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# An explicit construction of non-tempered cusp forms on $O(1,8n+1)$

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**Abstract**

This short note is a write-up of the results presented by the second named author at RIMS workshop "Analytic and arithmetic theory of automorphic forms". The main result is an explicit construction of the real analytic cusp forms on $O(1,8n+1)$ by a lifting from Maass cusp forms of level one. The lifting is proved to be Hecke-equivariant. Our results include an explicit formula for Hecke eigenvalues of the lifts and explicit determination of the cusidal representations generated by them. This leads to showing the nontemperedness of the cuspidal representations at every finite place, namely our explicit construction provides "real analytic counterexamples to Ramanujan conjecture".

## 1 Statement of the results

Let $\mathfrak{h} := \{u+\sqrt{-1}v \in \mathbb{C} \mid v>0\}$and $\Delta := v^2(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2})$ be the hyperbolic Laplacian on $\mathfrak{h}$. We then first review the definition of Maass cusp forms on $\mathfrak{h}$.

**Definition 1.1** *A $C^\infty$-function $f:\mathfrak{h}\to\mathbb{C}$ is called a Maass cusp form (of level one) if it satisfies the following:*

1. $f(\gamma(\tau)) = f(\tau) \quad \forall\gamma\in SL_2(\mathbb{Z})$,
2. $\Delta\cdot f = -(\frac{1}{4}+\frac{r^2}{4})f \ (r\in\mathbb{R})$,
3. *The Fourier expansion of $f$ has no constant term:*

$$f(\tau) = \sum_{n\neq 0} c_f(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v)\exp(2\pi\sqrt{-1}nu) \ (\tau = u+\sqrt{-1}v).$$

We next introduce Maass forms on real hyperbolic spaces (of higher dimension). Let $H_n := \{(\boldsymbol{x},y) \mid \boldsymbol{x}\in\mathbb{R}^n,\ y>0\}$ be the $n+1$-dimensional real hyperbolic space, which can be identified with $O(1,n+1)(\mathbb{R})/O(1,n+1)(\mathbb{R})\cap O(n+2)(\mathbb{R})$. For an arithmetic subgroup $\Gamma\subset O(1,n+1)(\mathbb{R})$ we introduce the following:

**Definition 1.2** *A $C^\infty$-function $F:H_n\to\mathbb{C}$ is called a Maass form on $H_n$ with respect to $\Gamma$ if it satisfies the following:*

1. $F(\gamma(z)) = F(z) \quad \forall(\gamma,z)\in\Gamma\times H_n$.

*2.* $\Omega \cdot F = \frac{1}{2n}(\rho^2 - \frac{n^2}{4})F$ ($\rho \in \mathbb{C}$) (*$\Omega$:Casimir operator).*

*3. $F$ is of moderate growth.*

*We denote by $M(\Gamma, \rho)$ the space of Maass forms above.*

In what follows, let $f$ be a Maass cusp form of level one and $(\mathbb{Z}^{8n}, S)$ be an even unimodular lattice with the quadratic form defined by a positive definite symmetric matrix $S$. We further let $O(Q)(\simeq O(1, 8n+1))$ be the orthogonal group defined by $Q := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$. We then see that $\{(x, y) \mid x \in \mathbb{R}^{8n},\ y > 0\}$ is $8n+1$-dimensional real hyperbolic space, which can be identified with $O(Q)(\mathbb{R})/O(Q)(\mathbb{R}) \cap O(8n+2)(\mathbb{R})$. We let $\Gamma_S := \{\gamma \in O(Q)(\mathbb{Q}) \mid \gamma\mathbb{Z}^{8n+2} = \mathbb{Z}^{8n+2}\}$.

For a Maass cusp form $f$ we now Introduce a function on the $8n+1$-dimensional hyperbolic space as follows:

$$F_f((x, y)) = \sum_{\lambda \in \mathbb{Z}^{8n} \setminus \{0\}} C_\lambda y^{4n} K_{\sqrt{-1}r}(4\pi|\lambda|_S y) \exp(2\pi\sqrt{-1}{}^t\lambda S x),$$

where $|\lambda|_S := \sqrt{\frac{1}{2}{}^t\lambda S \lambda}$. Here, with the greatest common divisor $d_\lambda$ of $\lambda \in \mathbb{Z}^{8n} \setminus \{0\}$,

$$C_\lambda := |\lambda|_S \sum_{d|d_\lambda} c(-\frac{|\lambda|_S^2}{d^2}) d^{4n-2}$$

We are ready to state our first result:

**Theorem 1.3** *(1) $F_f$ is a Maass cusp form in $M(\Gamma_S, \sqrt{-1}r)$, where $r \in \mathbb{R}$ is the parameter of the $\Delta$-eigenvalue for $f$.*
*(2) $f \not\equiv 0 \Rightarrow F_f \not\equiv 0$.*
*(3) $f$ is a Hecke eigenform$\Rightarrow$ so is $F_f$.*

Our next result concerns cuspidal representations generated by $F_f$. To this end we adelized $F_f$ as an automorphic form on $O(Q)(\mathbb{A})$ by

$$F_f(g) := \sum_{\lambda \in \mathbb{Q}^{8n} \setminus \{0\}} A_\lambda(g_f) W_{\lambda,\infty}(g_\infty) \quad (g = g_f g_\infty \in O(Q)(\mathbb{A}_f) \times O(Q)(\mathbb{R}) = O(Q)(\mathbb{A}))$$

where

$$A_\lambda(\begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix}) = \delta(\lambda \in L_h)|\lambda| \sum_{d|d_\lambda} d^{4n-2} c(-\frac{|\lambda|^2}{d^2})\ \forall h \in O(S)(\mathbb{A}_f),$$

$$A_\lambda(\begin{pmatrix} \alpha & & \\ & h & \\ & & \alpha^{-1} \end{pmatrix}) = ||\alpha||_{\mathbb{A}}^{4n} A_{||\beta||_{\mathbb{A}}^{-1}\lambda}(\begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix}),\ \forall(\alpha, h) \in \mathbb{A}_f^\times \times O(S)(\mathbb{A}_f),$$

$$A_\lambda(n(x)lk) = \Lambda({}^t\lambda S x) A_\lambda(l)\ \forall(x, l, k) \in \mathbb{A}_f^{8n} \times \mathcal{L}(\mathbb{A}_f) \times K_f,$$

with the Levi subgroup $\mathcal{L} \simeq \mathbb{G}_m \times O(S)$ and the archimedean Whittaker function $W_{\lambda,\infty}$ appearing in the non-adelic Fourier expansion.

Let $\pi_{F_f}$ be a cuspidal representation of $O(1, 8n+1)(\mathbb{A})$ generated by $F_f$. Our next result is stated as follows:

**Theorem 1.4** *(1) Let $f$ be a Hecke-eigen cusp form with a Hecke eigenvalue $\lambda_p$ at each prime $p$. Then $\pi_{F_f}$ is irreducible and thus decomposes into the restricted product $\otimes'_{v\leq\infty}\pi_v$. Every $\pi_p$ for $v = p < \infty$ is explicitly determined by the Satake parameter*

$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p^{4n-1}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-(4n-1)}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right).$$

*(2) For $p < \infty$, $\pi_p$ is non-tempered while $\pi_\infty$ is tempered (i.e. counterexample to the Ramanujan conjecture).*
*3) The standard L-function $L(\pi_{F_f}, \mathrm{std}, s)$ has the following coincidence:*

$$L(\pi_{F_f}, \mathrm{std}, s) = L(\mathrm{sym}^2(f), s) \prod_{i=0}^{8n-2} \zeta(s + (i - (4n-1))).$$

In what follows, we overview the proofs of the two theorems above. For the detailed proof see [3].

## 2 Outline of the proof for the first theorem

In this section we explain mainly of the $\Gamma_S$-automorphy of $F_f$. Our original idea was to use the converse theorem by Maass [4] (cf. [5], [8]). A basic limitation of the Maass converse theorem is that it provides automorphy only with respect to a discrete subgroup generated by translations and one inversion. For the case of $n > 1$, it seems difficult to determine the generators of $\Gamma_S$. Hence, the Maass converse theorem method, though applicable, does not give automorphy with respect to all of $\Gamma_S$. Instead we use the notion of a theta lifting. For this we remind the readers that $SL_2 \times O(1, m)$ forms a dual pair. It is natural to expect that our lift $f \mapsto F_f$ is a theta lift to $O(1, 8n+1)$. This new idea has enabled us to overcome the difficulty.

**Theorem 2.1** *With a suitable choice of a theta kernel $\Theta(\tau, (x, y))$ we have*

$$F_f((x, y)) = \int_{SL_2(\mathbb{Z})\backslash\mathfrak{h}} f(\tau)\overline{\Theta(\tau, (x, y))} v^{4n-\frac{3}{2}} du dv,$$

*namely, $F_f$ is a theta lift from $f$.*

We follow the formulation of the theta lift by Borcherds [1]. The theta integral kernel is given as $\Theta(\tau, \nu) := \sum_{\lambda \in L} \exp(-\frac{\Delta_n}{8\pi v}) P(\iota_\nu(\lambda_\nu)) e^{\pi\sqrt{-1}(q_Q(\lambda_\nu^-)\bar{\tau} + q_Q(\lambda_\nu^+)\tau)}$, where

- $\Delta_n = \sum_{i=0}^{8n+1} \frac{\partial_i^2}{\partial x_i^2}$ denotes the standard Laplacian on $\mathbb{R}^{8n+2}$,

- $q_Q$ denotes the quadratic form defined by $Q$.
- $H_{8n+1}$ is viewed as the Grassmanian $\mathcal{D}_{8n+1}$ of positive oriented lines in $(\mathbb{R}^{8n+2}, Q)$. Each $\nu \in \mathcal{D}_{8n+1} \simeq H_{8n+1}$ defines an isometry

$$\iota_\nu : (\mathbb{R}^{8n+2}, Q) \ni \lambda \mapsto ({}^t\lambda Q\nu, \lambda_\nu^-) \in \mathbb{R}^{1,0} \oplus \nu^\perp \simeq \mathbb{R}^{1,8n+1}$$

  with $\lambda_\nu^- := \lambda - \lambda_\nu^+$ with $\lambda_\nu^+ := ({}^t\lambda Q\nu) \cdot \nu$.
- $P(x_0, x_1, \cdots, x_{8n}, x_{8n+1}) := 2^{-n/4-3}x_0^{4n}$, which is a non-harmonic homogeneous polynomial.

Since $\Theta(\tau, (x, y))$ is $\Gamma_S$-invariant, the automorphy of $F_f$ follows. As the Fourier expansion of $F_f$ indicates, $F_f$ is a cusp form. Recall now that there has been the assertion of non-vanishing of $F_f$. We verify this by the argument similar to [5, Theorem 4.4]. For this, note that the set of the $\Gamma_S$-cusps are in bijection with the equivalence classes of even unimodular lattices of rank $8n$. To show the non-vanishing, we use the Fourier expansion of the $\Gamma_S$-cusp corresponding to $E_8^n$, where $E_8$ denotes the even unimodular lattice of rank 8 called the $E_8$-lattice. It is worth while to remark that the representability of every integer by the $E_8$-lattice is one essential point for the proof of the non-vanishing.

## 3 Outline of the proof for the second theorem

We discuss the Hecke theory of our lifting, which leads to overview of the proof for the second theorem. In fact, we will show that if $f$ is a Hecke eigenform then so is the lift $F_f$. We can compute the Hecke eigenvalues of $F_f$ explicitly in terms of those of $f$, which yields the theorem. The method is to use the non-archimedean local theory by Sugano [11, Section 7] for the Jacobi form formulation of the Oda-Rallis-Schiffmann lifting [7], [9].

To review the setting of Sugano's local theory we first introduce the notation on groups and lattices:

- $F$:non-archimedean local field of char.$\neq 2$ with integer ring $\mathfrak{o}$.
- $Q_m := \begin{pmatrix} & & J_m \\ & S_0 & \\ J_m & & \end{pmatrix}$ with

$$J_m := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}, \quad S_0 : \text{anisotropic part } (\mathrm{rank}(S_0) = n_0 \leq 4).$$

- $O(Q_m)$: the orthogonal group defined by $Q_m$ over $F$.
- $L_m := \mathfrak{o}^{2m+n_0}$: assumed to be a maximal lattice w.r.t. $Q_m$.
- $G_m := O(Q_m)(F) \supset K_m := \{k \in G_m \mid kL_m = L_m\}$.

We now review the non-archimedean local "Whittaker functions" (nowadays known as "Special Bessel model") studied by Sugano [11, Section 7]. To this end we need further notation. Let $G_m \supset P = LN$ be the standard maximal parabolic subgroup with the Levi subgroup $L$ of split rank one and the abelian unipotent radical $N$. For a "reduced character" $\chi$ of $N$ put $H_\chi := \mathrm{Stab}_{O(2(m-1))}(\chi) \subset L$, where we say that $\chi$ is reduced if it comes from a reduced element in $\mathbb{Q}_p^{2m-2}$. For the precise definition see [11, Section 7, p44, p47]. We introduce

$$\mathcal{W}_\chi := \left\{ W \in C^\infty(O(2m) \;\middle|\; \begin{array}{c} W(hngk) = \chi(n)^{-1}W(g) \\ \forall (h,n,k) \in H_\chi \times N \times K_{2m,p} \end{array} \right\},$$
$$\mathcal{W}_\chi^{\mathcal{M}} := \{W \in \mathcal{W}_\chi \mid W \text{ satisfies the local Maass relation.}\}.$$

Here, for the definition of the local Maass relation, see [11, (7.48), (7.49)].

What is crucial for the Hecke theory of our lifting is the Hecke module structure for $\mathcal{W}_\chi^{\mathcal{M}}$. We now introduce the Hecke operators $\{C_m^{(i)}\}_{1\le i\le m}$ which form a generator of the Hecke algebra $\mathcal{H}_m$ for $(G_m, K_m)$:

- $\{C_m^{(i)} := \mathrm{char}(K_m c_m^{(i)} K_m)\}_{1\le i\le m}$, where

$$c_m^{(i)} := \mathrm{diag}(\underbrace{p,\cdots,p}_{\mathrm{i}},1,\cdots,1,\underbrace{p^{-1},\cdots,p^{-1}}_{\mathrm{i}}).$$

To describe the Hecke module structure of $\mathcal{W}_\chi^{\mathcal{M}}$ explicitly we need the following:

- $q := \sharp k_F$ ($k_F := \mathfrak{o}/\mathfrak{p}$:residue field of $F$)
- $\partial := \dim_{k_F}(L'_{m-1}/L_{m-1})$, $L'_{m-1} := \{\lambda \in \hat{L_{m-1}} \mid \frac{1}{2}{}^t\lambda S \lambda \in \mathfrak{p}^{-1}\}$ ($\hat{L_{m-1}}$: dual lattice).
-

$$R_m^{(i)} := \sharp(K_m/c_m^{(i)}K_m(c_m^{(i)})^{-1} \cap K_m) = \prod_{i=1}^{m} f_{m,i},$$
$$f_{m,i} := \frac{q^{i-1}(q^{m-i+1}-1)(q^{m-i+n_0}+q^\partial)}{q^i-1}.$$

**Proposition 3.1 (Sugano)** *(1) The Whittaker spaces $\mathcal{W}_\lambda$ and $\mathcal{W}_\lambda^{\mathcal{M}}$ are $\mathcal{H}_m$-stable.*
*(2) On $\mathcal{W}_\lambda$ the Hecke operators $C_m^{(i)}$ for $i \ge 3$ acts by*

$$C_m^{(i)} = R_{m-2}^{(i-2)}(C_m^{(2)} - \frac{q^{i-2}-1}{q^{i-1}-1}\cdot f_{m-2,1}\cdot C_m^{(1)} + \frac{q^{i-2}-1}{q(q^i-1)} f_{m-2,1} f_{m+2,2}).$$

*(3) On $\mathcal{W}_\lambda^{\mathcal{M}}$ the Hecke operator $C_m^{(i)}$s satisfies*

$$\begin{aligned} C_m^{(2)} &= f_{m-2,1}C_m^{(1)} + q^4 f_{m-4,1}f_{m-4,2} + q^3 f_{m-4,1}^2 \\ &\quad - q^2(q^{m-4}-(q-2)q^\partial)f_{m-4,1} + (q-1)q^\partial f_{m-2,1} - q(q^{m-4}+q^\partial)^2. \end{aligned}$$

Though the Hecke module structure $\mathcal{W}_\lambda^{\mathcal{M}}$ looks complicated as we have seen just above it turns out to be quite simple as follows:

**Proposition 3.2** *On $\mathcal{W}_\lambda^{\mathcal{M}}$,*

$$C_m^{(i)} = R_{m-1}^{(i-1)}(C_m^{(1)} - \frac{q^{i-1}-1}{q^i-1} f_{m,1}) \quad (i \geq 2).$$

We apply the above results to the situation as follows:

$$F = \mathbb{Q}_p, \quad m = 4n+1, \quad n_0 = \partial = 0,$$

for which note that $O(Q)$ is isomorphic to $O(8n+2)$ over $\mathbb{Q}_p$. It can be shown that the adelic Fourier coefficient $A_\lambda$ can be viewed as a local Whittaker functions on $O(8n+2)(\mathbb{Q}_p)$ for each prime $p$. With the notation above we thus have the results on the Hecke theory stated as follows:

**Proposition 3.3** *(1) If $f$ is a Hecke eigenform, $F_f$ is also a Hecke eigenfunction.*
*(2) Let $\lambda_p$ be the Hecke eigenvalue of $f$ at $p < \infty$. Then $C_{4n+1}^{(i)} \cdot F_f = \mu_i F_f$, where*

$$\mu_i = \begin{cases} p^{4n}(\lambda_p^2 + p^{4n-1} + \cdots + p + p^{-1} + \cdots + p^{-(4n-1)}) & (i=1), \\ R_{4n}^{(i-1)}(\mu_1 - \frac{p^{i-1}-1}{p^i-1} f_{4n+1,1}) & (i > 1). \end{cases}$$

As a corollary to this proposition we have the following consequence on the cuspidal representation $\pi_{F_f}$.

**Corollary 3.4** *Suppose that $f$ is a Hecke eigenform.*
*(1) The cuspidal representation $\pi_{F_f}$ is irreducible and decomposes into $\pi_{F_f} \simeq \otimes'_{v \leq \infty} \pi_v$.*
*(2) With $\beta_p := \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}$, the Satake parameter of $F_f$ at $p < \infty$ is*

$$\mathrm{diag}((\beta_p^2, p^{4n-1}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-(4n-1)}, \beta_p^{-2}),$$

*which means the explicit determination of $\pi_p$ of $\pi_{F_f}$.*
*(3) The local components $\pi_p$ is non-tempered at every $v = p < \infty$ while $\pi_\infty$ is tempered at $v = \infty$.*

To show the irreducibility of $\pi_F$ we prove that $\pi_\infty$ is given explicitly as an irreducible spherical principal series representation, and can then apply [6, Theorem 3.1] to $\pi_{F_f}$ in order to show its irreducibility. The second assertion is a consequence from Proposition 3.3. For the third assertion we remark that the Satake parameter at $v = p < \infty$ include that for the trivial representation of $G_{4n} = O(8n)(\mathbb{Q}_p)$. We can deduce the non-temperedness of $\pi_p$ from this. The temperedness of $\pi_\infty$ is verified by the explicit description of $\pi_\infty$ mentioned above.

## 4 Concluding remarks

1. Sugano's non-archimedean local theory [11] is originally motivated by studying the non-archimedean local aspect of Oda-Rallis-Schiffimann lifting [7], [9]. One therefore naturally expects that the results similar to our two theorems hold also for this lifting. In the appendix of [3] we have included such results on the Oda-Rallis-Schiffmann lifting for the

orthogonal group $O(2, 8n+2)$ defined by $\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -S & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$, where $S$ denotes an even unimodular matrix.

2. Our non-archimedean local theory needs the theory of unramified principal series representations of $p$-adic reductive groups. For this we should note that many relevant references assume the connected-ness of the reductive groups. For the orthogonal groups in our setting, we have justified the useful-ness of the known theory on the unramified principal series based on [10] and [2] though the orthogonal group is not connected.

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