}$ and $s > 0$. Then there exists $\delta > 0$ such that if $\varphi \in H^1(\mathbb{R}^3)$ satisfies*

$$||\varphi||_{H^1(H^s_\theta)} := ||\varphi||_{L^2_r(H^s_\theta)} + ||\nabla\varphi||_{L^2_r(H^s_\theta)} < \delta \tag{1.4}$$

then we have a unique global solution u of (1.1) *satisfying $u(0) = \varphi$ and*

$$u \in C_t(\mathbb{R}; H^1(H^s_\theta)) \cap L^2_t(\mathbb{R}; L^\infty). \tag{1.5}$$

In the case of $m = 0$, we may replace the above norm of $H^1(H^s_\theta)$ with its homogeneous version, namely $||\nabla\varphi||_{L^2_r(H^s_\theta)}$.

Remark 1.2. This Theorem implies global existence of solution with small radial data in H^1. We prove Theorem by the standard fixed point arguments using the endpoint estimates that hold uniformly on any time interval. Hence we can easily obtain global wellposedness and scattering for small data, as well as local existence for large data by the standard arguments (see, e.g., [3]).

The rest of this note is organized as follows. In Section 2, we introduce the notation and basic estimates on the fractional Sobolev spaces on the sphere S^2. In

Section 3, we prove our endpoint Strichartz estimates. In Section 4, we prove the global wellposedness for the nonlinear Dirac equation.

Throughout this note, we often use the notation $A \lesssim B$ and $D \sim E$ which mean $A \le CB$ and $D/C \le E \le CD$, respectively, where C is some positive constant. We denote $\langle x\rangle := (1+|x|^2)^{1/2}$. We identify any set with its characteristic function. Thus for any set A, $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise.

2. Fractional Sobolev spaces on the sphere

In this section, we recall some basic facts that we need on the fractional Sobolev spaces on the unit sphere S^2. See [14, 17] for more general information. We denote the polar coordinates $x = r\theta$, $r = |x|$ and $\theta \in S^2$. Let Δ_θ denote the Laplace-Beltrami operator on S^2. For any function $f(r\theta)$, we have

$$\Delta_\theta f(x) = |x \times \nabla|^2 f(x). \tag{2.1}$$

The Lebesgue and Sobolev spaces on S^2 are defined by the norms

$$\|f\|_{L^p_\theta} = \left(\int_{S^2} |f(\theta)|^p d\theta\right)^{1/p}, \quad \|f\|_{H^{s,p}_\theta} = \|(1-\Delta_\theta)^{s/2} f\|_{L^p_\theta}. \tag{2.2}$$

Throughout this note, we will use these norms in the mixed form:

$$\|f(x)\|_{L^p_r(X_\theta)} = \left(\int \|f(r\theta)\|^p_{X_\theta} r^2 dr\right)^{1/p}. \tag{2.3}$$

The fractional power of Δ_θ can be written explicitly by introducing the spherical harmonics. Let $F^k_\nu(x)$ be a homogeneous polynomial of degree ν satisfying $\Delta F^k_\nu(x) = 0$, such that $\{F^k_\nu(\theta)\}_{\nu,k}$ makes a complete orthonormal basis of $L^2(S^2)$. Then any function $f(r\theta)$ can be decomposed as

$$f(r\theta) = \sum_{\nu=0}^{\infty} \sum_{k=1}^{N(\nu)} a^k_\nu(r) F^k_\nu(\theta), \tag{2.4}$$

where $a^k_\nu(r)$ are determined by f, and

$$(1-\Delta_\theta)^{s/2} f = \sum_{\nu,k} (1+\nu(\nu+1))^{s/2} a^k_\nu(r) F^k_\nu(\theta), \tag{2.5}$$

where we used $\Delta_\theta F^k_\nu(\theta) = -\nu(\nu+1)F^k_\nu(\theta)$. In the case $p = 2$, we may use the orthogonality to deduce that

$$\|f\|^2_{L^2_r(H^{s,2}_\theta)} \sim \sum_{\nu,k} \langle\nu\rangle^s \|a^k_\nu\|^2_{L^2_r}. \tag{2.6}$$

For nonlinear estimates, we use the equivalent norms defined through local coordinates. Let $\{(O_j, \Psi_j)\}_{j=1}^N$ be a system of coordinate neighborhoods, and $\{\lambda_j\}$ be a smooth partition of unity subordinate to $\{O_j\}$. Let $\{\chi_j\} \subset C^\infty_0(\mathbb{R}^2)$ satisfy $\chi_j = 1$

on $\Psi_j(\operatorname{supp}\lambda_j)$ and $\operatorname{supp}\chi_j \subset \Psi_j(O_j)$. Then, for any functions $f : S^2 \to \mathbb{C}$ and $h : (\mathbb{R}^2)^N \to \mathbb{C}$, we define $Sf : (\mathbb{R}^2)^N \to \mathbb{C}$ and $Rh : S^2 \to \mathbb{C}$ by

$$(Sf)_j(x) := (\lambda_j f)(\Psi_j^{-1}(x)), \quad Rh(y) := \sum_{j=1}^{N} (\chi_j h)(\Psi_j(y)). \tag{2.7}$$

Then we can define the Sobolev norms by

$$\|f\|_{H^{s,p}(S^2)} = \|Sf\|_{(H^{s,p}(\mathbb{R}^2))^N}. \tag{2.8}$$

This gives an equivalent norm of $H^{s,p}_\theta$ for $1 < p < \infty$ (see [17]). We do not deal with the cases $p = 1$ or ∞ in this note.

It is easily seen that $RSf = f$ and SR is bounded from $(H^{s,p}(\mathbb{R}^2))^N$ into itself, and so, R is a retraction from $(H^{s,p}(\mathbb{R}^2))^N$ to $H^{s,p}(S^2)$ with a coretraction S. Therefore we have the same embeddings and interpolations for $H^{s,p}(S^2)$ as on $\mathbb{R}^2$. We may introduce another equivalent norm

$$(S'f)_j(x) := \chi_j(x) f(\Psi_j^{-1}(x)), \quad \|S'f\|_{(H^{s,p}(\mathbb{R}^2))^N} \sim \|Sf\|_{(H^{s,p}(\mathbb{R}^2))^N}. \tag{2.9}$$

Then the Hölder inequality and the Leibniz rule easily transfers from the Euclidean case as follows. Let $s \geq 0$ and $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2, 1 < p < \infty, q_1 \neq \infty, r_2 \neq \infty$. We have

$$\begin{aligned} \|fg\|_{H^{s,p}(S^2)} &\sim \sum_j \|(Sf)_j (S'g)_j\|_{H^{s,p}(\mathbb{R}^2)} \\ &\lesssim \sum_j \left(\|(Sf)_j\|_{H^{s,q_1}(\mathbb{R}^2)} \|(S'g)_j\|_{L^{r_1}(\mathbb{R}^2)} + \|(Sf)_j\|_{L^{q_2}(\mathbb{R}^2)} \|(S'g)_j\|_{H^{s,r_2}(\mathbb{R}^2)} \right) \\ &\lesssim \|f\|_{H^{s,q_1}(S^2)} \|g\|_{L^{r_1}(S^2)} + \|f\|_{L^{q_2}(S^2)} \|g\|_{H^{s,r_2}(S^2)}, \end{aligned} \tag{2.10}$$

where we used the standard estimate on pointwise multiplication on $\mathbb{R}^2$ on the second line.

Finally we check the equivalence of the following norms,

$$\|(1 - \Delta_\theta)^{s/2} f\|_{H^1} \sim \|f\|_{H^1(H^s_\theta)}, \tag{2.11}$$

where the right hand side was introduced in (1.4). Note that ∇ and Δ_θ are not commutative. Since (2.11) is obvious if we replace H^1 by L^2, it suffices to prove the homogeneous version, i.e., for $\dot{H}^1_x$. Since $|\nabla| = \sqrt{-\Delta}$ commutes with Δ_θ, the above equivalence (2.11) reduces to the following one:

$$\||\nabla| f\|_{L^2_r(H^s_\theta)} \sim \|\nabla f\|_{L^2_r(H^s_\theta)}, \tag{2.12}$$

which is equivalent to the boundedness of the Riesz operators:

$$\nabla/|\nabla| : L^2_r(H^s_\theta) \to L^2_r(H^s_\theta) \quad \text{bounded}. \tag{2.13}$$

This is easily checked when s is an (even) integer by computing the commutators of $x \times \nabla$ and ∇. Then the remaining case is covered by interpolation.

3. Endpoint Strichartz estimates

In this section we consider the following free Klein–Gordon equation with $m \geq 0$ in three space dimension:

$$\begin{aligned} \partial_t^2 u - \Delta u + m^2 u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ u(0,x) = f(x),\ \partial_t u(0,x) = g(x), \quad x \in \mathbb{R}^3. \end{aligned} \tag{3.1}$$

We give the endpoint Strichartz estimate.

Theorem 3.1. *Let $n = 3$. For any $m \geq 0$, any $1 \leq p < \infty$, we have for any solution u of* (3.1),

$$\|u\|_{L_t^2 L_r^\infty L_\theta^p} \lesssim \|f\|_{H^1} + \|g\|_{L^2}. \tag{3.2}$$

The rest of this section is devoted to the proof of (3.2). Although one might expect that the estimates (3.2) were easier for the Klein-Gordon ($m > 0$) because of the faster decay ($t^{-3/2}$), the estimate for the Klein-Gordon actually implies that for the wave. In fact, suppose that we have an estimate (3.2) for a fixed $m = m_0 > 0$. Then we obtain the same estimate for all $m > 0$ just by rescaling $u \mapsto u(tm/m_0, xm/m_0)$. Taking the limit $m \to 0$, we obtain the same estimate for $m = 0$ as well. On the other hand, it is not trivial to extend such an estimate from $m = 0$ to $m > 0$.

3.1. TT^* **argument.** First of all, we convert them into the TT^* versions. Our desired estimates can be rewritten as

$$\|\omega_m^{-1} e^{\pm i\omega_m t}\varphi\|_{L_t^2 L_r^\infty L_\theta^p} \lesssim \|\varphi\|_{L_x^2}, \quad \omega_m := \sqrt{m^2 - \Delta}. \tag{3.3}$$

We apply the TT^* argument to the operators $T_\pm := \omega_m^{-1}(e^{i\omega_m t} \pm e^{-i\omega_m t})$. We have

$$T_\pm T_\pm^* u = 2\int_{\mathbb{R}} \omega_m^{-2}\{\cos(\omega_m(t-s)) \pm \cos(\omega_m(t+s))\} u(s)ds. \tag{3.4}$$

Hence, by time reversibility, it suffices to prove

$$\left\| \int_{\mathbb{R}} \omega_m^{-2} \cos\omega_m(t-s) u(s) ds \right\|_{L_t^2 L_r^\infty L_\theta^p} \lesssim \|u\|_{L_t^2 L_r^1 L_\theta^{p'}}, \tag{3.5}$$

where $p' = p/(p-1)$ is dual exponent. It is important for our later argument that we do not have 'sin' but 'cos' above. We denote the operator in (3.5) by $L_m(t) := \omega_m^{-2}\cos(\omega_m t)$ and its kernel function by

$$\mathcal{L}_m(t,x) = \mathcal{F}^{-1}\langle\xi\rangle_m^{-2} \cos\langle\xi\rangle_m t, \quad \langle\xi\rangle_m := \sqrt{|\xi|^2 + m^2}. \tag{3.6}$$

We use the following TT^* version of the Hardy–Littlewood maximal operator as the key estimate on (t,r). In the lemma below, we forget about the polar coordinates and so L_r^p denotes the standard $L^p((0,\infty); dr)$ without weights.

Lemma 3.2. *Let $g(r)$ be a nonnegative nonincreasing integrable function on $(0, \infty)$. Then the following estimate holds*

$$\left\| \int_0^\infty \int_{\mathbb{R}} \frac{1}{r \vee l} g\Big(\frac{|t-s|}{r \vee l}\Big) h(s,l) ds dl \right\|_{L_t^2 L_r^\infty} \lesssim \|g\|_{L_r^1} \|h\|_{L_t^2 L_r^1}. \tag{3.7}$$

where $r \vee l = \max(r, l)$.

Proof. The Hardy-Littlewood maximal function theorem shows the boundedness of the operator

$$M\varphi(t,r) = \frac{1}{r} \int_{|t-s|<r} \varphi(s) ds : L_t^2 \to L_t^2 L_r^\infty. \tag{3.8}$$

So MM^* is bounded

$$MM^* : L_t^2 L_r^1 \to L_t^2 L_r^\infty, \tag{3.9}$$

and it is written explicitly by

$$MM^* h(t,r) = \int_0^\infty \int_{\mathbb{R}} \frac{1}{rl} I(|t-s|, r, l) h(s,l) ds dl, \tag{3.10}$$

where

$$I(t,r,l) = \begin{cases} 2\min(r,l), & (t < |r-l|), \\ r+l-t, & (|r-l| < t < r+l), \\ 0, & (r+l < t). \end{cases} \tag{3.11}$$

Denote the operator in (3.7) by $\mathcal{M}(g,h)$. Since

$$\frac{1}{rl} I(t,r,l) \geq \frac{1}{r \vee l} \{0 < t < r \vee l\}, \tag{3.12}$$

the boundedness of MM^* implies the desired estimate for $\mathcal{M}([0,1], h)$, and by rescaling, for any interval $\mathcal{M}([0,a], h)$. (Remember that we identify any set with its characteristic function.) Then the general case follows by slicing g into intervals:

$$\begin{aligned} \|\mathcal{M}(g,h)\|_{L_t^2 L_r^\infty} &= \left\| \int_0^\infty -g'(a) \mathcal{M}([0,a], h) da \right\|_{L_t^2 L_r^\infty} \\ &\lesssim \int_0^\infty -g'(a) a \|h\|_{L_t^2 L_r^1} da = \|g\|_{L_r^1} \|h\|_{L_t^2 L_r^1}. \end{aligned} \tag{3.13}$$

□

3.2. **L_θ^p estimate** (3.3) **for the wave.** We fix t and estimate $L_0(t)$ pointwise. By symmetry, we may assume that $t > 0$. Using the well known formula for the fundamental solution, we obtain

$$\begin{aligned} L_0(t) &= \int_t^\infty \omega_0^{-1} \sin \omega_0 s ds, \\ \mathcal{L}_0(t,x) &= \int_t^\infty \frac{1}{4\pi s} \delta(s-r) ds = \frac{1}{4\pi r} \{t < r\}. \end{aligned} \tag{3.14}$$

Here again we identify the set with its characteristic function. Using the polar coordinates we may write it as

$$L_0(t)\varphi = \int_0^\infty \Omega[\varphi(l\theta)l^2]dl, \tag{3.15}$$

where Ω is an operator on S^2 defined by

$$\Omega\varphi(\theta) = \int_{S^2} F(|r\theta - l\alpha|)\varphi(\alpha)d\alpha, \quad F(r) = (4\pi r)^{-1}\{t < r\}. \tag{3.16}$$

We estimate the L^p_θ norm of Ω as follows. First we have the trivial L^∞_θ bound:

$$\|\Omega\varphi\|_{L^\infty_\theta} \le \|F(|r\theta - l\alpha|)\|_{L^\infty_\alpha}\|\varphi\|_{L^1_\theta} \lesssim t^{-1}\{t < r+l\}\|\varphi\|_{L^1_\theta}. \tag{3.17}$$

For the L^2_θ estimate, we apply the Young inequality for the convolution on $SO(3)$. Using the identity

$$\int_{S^2} f(\theta)d\theta = C\int_{SO(3)} f(Ae)dA, \quad e \in S^2, \tag{3.18}$$

we estimate

$$\begin{aligned} \|\Omega\varphi\|_{L^2_\theta} &\sim \left\| \int_{SO(3)} F(|re - lBe|)\varphi(ABe)dB \right\|_{L^2_A} \\ &\lesssim \|\varphi(Ae)\|_{L^2_A} \int_{SO(3)} F(|re - lBe|)dB \\ &\sim \|\varphi\|_{L^2_\theta} \int_{S^2} F(|re - l\theta|)d\theta, \end{aligned} \tag{3.19}$$

where we changed the variables as $\theta \mapsto Ae$ and $\alpha \mapsto ABe$. The last integral of F is dominated by

$$\{t < r+l\}\int_{S^2} |re - l\theta|^{-1}d\theta \lesssim \{t < r+l\}(r \vee l)^{-1}. \tag{3.20}$$

Interpolating these estimates, we obtain for $2 \le p \le \infty$

$$\|\Omega\varphi\|_{L^p_\theta} \lesssim t^{2/p-1}(r \vee l)^{-2/p}\{t < r+l\}\|\varphi\|_{L^{p'}_\theta}. \tag{3.21}$$

Plugging this estimate into $L_0(t)$, we obtain

$$\|\mathcal{L}_0 * f(t, r\theta)\|_{L^p_\theta} \lesssim \int_{\mathbb{R}}\int_0^\infty \frac{1}{r \vee l} g_p\left(\frac{|t-s|}{r \vee l}\right) \|f(s, l\theta)l^2\|_{L^{p'}_\theta} dlds, \tag{3.22}$$

where

$$g_p(t) = t^{2/p-1}\{0 < t < 2\}. \tag{3.23}$$

Then the desired L^p_θ estimate (3.3) for $m = 0$ follows from Lemma 3.2 together with the estimate $\|g_p\|_{L^1} \lesssim p$. The case $p < 2$ is covered by the embedding $L^2_\theta \hookrightarrow L^p_\theta$.

3.3. L^p_θ **estimate** (3.3) **for the Klein-Gordon.** Next we extend the above result to the Klein-Gordon $m > 0$. Since our estimate is global in time and the large time behavior is essentially different between the wave and the Klein-Gordon, it seems meaningless to approximate the latter by the former. Nevertheless, we will show that the TT^* operator $L_m(t)$ for the Klein-Gordon can be dominated by the wave correspondence and a "dispersive" part, which is smooth and decays fast in time.

By the rescaling argument, it suffices to prove the estimate for $m = 1$. We may assume $t > 0$ by symmetry. We calculate the kernel $\mathcal{L}_m$ by writing the Fourier transform in the polar coordinates as

$$\begin{aligned}\mathcal{L}_m(t,x) &= C\int_0^\infty \int_{S^2} \langle\rho\rangle_m^{-2} \cos(t\langle\rho\rangle_m) e^{ir\theta\cdot\rho\alpha}\rho^2 d\alpha d\rho \\ &= C\int_0^\infty \int_0^1 \langle\rho\rangle_m^{-2} \cos(t\langle\rho\rangle_m)\cos(r\rho\lambda)\rho^2 d\lambda d\rho \\ &= C\int_0^\infty \cos(r\nu)\int_\infty^{t\langle\nu\rangle_m} \frac{\cos l}{l} dl d\nu,\end{aligned} \tag{3.24}$$

where we changed the variables as $\lambda = \cos(\theta\cdot\alpha)$, $\nu = \rho\lambda$ and $l = t\langle\rho\rangle_m$. Then we obtain a uniform bound

$$|\mathcal{L}_1(t,x) - \mathcal{L}_0(t,x)| \lesssim \int_0^\infty \int_{t\nu}^{t\langle\nu\rangle} \frac{dl}{l}\, d\nu \lesssim 1. \tag{3.25}$$

Integrating by parts after changing the variable $l \mapsto l/\langle\nu\rangle_m$, we further rewrite (3.24) as

$$\mathcal{L}_m(t,x) = Ct^{-1}\mathcal{K}_m(t,x) + C\int_\infty^t \mathcal{K}_m(l,x)l^{-2}dl, \tag{3.26}$$

where $\mathcal{K}_m(t)$ denotes the one-dimensional fundamental solution of the Klein-Gordon. When $m = 1$, we have

$$\begin{aligned}\mathcal{K}_1(t,r) &= C\int_0^\infty \langle\nu\rangle^{-1}\sin(t\langle\nu\rangle)\cos(r\nu)d\nu = CJ_0(\sqrt{t^2-r^2})\{r<t\} \\ &\lesssim \left\langle\sqrt{t^2-r^2}\right\rangle^{-1/2}\{r<t\},\end{aligned} \tag{3.27}$$

where J_0 is the Bessel function of order 0 and we used the estimate $|J_0(s)| \lesssim \langle s\rangle^{-1/2}$ [12, p. 98]. Hence we have for $t < r$,

$$|\mathcal{L}_1(t,x)| \lesssim \int_r^\infty (l^2-r^2)^{-1/4}l^{-2}dl \lesssim r^{-3/2}. \tag{3.28}$$

When $t/2 < r < t$, we estimate $|\mathcal{K}_1(t,r)| \lesssim 1$ and

$$|\mathcal{L}_1(t,x)| \lesssim t^{-1} + \int_t^\infty l^{-2}dl \lesssim t^{-1} \lesssim r^{-1}. \tag{3.29}$$

When $r < t/2$, we have $\sqrt{t^2 - r^2} \gtrsim t$ and so

$$|\mathcal{L}_1(t,x)| \lesssim t^{-3/2} + t^{-1/2} \int_t^\infty l^{-2} dl \lesssim t^{-3/2}. \tag{3.30}$$

Gathering the estimates (3.14), (3.25), (3.29) and (3.30), we conclude

$$|\mathcal{L}_1(t,x)| \lesssim \mathcal{L}_0(t/2, x) + \langle t\rangle^{-3/2}. \tag{3.31}$$

Thus we have reduced the desired estimate for $m = 1$ to that for $m = 0$ and the $L_t^2 L_x^\infty$ estimate for the dispersive part $\langle t\rangle^{-3/2}$, which follows simply from the Young inequality.

4. Global solutions for the nonlinear Dirac equation

In this section, we prove Theorem 1.1. We rewrite the equation (1.1) as the following integral equation:

$$u = U_m(t)\varphi + \int_0^t U_m(t-s) F(u(s)) ds, \tag{4.1}$$

where $F(u) = -i\lambda\gamma^0(\gamma^0 u, u)u$ and $U_m(t)$ denotes the propagator of the free Dirac equation given by

$$U_m(t) = \cos(\omega_m t) - \gamma^0\Big(\sum_{j=1}^{3} \gamma^j \partial_j + im\Big)\omega_m^{-1} \sin(\omega_m t), \tag{4.2}$$

where $\omega_m = \sqrt{m^2 - \Delta}$. We set $\Phi u =$ R.H.S of (4.1) and apply the contraction mapping theorem.

For the linear term, we use the Strichartz estimates (3.3). We see from (4.2) that $\omega_m^{-1} U_m(t)$ is a linear combination of $\omega_m^{-1} e^{\pm i\omega_m t}$ with bounded Fourier multipliers. So we have estimates for $m \ge 0, 1 \le p < \infty$ as

$$\|U_m(t)\varphi\|_{L_t^2 L_r^\infty L_\theta^p} \lesssim \|\varphi\|_{H^1}. \tag{4.3}$$

Moreover, from the fact that Δ is commutative with Δ_θ, it follows that

$$\|U_m(t)\varphi\|_{L_t^2 L_r^\infty H_\theta^{s,p}} \lesssim \|(1-\Delta_\theta)^{s/2}\varphi\|_{H^1} \sim \|\varphi\|_{H^1(H_\theta^s)}. \tag{4.4}$$

Therefore putting $X = L_t^\infty H^1(H_\theta^s) \cap L_t^2 L_r^\infty H_\theta^{s,p}$ with p sufficiently large as $p > 2/s$, we have

$$\begin{aligned} \|\Phi u\|_X &\lesssim \|\varphi\|_{H^1(H_\theta^s)} + \int_0^\infty \|U_m(t-s)F(u(s))\|_X ds \\ &\lesssim \|\varphi\|_{H^1(H_\theta^s)} + \|F(u)\|_{L_t^1 H^1(H_\theta^s)}. \end{aligned} \tag{4.5}$$

By (2.10), we estimate the nonlinear term $F(u)$ as

$$\begin{aligned} \|F(u)\|_{H_\theta^s} &\lesssim \|u\|_{L_\theta^\infty}^2 \|u\|_{H_\theta^s}, \\ \|\nabla F(u)\|_{H_\theta^s} &\lesssim \|u\|_{H_\theta^{s,p}} \|u\|_{L_\theta^\infty} \|\nabla u\|_{L_\theta^q} + \|u\|_{L_\theta^\infty}^2 \|\nabla u\|_{H_\theta^s} \end{aligned} \tag{4.6}$$

with $1/p + 1/q = 1/2$. By the embeddings $H_\theta^{s,p} \hookrightarrow L_\theta^\infty$ for $s > 2/p$, $H_\theta^s \hookrightarrow L_\theta^q$ for $s \geq 2/p$, and the Hölder inequality for variables t and r, we have

$$\|F(u)\|_{L_t^1 H^1(H_\theta^s)} \lesssim \|u\|^2_{L_t^2 L_r^\infty H_\theta^{s,p}} \|u\|_{L_t^\infty H^1(H_\theta^s)}. \tag{4.7}$$

Analogously we have

$$\|\Phi u - \Phi v\|_X \lesssim (\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X. \tag{4.8}$$

Therefore Φ is a contraction map on a small closed ball in X.

For the uniqueness of solutions in the class of (1.5), we consider the $L_t^\infty L_x^2$ metric. By the L^2 invariance of $U(t)$, we have

$$\|u - v\|_{L_t^\infty L_x^2} \lesssim (\|u\|^2_{L_t^2 L_x^\infty} + \|u\|^2_{L_t^2 L_x^\infty})\|u - v\|_{L_t^\infty L_x^2}. \tag{4.9}$$

We can conclude $u = v$ time locally, so that for the entire time interval by the repetition.

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