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Turaev surfaces and invariants of knots and links

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1 Introduction

Turaev constructed to each knot diagram a closed, orientable surface on which the knot projects [Tur87]. In this short survey, we briefly recall the construction, and then discuss how Turaev surfaces relate to knot invariants. Most results extend to links as well.

1.1 Turaev surfaces

Kauffman defined an A-splicing and a B-splicing at a crossing of a knot diagram that are depicted in Figure 1.

Figure 1: A-splicing (middle) and B-splicing (right) of a crossing

For a diagram of a knot one can define a closed oriented surface: Let $\Gamma \subset S^2$ be the planar, 4–valent graph of the knot diagram. Thicken the projection plane to a slab $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings), the surface will intersect this slab in $\Gamma \times [-1, 1]$. In the neighborhood of each vertex, we insert a saddle, positioned so that the boundary circles on $S^2 \times \{1\}$ are the state circles of the all-A resolution, and the boundary circles on $S^2 \times \{-1\}$ are the state circles of the all-B resolution. (See Figure 2.) The boundary circles are capped off with disks, obtaining a closed oriented surface Σ.

It is shown in [DFKLS08] that the surface Σ is unknotted. In other words, $S^3 \setminus \Sigma$ is a disjoint union of two handlebodies. Armond, Druivenga and Kindred [ADK15] further studied the Heegaard splittings given by the Turaev surfaces, and showed that there is a one-to-one correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and Heegaard diagrams with certain conditions.

Define the Turaev genus $g_T(K)$ of a knot K to be the minimal genus of the Turaev genus of all diagrams of the knot. The knot projects alternatingly on this surface [DFKLS08], and it follows that the Turaev genus is 0 if and only if the knot is alternating.

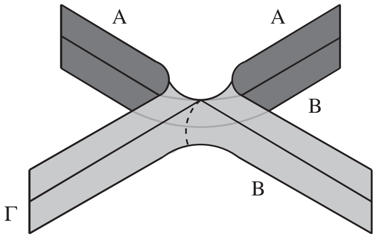

Figure 2: Near each crossing of the diagram, a saddle surface interpolates between state circles of the all-A resolution and state circles of the all-B resolution.

Turaev [Tur87] used the Turaev surfaces to show that the span of the Jones polynomial of a knot K with crossing number $c(K)$ is bounded from above by

$$\operatorname{span} V_K(t) \leq c(K) - g_T(K).$$

2 The Jones polynomial as an evaluation of the Bollobás-Riordan-Tutte polynomial

Thistlethwaite [Thi87] gave an interpretation of the Kauffman bracket of an alternating knot projection as an evaluation of the Tutte polynomial of a checkerboard graph of the knot. To extend this approach naturally to non-alternating knots the following generalization is useful: Instead of working with alternating knot projections on the plane one uses the alternating projections on a Turaev surfaces. Those projections admit a checkerboard coloring and thus the two checkerboard graphs are ribbon graphs, i.e. graphs with an embedding on a surface such that every face is a disk. Bollobás and Riordan [BR01, BR02] extended the two variable Tutte polynomial to a three variable polynomial for ribbon graphs. In [DFKLS08] it is shown that the Kauffman bracket is a suitable evaluation of this Bollobás-Riordan-Tutte polynomial.

More specifically, the surface minus the knot projection can be shaded in black and white. Here we suppose that the white colored faces correspond to the all-A resolution. The vertices of the graph correspond to the white regions and two vertices are connected by an edge if and only if the two faces meet at a crossing of the knot projection. This graph $\mathbb{D}$ is a ribbon graph, i.e. a graph embedded on a surface such that every face is a disk. Note, that ribbon graphs can be represented by graphs together with a cyclic orientation of the edges at every vertex. The ribbon graph will be frequently referred to as the all-A ribbon graph. Figure 3 shows the knot 8_{21}, its all-A resolution, and the associated ribbon graph.

The dual graph on the surface is the all-B ribbon graph. Bollobás and Riordan defined an extension of the Tutte polynomial to ribbon graphs. The Bollobás-Riordan-Tutte polynomial is given by

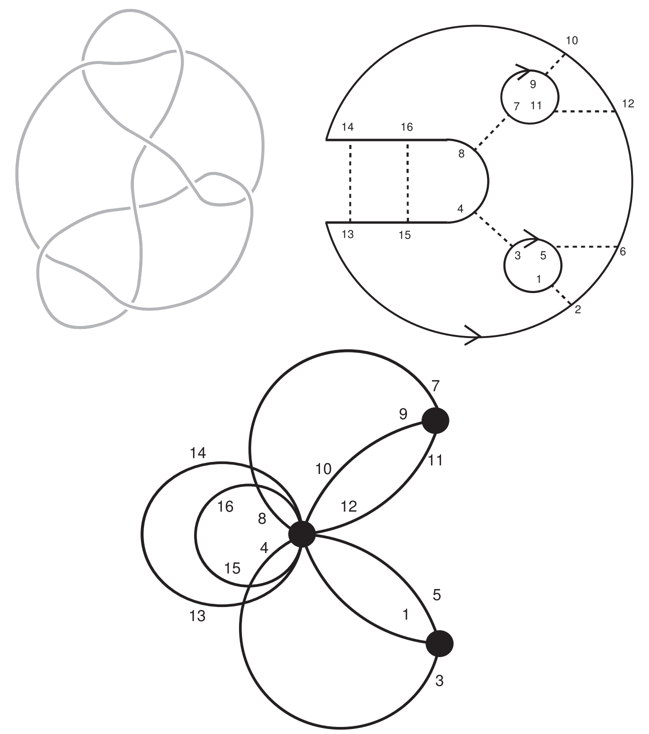

Figure 3: The knot 8_{21} with all-A resolution and associated ribbon graph.

$$C(\mathbb{D}; X, Y, Z) = (X-1)^{-k(\mathbb{D})} \sum_{\mathbb{H} \subset \mathbb{D}} (X-1)^{k(\mathbb{H})} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}$$

Here, $\mathbb{H}$ ranges over all spanning subribbon graphs of $\mathbb{D}$, i.e. the set of vertices in $\mathbb{H}$ and $\mathbb{D}$ are the same. Let $v(\mathbb{H})$ denote the number of vertices and $e(\mathbb{H})$ denote the number of edges in $\mathbb{H}$. The number of components of $\mathbb{H}$ is denoted by $k(\mathbb{H})$, the nullity, i.e. $e(\mathbb{H}) - v((\mathbb{H}) + k((\mathbb{H})$, by $n((\mathbb{H})$ and the genus is the genus of a minimal genus orientable surface on which ($\mathbb{H}$ embeds. The following relation to the Kauffman bracket was discovered in [DFKLS08], and in particular one obtains a topological description of the Jones polynomial in terms of the Turaev surface:

Theorem 1 ([DFKLS08]) *For a connected link projection P, and $\mathbb{D}$ the all-A ribbon graph of P as constructed above, the Kauffman bracket $\langle P \rangle$ of P is given by:*

$$A^{-e(\mathbb{D})} \langle P \rangle = A^{2-2v(\mathbb{D})} C(\mathbb{D}; -A^4, A^{-2}\delta, \delta^{-2}), \text{ where } \delta := (-A^2 - A^{-2}).$$

For an extension by Chmutov to virtual links and for further discussions see [Chm09, CK14, EMM13].

Of particular interest are subgraphs of the ribbon graphs that are called quasi-trees [DFKLS10]. More explicitly, a spanning subgraph $\mathbb{H}$ of a ribbon graph $\mathbb{D}$ is called j-quasi-tree if $\mathbb{H}$ is of genus j and he complement of the embedding of $\mathbb{H}$ on the surface is connected. Quasi-trees are a generalization of spanning trees for planar graphs. The 0-quasi-trees in a ribbon graph are the spanning trees in the ribbon graph. The general advantage of developing a theory of the Jones polynomial from the ribbon graph approach is that one has one additional parameter at hand, the genus. To show an application where the genus determines the sign: Let $s(\mathbb{D}, j)$ be the number of j-quasi-trees in the all-A ribbon graph of diagram D of the knot K. Then the determinant $\det(K)$ of the knot can be computed by [DFKLS10]:

$$\det(K) = \left| \sum_{j=0} (-1)^j s(\mathbb{D}, j) \right|$$

3 Turaev surfaces, and knot invariants

The approach to knot and link theory via projections on Turaev surfaces turns out to be quite fruitful for the study of knot invariants:

3.1 Knot Homology Theories

By its very nature every knot homology theory expresses the coefficients of the underlying knot polynomial of a knot K as Euler characteristic of a homology. Thus, it gives a splitting of the coefficients into an alternating sum of $\mathbb{Z}$-valued knot invariants. Khovanov homology [Kho00] can be naturally defined via the Turaev surface approach. One can construct a homology theory for ribbon graphs that, if applied to the all-A ribbon graph of a link diagram, yields the (reduced) Khovanov homology of the corresponding link (see [DL14] for details on the construction). The ribbon graph chain complex retracts onto a complex with generators that correspond to spanning quasi-trees of the ribbon graph. More specifically, the two gradings are determined by the polynomial grading and (up to a shift) the genus of the quasi-trees. Thus the homological degree is interpreted in a purely topological way.

For (reduced) Khovanov homology it was shown by Manturov [Man05] and Champanerkar, Kofman and Stoltzfus [CKS11] that the number of terms in this splitting is generally bounded from above by the Turaev genus $g_T(K)$ increased by 1. More specifically, the non-trivial groups in the bi-graded Khovanov homology $H^{i,j}_{Kh}(K)$ of the knot K lie on slope-one lines with respect to the (i, j)-grading, and their width w_{Kh} is bounded from above by $g_T(K) + 1$.

Surprisingly, Lowrance was able to show that this result also holds for the width $w_{HF}(K)$ of the Ozsváth-Szabó knot Floer homology [Low08]:

$$w_{HF}(K) \leq g_T(K) + 1.$$

3.2 The signature of a knot, and the τ and s-invariants

For knots of Turaev genus 0, i.e. alternating knots, the knot signature σ, the Ozsváth-Szabó τ-invariant [OS03] and the Rasmussen s-invariant [Ras10] satisfy:

$$2\tau(K) = s(K) = -\sigma(K). \tag{1}$$

Traczyk [Tra04] (see also [GL78]) proved that if D is a reduced alternating diagram of an alternating knot K then

$$\sigma(K) = s_A(D) - n_+(D) - 1, \tag{2}$$

where $s_A(D)$ is the number of components in the all-A Kauffman resolution of D and $n_+(D)$ is the number of positive crossings in D.

Equation (1) can be generalized using the Turaev genus [DL11]:

$$\left|\tau(K) + \frac{\sigma(K)}{2}\right| \leq g_T(K),$$

$$\frac{|s(K) + \sigma(K)|}{2} \leq g_T(K), \text{ and}$$

$$\left|\tau(K) - \frac{s(K)}{2}\right| \leq g_T(K).$$

For a diagram D of a knot K equation (2) generalizes to [DL11]:

$$s_A(D) - n_+(D) - 1 \leq \sigma(K) \leq (s_A(D) - n_+(D) - 1) + 2g_T(D). \tag{3}$$

4 Further aspects

4.1 Adequate knots

Adequate knots are knots that admit a Turaev surface where neither of the two checkerboard ribbon graphs contain a loop, i.e. an edge that connects a vertex with itself. Alternating knots are examples of adequate knots. Khovanov studied the Khovanov homology of adequate knots [Kho03]. Using Khovanov's results, Abe [Abe09b] was able to determine the Turaev genus of adequate knots: It is realized by an adequate diagram of the knot. Moreover, for an adequate knot K of crossing number $c(K)$ and Turaev genus $g_T(K)$ the span of the Jones polynomial satisfies:

$$\operatorname{span} V_K = c(K) - g_T(K).$$

4.2 Knots with small Turaev genus

Knots with Turaev genus 0, i.e. alternating knots, are well understood. Menasco [Men84] showed that prime alternating knots are hyperbolic, unless they are $(2, k)$-torus knots. Moreover, Tait's flyping theorem [MT93] gives a classification of alternating diagrams of an alternating knot: Given an oriented, prime alternating knot K with reduced alternating diagrams D_1 and D_2, D_1 can be transformed to D_2 by a finite sequence of moves, called flypes.

Armond and Lowrance [AL15] classify link diagrams whose Turaev surface has genus one or two, and prove that similar classification theorems exist for all genera. A similar result was independently and with different methods proven by Seungwon Kim [Kim18] for Turaev genus one and two.

For knots with Turaev genus 1 Equation (3) strengthens to

Theorem 2 *[DL16] Let K be a knot with diagram D whose Turaev surface has genus one. The signature of K is determined by*

$$\sigma(K) = s_A(D) - c_+(D) \pm 1 \text{ and } \sigma(K) = \det(K) - 1 \bmod 4.$$

By using results from Adams [Ada94] and Abe [Abe09a] the result of Menasco [Men84] on the hyperbolicity of prime, alternating knots that are not torus knots, extends to:

Theorem 3 *[DL16] If K is a prime knot of Turaev genus one, then K is either hyperbolic or a torus pretzel knot.*

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