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ARITHMETIC DEGREES OF SPECIAL CYCLES AND DERIVATIVES OF SIEGEL EISENSTEIN SERIES

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ABSTRACT. We report on recent joint work with Tonghai Yang [BY] on a conjecture of Kudla relating the arithmetic degrees of top degree special cycles on an integral model of an orthogonal Shimura variety to the coefficients of the central derivative of an incoherent Siegel Eisenstein series.

The classical Siegel-Weil formula connects the arithmetic of quadratic forms with Eisenstein series for symplectic groups. It also has important geometric applications. For instance, it leads to formulas for the degrees and intersection numbers of special cycles on orthogonal Shimura varieties in terms of Fourier coefficients of Eisenstein series. We begin by recalling some of these results in order to motivate the analogous results in the arithmetic setting and to set up some notation.

Let (V, Q) be a rational quadratic space of signature $(m, 2)$. To simplify the exposition, we assume throughout that m is even and refer to [BY] for the general case. Let $H = \mathrm{GSpin}(V)$, and write

$$\mathcal{D} = \{z \in V_{\mathbb{C}} :\ (z, z) = 0,\ (z, \bar{z}) < 0\}/\mathbb{C}^\times$$

for the corresponding hermitian symmetric space. We fix an even lattice $L \subset V$ and consider the arithmetic subgroup $\mathrm{GSpin}(L) \subset H$. The quotient

$$X = \Gamma\backslash\mathcal{D}$$

is the complex space of a Shimura variety associated with H. It has dimension m. For small m, these Shimura varieties include several classes of classical examples such as Shimura curves and Hilbert modular surfaces.

There are important families of special cycles on X coming from embedded quadratic spaces of smaller dimensions. To define these in our setting we fix a positive integer n. Let $x = (x_1, \dots, x_n) \in V^n$ and assume that the moment matrix $Q(x) = \frac{1}{2}((x_i, x_j))$ is positive semi-definite. Then

$$\mathcal{D}_x = \{z \in \mathcal{D} \mid\ (z, x_i) = 0 \text{ for } i = 1, \dots, n\},$$

is an analytic subspace of $\mathcal{D}$ of codimension $\mathrm{rank}(T)$. If $T \in \mathrm{Sym}_n(\mathbb{Q})$ is positive definite we obtain an algebraic special cycle of codimension n on X by taking the image of

$$Z(T) = \sum_{\substack{x \in L^n \\ Q(x)=T}} \mathcal{D}_x \tag{1}$$

under the quotient map. We denote by $[Z(T)]$ its class in in the cohomology group $H^{2n}(X)$. We can extend this definition to classes for positive *semi*-definite matrices T by intersecting the naive class given by (1) with the power $(\mathcal{L}^\vee)^{n-\mathrm{rank}(T)}$ of the co-tautological bundle $\mathcal{L}^\vee$ on X.

It is natural to ask for the relations among the classes $[Z(T)]$ of these cycles in $H^{2n}(X)$. Such relations can be described in an elegant way by looking at the formal generating series

$$A_n^{\mathrm{coh}} = \sum_{T\geq 0} [Z(T)] \cdot q^T, \tag{2}$$

where T runs through the positive semi-definite half integral symmetric matrices T. Generalizing the fundamental work of Hirzebruch and Zagier in the case of Hilbert modular surfaces, Kudla and Millson proved the following result, see [KM].

Theorem 1 (Kudla-Millson). *The generating series (2) is the Fourier expansion of a Siegel modular form $A_n^{\mathrm{coh}}(\tau)$ of weight $\kappa = 1 + m/2$ and genus n for a congruence subgroup $\Gamma' \subset \mathrm{Sp}_n(\mathbb{Z})$, taking values in $H^{2n}(X)$. Here q^T has to be interpreted as $e^{2\pi i\,\mathrm{tr}(T\tau)}$ with τ in the Siegel upper half plane $\mathbb{H}_n$ of genus n.*

The level of Γ' is given by the level of L. An analogous statement for classes in Chow groups was obtained in [Zh], [BW].

In the case when $n = m$, that is, for special cycles of top degree, it is possible to describe the resulting Siegel modular forms precisely by means of the following geometric Siegel-Weil formula [Ku2].

Theorem 2 (Kudla). *Assume that X is compact such that $H^{2n}(X) \cong \mathbb{C}$ via the degree map. Then*

$$\deg(A_n^{\mathrm{coh}})(\tau) = \sum_{T\geq 0} \deg[Z(T)] \cdot q^T$$

is (up to a non-zero constant factor) equal to the Siegel Eisenstein series

$$E(\tau, s_0, \lambda(\varphi_L) \otimes \Phi_\kappa)$$

of genus n and weight κ for Γ'. Here $\lambda(\varphi_L)$ denotes the section of the induced representation $I(s, \chi_V)$ of $\mathrm{Sp}_n(\mathbb{A}_f)$ associated with the characteristic function φ_L of $\hat{L}^n \subset V_{\mathbb{A}_f}^n$, and Φ_κ denotes the standard section of weight κ of the induced representation of $\mathrm{Sp}_n(\mathbb{R})$. In the spectral parameter s the value is taken at $s_0 = 1/2$, see [BY] for details.

If X is non-compact, the Eisenstein series is usually non-holomorphic and the treatment of the non-holomorphic contributions needs extra care, see e.g. [FM].

Kudla initiated a program connecting the Arakelov geometry of special cycles on integral models of orthogonal (and unitary) Shimura varieties to Siegel (Hermitian) modular forms, see [Ku1], [KRY2]. It is conjectured (and in a few low dimensional cases known) that there should be results which parallel the above theorems. In particular, it is expected that arithmetic degrees of special cycles are connected to derivatives of Siegel Eisenstein series.

The main objects in this setting are arithmetic cycles in the sense of Gillet-Soulé [GS], which are given by pairs consisting of a cycle on an integral model of X and a Green current for the cycle. For $x \in V_{\mathbb{R}}$, Kudla defined a Green function for the divisor $\mathcal{D}_x \subset \mathcal{D}$ by

$$\xi_0(x,z) = -\operatorname{Ei}(2\pi(x_z, x_z))$$

for $z \in \mathcal{D}$. Here $\operatorname{Ei}(w) = \int_{-\infty}^{w} e^t \frac{dt}{t}$ denotes the exponential integral, and x_z means the orthogonal projection of x to the negative definite plane $\mathbb{R}\Re(z) + \mathbb{R}\Im(z) \subset V_{\mathbb{R}}$. This Green function has a logarithmic singularity along $\mathcal{D}_x$. More generally, if $x = (x_1, \dots, x_n) \in V_{\mathbb{R}}^n$ with invertible moment matrix $Q(x)$, one obtains a Green current for the cycle $\mathcal{D}_x$ by taking the star product

$$\xi_0^n(x,z) = \xi_0(x_1, z) * \cdots * \xi_0(x_n, z)$$

in the sense of Gillet-Soulé. It defines a current on compactly supported differential forms. If $\mathcal{D}_x$ is compact, then $\xi_0^n(x,z)$ is rapidly decreasing and extends to test forms with moderate growth. For the rest of this exposition we assume that $T \in \operatorname{Sym}_n(\mathbb{Q})$ is invertible. Then we obtain a Green current for the cycle $Z(T)$ on X by

$$G_T(v,z) = \sum_{\substack{x \in L^n \\ Q(x)=T}} \xi_0^n(xv^{1/2}, z),$$

where the additional parameter v is any positive definite matrix in $\operatorname{Sym}_n(\mathbb{R})$ (for which we will later insert the imaginary part of $\tau \in \mathbb{H}_n$), and $v^{1/2}$ denotes its positive square root in $\operatorname{Sym}_n(\mathbb{R})$.

To describe the integral models of X and the special cycles we are working with, we assume for convenience that V contains an unimodular even lattice L. (This assumption can be relaxed if one works over the localisation $\mathbb{Z}_{(p)}$ at a prime p, see [BY]). By work of Kisin, Vasiu, and Madapusi Pera, the Shimura variety X has a canonical integral model $\mathcal{X}$, which is a smooth stack over $\mathbb{Z}$, see [Ki], [MP].

There is a polarized abelian scheme $A \to \mathcal{X}$ of relative dimension 2^{m+1} over $\mathcal{X}$, which is equipped with an action of the Clifford algebra $C(L)$ of L. For any $\mathbb{Z}$-scheme S and any S-valued point $\alpha : S \to \mathcal{X}$ there is a space of special endomorphisms

$$V(A_\alpha) \subset \operatorname{End}_{C(L)}(A_\alpha)$$

on the pull-back A_α of A, which is endowed with a positive definite even quadratic form Q, see [AGHM, Section 4]. It can be used to define an integral model of $Z(T)$ as the sub-stack of $\mathcal{X}$ whose S-valued points are given by

$$\mathcal{Z}(T)(S) = \{(\alpha, x) : \ \alpha \in \mathcal{X}(S),\ x \in V(A_\alpha)^n,\ Q(x) = T\}.$$

The pair

$$\widehat{\mathcal{Z}}(T,v) = \big(\mathcal{Z}(T), G_T(v)\big)$$

determines a class in an arithmetic Chow group of $\mathcal{X}$. Through the Green current it depends on v. In analogy with the geometric situation described earlier, we would like to understand the classes of these cycles and their relations.

Again we focus on the case of top degree cycles, which is here the case when $n = m+1$. If T is not positive definite, then $\mathcal{Z}(T)$ vanishes, but the arithmetic cycle $\widehat{\mathcal{Z}}(T,v)$ has non-trivial current part. On the other hand, if T is positive definite, then $\widehat{\mathcal{Z}}(T,v)$ has trivial current part, and the cycle is entirely supported in positive characteristic. In fact, if it is non-trivial then it is supported in the fiber above one single prime p. The dimension of the irreducible components were determined by Soylu [So]. In particular, he showed that $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ is finite if and only if the reduction of T modulo p is of rank $n-1$, $n-2$, or of rank $n-3$ (plus a technical condition).

Here we consider the cases when either T is not positive definite, or T is positive definite and $\mathcal{Z}(T)$ has dimension 0. Then $\widehat{\mathcal{Z}}(T,v)$ defines a class in the arithmetic Chow group $\widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\overline{\mathcal{X}})$ of a toroidal compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$. Recall that there exists an arithmetic degree map

$$\widehat{\deg}:\widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\overline{\mathcal{X}})\longrightarrow\mathbb{C}$$

which is given as a sum of local degrees

$$\widehat{\deg}(\mathcal{Z},G)=\sum_{p\leq\infty}\widehat{\deg}_p(\mathcal{Z},G),$$

where

$$\widehat{\deg}_p(\mathcal{Z},G)=\begin{cases}\displaystyle\sum_{x\in\mathcal{Z}(\bar{\mathbb{F}}_p)}\frac{\mathrm{ht}_p(x)}{|\operatorname{Aut}(x)|}\cdot\log(p), & \text{if } p<\infty,\\ \displaystyle\frac{1}{2}\int_{\overline{\mathcal{X}}(\mathbb{C})}G, & \text{if } p=\infty.\end{cases}$$

Here $\mathrm{ht}_p(x)$ denotes the length of the étale local ring $\mathcal{O}_{\mathcal{Z},x}$ of $\mathcal{Z}$ at the point x. Kudla conjectured the following description of the arithmetic degrees of special cycles in terms of derivatives of Siegel Eisenstein series of genus n, see [Ku1],[Ku3].

Conjecture 3 (Kudla). *Assume that $n=m+1$ and that $T\in\mathrm{Sym}_n(\mathbb{Q})$ is invertible. Then*

$$\widehat{\deg}\big(\widehat{\mathcal{Z}}(T,v)\big)\cdot q^T=C\cdot E_T'(\tau,0,\lambda(\varphi_L)\otimes\Phi_\kappa),$$

where C denotes an explicit constant which is independent of T, $E_T(\tau,s,\Phi)$ denotes the T-th Fourier coefficient of a Siegel Eisenstein series $E(\tau,s,\Phi)$, and the derivative is taken with respect to s.

The ideal statement of the conjecture would involve a suitable generalization of the arithmetic degrees of the $\widehat{\mathcal{Z}}(T,v)$ to *all* half integral $T\in\mathrm{Sym}_n(\mathbb{Q})$. The generating series of these arithmetic degrees should be given by the central derivative of the Eisenstein series $E(\tau,s,\lambda(\varphi_L)\otimes\Phi_\kappa)$, in analogy with Theorem 2. Such an identity could be viewed as an *arithmetic Siegel-Weil formula.* The full conjecture is known for $m=0$ and for the $m=1$ case of Shimura curves, see [KRY1], [KRY2].

To state our results on Conjecture 3, we let $\mathcal{C}=\bigotimes_{p\leq\infty}\mathcal{C}_p$ be the incoherent quadratic space over $\mathbb{A}$ for which $\mathcal{C}_f=\otimes_{p<\infty}\mathcal{C}_p\cong V_{\mathbb{A}_f}$ and $\mathcal{C}_\infty$ is positive definite of dimension $m+2$. The Eisenstein series appearing in Conjecture 3 is naturally associated with the Schwartz function on $S(\mathcal{C}^n)$ given by the tensor product of φ_L and the Gaussian on $\mathcal{C}_\infty^n$ via the intertwining operator λ to the induced representation. Hence it is incoherent and vanishes at the central point $s=0$. The conjecture gives a formula for the leading term of the Taylor expansion in s at this point. Define the 'Diff set' associated with $\mathcal{C}$ and T as

$$\mathrm{Diff}(\mathcal{C},T)=\{p\leq\infty:\ \mathcal{C}_p\text{ does not represent } T\}.$$

Then $\mathrm{Diff}(\mathcal{C}, T)$ is a finite set of odd order, and $\infty \in \mathrm{Diff}(\mathcal{C}, T)$ if and only if T is not positive definite.

Theorem 4 (See [BY], Theorem 1.2). *Assume that $T \in \mathrm{Sym}_n(\mathbb{Q})$ is invertible. Then Conjecture 3 holds in the following cases:*

(1) *If $|\mathrm{Diff}(\mathcal{C}, T)| > 1$. In this case both sides of the equality vanish.*
(2) *If $\mathrm{Diff}(\mathcal{C}, T) = \{\infty\}$. In this case T is not positive definite, and the only contribution comes from the archimedean place, i.e.,*

$$\widehat{\deg}\big(\widehat{\mathcal{Z}}(T, v)\big) \cdot q^T = \widehat{\deg}_\infty\big(\widehat{\mathcal{Z}}(T, v)\big) \cdot q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi_L) \otimes \Phi_\kappa).$$

(3) *If $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$ for a finite prime $p \neq 2$ and $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ has dimension 0. In such a case, the only contribution comes from the prime p, i.e,*

$$\widehat{\deg}\big(\widehat{\mathcal{Z}}(T, v)\big) \cdot q^T = \widehat{\deg}_p\big(\widehat{\mathcal{Z}}(T, v)\big) q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi_L) \otimes \Phi_\kappa).$$

To prove Theorem 4, we decompose the Fourier coefficients of the Eisenstein series into local factors. If $\Phi = \otimes_v \Phi_v$ is a factorizable section of the induced representation, then

$$E_T(g, s, \Phi) = \prod_{v \leq \infty} W_{T,v}(g, s, \Phi_v),$$

where $W_{T,v}(g, s, \Phi_v)$ is the local Whittaker function at v. It is a basic fact that the local Whittaker functions appearing in Theorem 4 vanish for every $p \in \mathrm{Diff}(\mathcal{C}, T)$. Since the arithmetic degree is also given by local degrees, the claimed identities can be reduced to local statements. Very roughly speaking they can be deduced from the classical local Siegel-Weil formula (and a geometric variant) together with the following *arithmetic* local Siegel-Weil formulas.

Theorem 5 (See [BY], Theorem 1.4). *Let $x \in V_{\mathbb{R}}^n$ such that the $Q(x) = T$ is invertible. Then the archimedean local height function*

$$\mathrm{ht}_\infty(x) = \frac{1}{2} \int_{\mathcal{D}} \xi_0^n(x, z)$$

is given by

$$\mathrm{ht}_\infty(x v^{1/2}) \cdot q^T = -B_{n,\infty} \det(v)^{-\kappa/2} \cdot W'_{T,\infty}(g_\tau, 0, \Phi_\kappa),$$

where $B_{n,\infty}$ is an explicit non-zero constant which is independent of x.

In the special case $m = 0$ Theorem 5 was proved in [KRY1], for $m = 1$ in [Ku1], and for $m = 2$ in [YZZ]. For the related case of Shimura varieties associated to unitary groups of signature $(m, 1)$ it was proved in [Liu].

The proof of Theorem 5 is given by an inductive argument. It relies on the classical local Siegel-Weil formula, the action of the Lie algebra of $\mathrm{Sp}_n(\mathbb{R})$ in the induced representation, and some asymptotic properties of archimedean Whittaker functions. While Theorem 5 is crucial for the proof of the second assertion of Theorem 4, the following non-archimedean analogue is required for the third assertion.

Theorem 6 (See [BY], Theorem 1.5). *Let $p \neq 2$ be a prime number and assume that $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ is finite. Then for $x \in \mathcal{Z}(T)(\bar{\mathbb{F}}_p)$, the local height $\mathrm{ht}_p(x)$ is independent of the choice of x and is given by*

$$\mathrm{ht}_p(x) \cdot \log p = \frac{W'_{T,p}(1,0,\lambda(\varphi_L))}{W_{T^u,p}(1,0,\lambda(\varphi_L))},$$

where T^u is any unimodular matrix in $\mathrm{Sym}_n(\mathbb{Z}_p)$ (i.e., $\det T^u \in \mathbb{Z}_p^\times$).

Similarly as in the archimedian case the proof of this result relies on an inductive argument. By a recursion formula for the local Whittaker function and a reduction formula for the local height function for special cycles on Rapoport-Zink spaces due to Li and Zhu, the assertion can be reduced to the Hilbert modular surface case (where $n = 3$), which was studied by Kudla and Rapoport [KR].

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