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# Paramodularity

Cris Poor

**Abstract**

We survey results motivated by the Paramodular conjecture of Brumer and Kramer. These include systematic computations of spaces of weight two paramodular cusp forms, constructions of nonlift newforms using Borcherds products, proofs that specific examples of abelian surfaces are paramodular, and counterexamples due to Calegari.

## Contents



## Introduction

The Paramodular conjecture asserts the modularity of abelian surfaces defined over $\mathbb{Q}$ with minimal endomorphisms and tells us exactly where to look for the corresponding nonlift paramodular newform. When the endomorphism ring is not minimal, modularity is already known. There is some evidence for the conjecture. In [20] spaces of paramodular cusp forms were studied for prime levels $p < 600$ and it was shown that weight two nonlifts could exist only for $p \in \{277, 349, 353, 389, 461, 523, 587\}$. Although abelian surfaces are known for each of these conductors, the existence of a nonlift was shown in [20] only for the level 277; the existence of the nonlift in all cases was completed in [12, 18] using the theory of Borcherds products.

In each of these cases the first few Euler factors of the abelian surface and the paramodular form match. For the levels $277, 353, 587^-$, enough eigenvalues have been computed in [6] to actually prove the equality of $L$-functions. Additionally, using the theory of Galois deformations, Berger and Klosin have proved modularity for an abelian surface of conductor 731 in [2], the first resut for a composite level. See the website siegelmodularforms.org for a heuristic table of paramodular newforms of level $N < 1000$, and whether a potential match has been found yet.

Infinite families exhibiting paramodularity have also been constructed. In [16] Johnson-Leung and Roberts constructed a theory of twisting paramodular forms consistent with the formation of $L$-functions, so that the modularity of one pair $(A, f)$ implies the modularity of all the twists $(A^{\chi}, f^{\chi})$, at least when the conductor of $\chi$ is prime to the level of $f$. Furthermore, Johnson-Leung and Roberts have a theory of lifting [15] Hilbert modular forms to paramodular forms so that when a Hilbert modular form $h$ shows the modularity of an elliptic curve $E$ over a real quadratic field $K$, which is not isogenous to its conjugate, the lifted paramodular form $\hat{h}$ shows the modularity of the abelian surface given by the Weil restriction of $E/K$. For imaginary quadratic $K$, Berger, Dembélé, Pacetti, and Sengun have a similar theory [1] lifting Bianchi eigenforms to paramodular eigenforms, so that the modularity of the Weil restriction of $E/K$ is known whenever the modularity of $E/K$ is known. All in all, we can say that when an abelian surface $A$ is known and the conductor $N$ is within the range of present theory or computation, evidence for a weight two nonlift paramodular newform $f$ has been found.

The converse direction of the Paramodular conjecture has been more troublesome. First, there is evidence for paramodular newforms where no matching abelian surface, or any arithmetic object, has yet been found, level 550 for example. Second, the original Paramodular conjecture overlooked the circumstance that some nonlift, weight two, paramodular newforms correspond to abelian fourfolds with quaternionic multiplication, as pointed out by Calegari. We include Brumer and Kramer's revised version of the Paramodular conjecture in section three.

In the above cases of Weil restriction, the abelian surface acquires extra endomorphisms over a quadratic extension. Our focus in this article is on the *typical* case, when the abelian surface retains minimal endomorphisms over $\bar{\mathbb{Q}}$. Counted among these are the levels $277, 353, 587^{-}$, where modularity has been proven by the direct construction of the paramodular newforms and a generalization of the method of Faltings-Serre to GSp(4). Our main goal is to give the flavor of these constructions of paramodular newforms in terms of Gritsenko lifts and Borcherds products. Finally, we can say that for $N = 277$ we have a complete proof of *one instance* of the Paramodular conjecture. It is proven that there is a single isogeny class of abelian surafces defined over $\mathbb{Q}$ with mininal endomorphisms of conductor 277, see [3]. It is proven that there is a single line of nonlift newforms in $S_2(K(277))$ with rational eigenvalues, see [20]. And it is proven that the associated Galois representations and $L$-functions match, see [6].

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## 1 All elliptic curves defined over $\mathbb{Q}$ are modular.

The Modularity Theorem relating elliptic curves and elliptic modular forms is the model for questions of modularity

**1.0.1 Modularity Theorem.** (Wiles; Wiles & Taylor; Breuil, Conrad, Diamond & Taylor.) *Let $N \in \mathbb{N}$. There is a bijection between*

*i) isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor $N$, and*

*ii) normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

*In this correspondence we have* $L(E, s, \text{Hasse}) = L(f, s, \text{Hecke})$.

Eichler proved the first examples of this in 1954, see [7]. From his work we conclude, for example,

$$L(X_0(11), s, \text{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \text{Hecke}).$$

Looking ahead to the Paramodular conjecture, the case of abelian surfaces and paramodular forms, we are still trying to be like Eichler. We are trying to prove specific examples of what is expected to be a general phenomenon. And we will indeed succeed in giving a few beautiful formulae that generalize Eichler's equality of $L$-functions above.

Returning to the Modularity Theorem, Shimura gave a construction from ii) to i). Weil added $N = N$, that the analytic conductor and the arithmetic conductor should be equal. We should credit Taniyama (1956) and Shimura ($\sim$ 1963) for important modularity conjectures. In its final form, however, the Modularity Theorem is a *classification* theorem. Cremona has led the classification of $E/\mathbb{Q}$ up to conductor $N \leq 400\,000$, (johncremona.github.io/ecdtata).

### 1.1 $L$-functions of elliptic curves over $\mathbb{Q}$.

We mention both the definition of the Hasse $L$-function and the manner in which it is typically computed. The local $p$-Euler factor is the characteristic polynomial of Frobenius acting on the Tate module $\mathbb{T}_\ell(E)$ of the elliptic curve $E$,

$$Q_p(E, t) = \det\left(I - t\,\mathrm{Frob}_p\,|\mathbb{T}_\ell(E)^{I_p}\right).$$

The local Zeta function can be computed by counting points on the elliptic curve $E$ over finite fields,

$$Z_p(E, t) = \exp\left(\sum_{n=1}^{\infty} \#\{\text{Points on } E/\mathbb{F}_{p^n}\}\frac{t^n}{n}\right) = \frac{Q_p(E, t)}{(1-t)(1-pt)}.$$

The global $L$-function is defined by an Euler product that converges for $\Re(s) > 3/2$,

$$L(E, s, \text{Hasse}) = \prod_{\text{primes } p} Q_p(E, p^{-s})^{-1}.$$

Here is an example of finding an Euler 2-factor. Take the elliptic curve over $\mathbb{Q}$ of conductor 11, $y^2 + y = x^3 - x^2$, which has label 11.a3 on the lmfdb, and write the projectivized set of points over a field $\mathbb{F}$ of the elliptic curve $E$ as

$$E[\mathbb{F}] = \{(x, y, z) \in \mathbb{P}^2(\mathbb{F}) : y^2z + yz^2 = x^3 - x^2z\}.$$

It is not too hard, even without a computer, to count the number of points over small finite fields.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\#E[\mathbb{F}_{2^n}]$ | 5 | 5 | 5 | 25 | 25 | 65 | 145 |

The Zeta function at $p = 2$ can be computed from the first two data points, but the consistency of all the data illustrates the rationality of the Zeta function,

$$Z_2(E,t) = \exp\left(5t + 5\frac{t^2}{2} + 5\frac{t^3}{3} + 25\frac{t^4}{4} + \cdots\right) = \frac{1+2t+2t^2}{(1-t)(1-2t)}.$$

From this we obtain the Euler 2-factor $Q_2(E,t) = 1 + 2t + 2t^2$. Continuing in this way, we can get the beginning of the global $L$-function of $E$,

$$L(E, s, \text{Hasse}) = \prod_{\text{primes } p} Q_p(E, p^{-s})^{-1} = 1 - \frac{2}{2^s} - \frac{1}{3^s} + \frac{2}{4^s} + \frac{1}{5^s} + \cdots.$$

### 1.2 Modular newforms.

On the automorphic side of the above example, the space $S_2(\Gamma_0(11))$ is one dimensional and $S_2(\Gamma_0(1))$ is trivial, and so there is a normalized elliptic modular newform $f_{11} \in S_2(\Gamma_0(11))$ that is necessarily a Hecke eigenform with rational coefficients. The $q$-expansion can be looked up on the lmfdb, where it was computed by the method of modular symbols,

$$f_{11}(\tau) = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots,$$

and we see that the Fourier coefficients of $f_{11}$ match the Dirichlet coefficients of the Hasse $L$-function. Instead of holding the $L$-functions as the primary object, a more sophisticated point of view is to consider the equality of the Galois representations associated to the abelian variety and to the modular form, but here we favor a computational point of view and focus on $L$-functions.

One could also expand $f_{11}(\tau) = \eta(\tau)^2\eta(11\tau)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$, or use theta series to construct the newform. The theta series $\vartheta[Q]$ of an even $m$-by-$m$ quadratic form is defined by $\vartheta[Q](\tau) = \sum_{n\in\mathbb{Z}^m} e\left(\frac{1}{2}Q[n]\tau\right)$. If $\ell Q^{-1}$ is also even then $\vartheta[Q] \in M_{m/2}(\Gamma_0(\ell), \chi)$ for some character $\chi$. The character is trivial when $\det(Q)$ is a square and $4 \mid m$, see [9], page 203. In this case we have

$$f_{11}(\tau) = \frac{1}{2}\vartheta\begin{bmatrix} 2&1&1&1\\ 1&2&0&1\\ 1&0&8&4\\ 1&1&4&8 \end{bmatrix}(\tau) - \frac{1}{2}\vartheta\begin{bmatrix} 2&0&1&0\\ 0&2&0&1\\ 1&0&6&0\\ 0&1&0&6 \end{bmatrix}(\tau).$$

The variety of ways to construct modular forms has always been part of the charm of the subject.

## 2 All abelian surfaces over $\mathbb{Q}$ with minimal endomorphisms are paramodular.

We state the Paramodular conjecture as it was given by Brumer and Kramer in 2009. The direction that i) implies ii) is still believed but Calegari pointed out a counterexample to the converse in 2018. Brumer and Kramer have given an amended version in [4] and we will also reproduce it here in section 4. It is still interesting to give the original version, however, both to discuss related work and to discuss counterexamples.

**2.0.1 Paramodular Conjecture** 1.0 (Brumer and Kramer 2009.) *Let $N \in \mathbb{N}$. There is a bijection between*

*i) isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor $N$ and endomorphisms* $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$,

ii) *lines of Hecke eigenforms* $f \in S_2(K(N))^{\rm new}$ *that have rational eigenvalues and are not Gritsenko lifts from* $J_{2,N}^{\rm cusp}$.

*In this correspondence we have* $L(A, s, \text{Hasse-Weil}) = L(f, s, \text{spin})$.

The subtle condition in the Paramodular conjecture is that the ring of endomorphisms *defined over* $\mathbb{Q}$ should be minimal: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. The Paramodular conjecture addresses the essential case, because when $\mathrm{End}_{\mathbb{Q}}(A) > \mathbb{Z}$, modularity is known by the theory of GL(2)-type. It is natural to ask why we did not see the condition on the endomorphism ring in the elliptic case. The answer is that $\mathrm{End}_{\mathbb{Q}}(E) = \mathbb{Z}$ for every elliptic curve $E/\mathbb{Q}$.

Just as certain abelian surfaces were excluded, we also exclude certain paramodular forms, the Gritsenko lifts. Gritsenko has an injective map [11] from the space of Jacobi cusp forms to the space of paramodular forms, $\mathrm{Grit} : J_{k,N}^{\rm cusp} \to S_k(K(N))$, and the $L$-functions of Gritsenko lifts, which are built up from GL(2) eigenforms, have poles and do not respect the Weil bounds. Furthermore, we insist that the paramodular forms be new. Notice that it is not even possible to state the Paramodular conjecture without the global theory of paramodular newforms due to Roberts and Schmidt in [21, 22]. Thus we exclude oldforms, which come from discrete subgroups of lower level, and Gritsenko lifts, which come from a group of lower rank.

We can learn something by comparing the Paramodular conjecture to Yoshida's earlier work. In 1980, see [25], Yoshida conjectured that *For every abelian surface* $A/\mathbb{Q}$ *there exists a Siegel modular form* $\mathcal{F}$ *of weight* 2 *of a suitable level such that* $L(s, A) = L(s, \mathcal{F})$, and gave examples with $\mathrm{End}_{\mathbb{Q}}(A) \neq \mathbb{Z}$. Three example that Yoshida gave were $J_0(p) = \mathrm{Jac}(X_0(p))$ with conductor $p^2$ and $\mathrm{End}_{\mathbb{Q}}(J_0(p)) = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, for $p = 23, 29, 31$, respectively. Yoshida proved that $L(J_0(p), s, \text{H-W}) = L(Y_p, s, \text{spin})$ where $Y_p \in S_2\left(\Gamma_0^{(2)}(p)\right)$ is the Yoshida lift of a newform $f_p \in S_2\left(\Gamma_0^{(1)}(p)\right)$ and its conjugate. It is instructive to trace, $\mathrm{Tr} : S_2\left(\Gamma_0^{(2)}(p)\right) \to S_2\left(K(p^2)\right)$, because a nonzero image would contradict the Paramodular conjecture. The point is that the endomorphisms of the associated abelian surface would be too big. Due to a lemma of Ralf Schmidt, however, a nonzero trace cannot occur in this situation because $S_k\left(K(N)\right)_{(\text{Yosh})} = \{0\}$, eigenforms of Yoshida type do not have paramodular fixed vectors, see [24].

### 2.1 Abelian varieties.

Let $K \subseteq \mathbb{C}$ be an algebraic number field.

**2.1.1 Definition.** *An abelian variety $A/K$ is a projective variety defined over $K$ with an algebraic group law also defined over $K$.*

In particular, the identity element of the group law is defined over $K$, so that $A/K$ always has at least one $K$-rational point. The group structure of an abelian variety is more visible in the holomorphic category. If we consider the complex points of $A$ to form a complex manifold, $A_{\rm hol}$, then $A_{\rm hol}$ is biholomorphic to a complex torus. When the torus has complex dimension one we are in the case of elliptic curves and $A_{\rm hol} \cong \mathbb{C}^1/\left(\mathbb{Z} + \tau\mathbb{Z}\right)$ for $\tau \in \mathcal{H}$. When the torus has complex dimension two we are in the case of abelian surfaces and

$$A_{\rm hol} \cong \mathbb{C}^2/\left(D\mathbb{Z}^2 + Z\mathbb{Z}^2\right) \text{ for } Z \in \mathcal{H}_2,$$

where $D = \operatorname{diag}(1, d)$ for $d \in \mathbb{N}$ gives the type of "polarization" of $A$. Here we use the *Siegel upper half space*, $\mathcal{H}_2 = \{Z \in M_{2\times 2}^{\mathrm{sym}}(\mathbb{C}) : \operatorname{Im} Z > 0\}$, which is also fundamental in the subject of Siegel modular forms. This modern definition of an abelian variety, however, hides the fact that, for $g > 1$, the equations defining $A$ inside a projective space are *complicated.* So for $g > 1$, it requires skill to produce examples of abelian varieties over $K$ at all.

### 2.2 Constructing examples of abelian surfaces.

In this section we content ourselves with constructing *some* abelian surfaces defined over $\mathbb{Q}$. If $\mathcal{C}$ is a curve of genus 2 defined over $\mathbb{Q}$, then

$$A = \operatorname{Jac}(\mathcal{C})$$

is an abelian surface defined over $\mathbb{Q}$ with a *principal* polarization, i.e., type $D = \operatorname{diag}(1, 1)$. Many examples of Jacobians can be found on the lmfdb. If a genus 3 curve $\mathcal{C}_3$ is a ramified degree $d$ cover of a genus 1 curve $\mathcal{C}_1$, simply branched at four points, then the abelian surface

$$A = \operatorname{Prym}(\mathcal{C}_3/\mathcal{C}_1) = \operatorname{Jac}(\mathcal{C}_3)/\operatorname{Jac}(\mathcal{C}_1)$$

has a natural polarization of type $D = \operatorname{diag}(1, d)$. These constructions, however, are by no means exhaustive and when the isogeny class of an $A/\mathbb{Q}$ does not contain a representative given by a common construction, a representative may be hard to find.

### 2.3 $L$-functions of abelian surfaces over $\mathbb{Q}$.

The local $p$-Euler factor is the characteristic polynomial of Frobenius acting on the Tate module $\mathbb{T}_\ell(A)$ of the abelian surface $A$,

$$Q_p(A, t) = \det\left(I - t \operatorname{Frob}_p |\mathbb{T}_\ell(A)^{I_p}\right).$$

The global Hasse-Weil $L$-function is defined by an Euler product and converges in the half-plane $\Re(s) > 3/2$,

$$L(A, s, \text{H-W}) = \prod_{\text{primes } p} Q_p(A, p^{-s})^{-1}.$$

In the special case when $A = \operatorname{Jac}(C)$ is the Jacobian of a genus two curve $C$, we have

$$L(A, s, \text{H-W}) = L(C, s, \text{H-W}).$$

The local Hasse-Weil $p$-Euler factors $Q_p(C, t)$ for $C$ are accessible by counting points

$$Z_p(C, t) = \exp\left(\sum_{n=1}^{\infty} \#\{\text{Points on } C/\mathbb{F}_{p^n}\}\frac{t^n}{n}\right) = \frac{Q_p(C, t)}{(1-t)(1-pt)}.$$

If an abelian surface has been given as a Prym, $A = \operatorname{Prym}(\mathcal{C}_3/\mathcal{C}_1)$, then we have $L(A, s, \text{H-W}) = L(\mathcal{C}_3, s, \text{H-W})/L(\mathcal{C}_1, s, \text{H-W})$, so this case is amenable to computation as well.

Here is one example. Consider the hyperelliptic curve $C_{277}$ of genus two given by the equation $y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x$ and its Jacobian $A_{277} = \operatorname{Jac}(C_{277})$ of conductor 277. The label on the lmfdb is 277.a.277.1. Magma will compute Hasse-Weil Euler factors of a curve, the general form of the input is $y^2 + G(x)y = F(x)$.

```
> G:=x^3 + x^2 + x + 1; F:=-x^2 - x;
> C:=HyperellipticCurve(F,G);
> J:=Jacobian(C);
> h2:=EulerFactor(J,GF(2)); h2;
4*x^4 + 4*x^3 + 4*x^2 + 2*x + 1
> h3:=EulerFactor(J,GF(3)); h3;
9*x^4 + 3*x^3 + x^2 + x + 1
> h5:=EulerFactor(J,GF(5)); h5;
25*x^4 + 5*x^3 - 2*x^2 + x + 1  .
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### 2.4 Paramodular forms.

We introduce Siegel modular forms, which are a natural generalization of elliptic modular forms. We begin with the symmetric space, the *Siegel upper half space,*

$$\mathcal{H}_n = \{Z \in M_{n\times n}^{\mathrm{sym}}(\mathbb{C}) : \operatorname{Im} Z > 0\}.$$

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acts transitively on the Siegel upper half space via linear fractional transformations:

$$\sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}_n(\mathbb{R}) \text{ acts on } Z \in \mathcal{H}_n \text{ by } \sigma \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

We let the action of the group transform complex functions on the Siegel upper half space according to powers of the Siegel factor of automorphy $\mu(\sigma, Z) = \det(CZ+D)$:

$$\text{For } f : \mathcal{H}_n \to \mathbb{C} \text{ and } \sigma \in \mathrm{Sp}_n(\mathbb{R}), \ \text{ let } (f|_k\sigma)(Z) = \det(CZ+D)^{-k} f(\sigma \cdot Z).$$

We select a discrete subgroup $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in $\Gamma$ and $\mathrm{Sp}_n(\mathbb{Z})$. We fix an integer $k$ and define the $\mathbb{C}$-vector space $M_k(\Gamma)$ of Siegel modular forms of *weight* $k$, automorphic with respect to $\Gamma$: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$. We define the subspace of cusp forms: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$. More precisely, the condition that $f$ be "bounded at the cusps" is that

$$\forall \sigma \in \mathrm{Sp}_n(\mathbb{Q}),\ \forall\, Y_o > 0,\ \ (f|_k\sigma) \text{ is bounded on } \{Z \in \mathcal{H}_n : \operatorname{Im} Z > Y_o\}.$$

For $n \ge 2$, this boundedness condition is redundant by the Koecher principle, but the boundedness condition is still a natural part of the definition and is necessary when $n = 1$. In any case, we need to mention the cusps to define what is meant by $f$ "vanishing at the cusps." Introduce the Siegel map $\Phi : M_k(\Gamma) \to \mathcal{O}(\mathcal{H}_{n-1})$ given by

$$(\Phi f)(Z_{n-1}) = \lim_{\lambda \to +\infty} f\left[\begin{smallmatrix} Z_{n-1} & 0 \\ 0 & i\lambda \end{smallmatrix}\right].$$

The condition that $f$ "vanish at the cusps" is that

$$\forall \sigma \in \mathrm{Sp}_n(\mathbb{Q}),\ \Phi(f|_k\sigma) = 0.$$

We now specialize to degree $n = 2$ and give the definition of Siegel paramodular forms. A *paramodular form* is a Siegel modular form for a paramodular group. In degree 2, the paramodular group of level $N$, is

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z}.$$

This definition of $K(N)$ in terms of divisibilities is nice for the computer. An intrinsic definition is that $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of the lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$, where the elements of the lattice are written as column vectors. Siegel, Christian, and Igusa studied paramodular groups because of their connection to moduli spaces. If we let ${}^T K(N)$ denote the transposed group, then ${}^T K(N) \backslash \mathcal{H}_2$ is naturally a moduli space for complex abelian surfaces with polarization type $(1, N)$. This property, however, is not what connects the paramodular groups to questions of modularity. The connection to modularity comes from the realization of $K(N)$ as the integral stable special orthogonal group (of spinor norm one) for the quadratic form $Q = \mathrm{antidiag}(1, 1, -2N, 1, 1)$, just as $\Gamma_0(N)$ corresponds to $Q = \mathrm{antidiag}(1, -2N, 1)$. The Tate module $\mathbb{T}(A)$ of an abelian variety gives rise to a symplectic Galois representation. According to the Langlands' program, the modularity of this Galois representation should be shown by an automorphic form on the dual group, and the dual groups of symplectic groups are orthogonal groups. In the general case of an abelian variety of degree $g$, an automorphic form on the split orthogonal group $O(g, g+1)$ will not be holomorphic for $g > 2$. So we should count ourselves lucky that for modularity in $g = 1$ and $g = 2$ we still get to work with holomorphic functions.

As in the elliptic case for $\Gamma_0(N)$, there is a Fricke involution $\mu_N$ that splits paramodular forms into plus and minus spaces.

$$\mu_N = \begin{bmatrix} F_N^* & 0 \\ 0 & F_N \end{bmatrix}, \text{ where } F_N = \tfrac{1}{\sqrt{N}} \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix};$$
$$S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^- .$$

We have described all the objects in the Paramodular conjecture except the Gritsenko lift. We do this in terms of the Fourier-Jacobi expansion.

### 2.5 Fourier-Jacobi expansions.

Every paramodular form $f \in M_k(K(N))$ has a Fourier expansion:

$$f(Z) = \sum_{t \geq 0} a(t; f) e(\langle Z, t \rangle),$$

where the sum is over $t \in \mathcal{X}_2^{\mathrm{semi}}(N) = \{\begin{bmatrix} n & r/2 \\ r/2 & Nm \end{bmatrix} \geq 0 : n, r, m \in \mathbb{Z}\}$, and where $\langle Z, t \rangle = \mathrm{tr}(Zt)$.

The Fourier expansion of a paramodular cusp form $f \in S_k(K(N))$ may be rearranged to give the Fourier-Jacobi expansion, setting $Z = \begin{bmatrix} \tau & z \\ z & \omega \end{bmatrix} \in \mathcal{H}_2$, and $q = e(\tau)$, $\zeta = e(z)$,

$$f(Z) = \sum_{j=1}^{\infty} \phi_j(\tau, z) e(Nj\omega), \tag{1}$$
$$\phi_j(\tau, z) = \sum_{n, r \in \mathbb{Z}:\ 4nNj > r^2} a\left(\begin{bmatrix} n & r/2 \\ r/2 & Nj \end{bmatrix}; f\right) q^n \zeta^r.$$

The Fourier-Jacobi expansion of a paramodular form is a more suggestive analogue to the elliptic case than the full Fourier expansion, even though the coefficients are now Jacobi forms. We recall the definition of a Jacobi form and the following subgroups, for rings $R \subseteq \mathbb{C}$,

$$P_{2,1}(R) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}(4,R); \quad GP_{2,1}(R) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{GSp}(4,R).$$

A *Jacobi form* $\phi \in J_{k,m}(\chi)$ of weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that the associated function $E_m\phi : \mathcal{H}_2 \to \mathbb{C}$ given by $(E_m\phi)(Z) = \phi(\tau,z)e(m\omega)$ transforms by a multiplier $\chi$ under $P_{2,1}(\mathbb{Z})$, and is bounded on domains of the type $\{Z \in \mathcal{H}_2 : \mathrm{Im}\, Z > Y_o\}$. The boundedness condition is essential and, given the other assumptions, is equivalent to a Fourier expansion for $\phi$ of the form $\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}:\ n\geq 0,\, 4nm \geq r^2} c(n,r;\phi)q^n\zeta^r$. For *Jacobi cusp forms* $\phi \in J_{k,m}^{\mathrm{cusp}}(\chi)$, we require $4mn > r^2$. For a *weakly holomorphic* $\psi \in J_{k,m}^{\mathrm{wh}}$ we drop the boundedness condition and require that $n$ be bounded from below.

The subgroup $K_\infty(N) = P_{2,1}(\mathbb{Q}) \cap K(N)$ stabilizes the Fourier-Jacobi expansion (1) term by term, so that each $\phi_j \in J_{k,Nj}^{\mathrm{cusp}}$ is a Jacobi form with trivial character and the Fourier coefficients of the $\phi_j$ are

$$c\,(n,r;\phi_j) = a\left(\begin{bmatrix} n & r/2 \\ r/2 & Nj \end{bmatrix}; f\right).$$

Following [8], we define the raising operator $V_\ell : J_{k,m} \to J_{k,m\ell}$ for $\ell \in \mathbb{N}$ by

$$(\phi|V_\ell)\,(\tau,z) = \sum_{\substack{a,d\in\mathbb{N}:\\ ad=\ell}} a^{k-1}\left(\frac{1}{d}\sum_{b \bmod d}\phi(\frac{a\tau+b}{d},az)\right),$$

or equivalently by

$$c(n,r;\phi|V_\ell) = \sum_{a|\gcd(n,r,\ell)} a^{k-1}c\left(\frac{n\ell}{a^2},\frac{r}{a};\phi\right).$$

The invariance properties of the raising operator, i.e., that it sends Jacobi forms to Jacobi forms, can be obtained by considering it as the Hecke operator $V_\ell = K_\infty(N)\,\mathrm{diag}(\ell,\ell,1,1)K_\infty(N)$ for the noncommutative Jacobi Hecke algebra for $K_\infty(N)$ inside $GP_{2,1}(\mathbb{Q})$, see [11].

Any Jacobi cusp form can be the leading Fourier-Jacobi coefficient of a paramodular form.

**2.5.1 Theorem.** (Gritsenko) *For $\phi \in J_{k,m}^{\mathrm{cusp}}$ the series* $\mathrm{Grit}(\phi)$ *converges and defines a map*

$$\mathrm{Grit} : J_{k,m}^{\mathrm{cusp}} \to S_k\,(K(m))^\epsilon, \quad \epsilon = (-1)^k,$$
$$\mathrm{Grit}(\phi)[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}] = \sum_{\ell\in\mathbb{N}}(\phi|V_\ell)(\tau,z)e(\ell m\omega).$$

## 2.6 Methods for constructing paramodular newforms.

We now turn to the task of actually constructing examples of paramodular newforms. We have seen that we can compute the Euler factors of an abelian surface defined over $\mathbb{Q}$ of conductor 277, but how can we make a newform in $S_2(K(277))$ that has a chance of having the same Euler factors? The construction of modular forms has a long history and we review some possibilities.

Eisenstein series are a standard way to construct automorphic forms but, for squarefree level $N$, we have $M_2(K(N)) = S_2(K(N))$ and so there are no Eisenstein series. Furthermore, there are no dimension formulas in weight 2 and degree 2, so we cannot directly use a dimension formula to show the existence of a nonlift. The Gritsenko lift, Grit : $J_{2,N}^{\text{cusp}} \to S_2(K(N))^+$, does give one way into the space of cusp forms; we do not want the $L$-series of a Gritsenko lift for modularity purposes but at least we are in the space. If we multiply by weight two Gritsenko lifts, we can put general weight two paramodular forms in the weight four space, where the dimension formulae of Ibukiyama [13] allow rigorous calculations for prime levels. For squarefree levels, Ibukiyama and Kitayama [14] provide dimension formulae for $k \geq 3$. Theta series will give us modular forms in $M_2\left(\Gamma_0^{(2)}(N)\right)$ and we can trace them over to $M_2\left(K(N)\right)$ but, when $N$ is prime, this gives zero. There is, at the moment, no theory of modular symbols for paramodular forms. Borcherds products, when they exist, provide an important way to construct paramodular forms of low weight. In conclusion, we will build our examples out of Borcherds products and Gritsenko lifts.

**2.6.1 Theorem.** (Borcherds, Gritsenko, Nikulin) *Given $\psi \in J_{0,N}^{\text{wh}}(\mathbb{Z})$, a weakly holomorphic weight zero, index $N$ Jacobi form with integral coefficients,*

$$\psi(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, n\geq -N_o} c(n,r)\, q^n \zeta^r,$$

*there is a weight $k' \in \mathbb{Z}$, a character $\chi$, and a* meromorphic *paramodular form* Borch$(\psi)$ *in $M_{k'}^{\text{mero}}(K(N))(\chi)$, defined by*

$$\text{Borch}(\psi)(Z) = q^A \zeta^B \xi^C \prod_{n,m,r\in\mathbb{Z}} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)},$$

*in the sense that this product converges, as an infinite product, in a neighborhood of infinity and defines* Borch$(\psi)$ *on $\mathcal{H}_2$ by analytic continuation.*

See [17] for an algorithm that works well when $kN < 600$ and that, given world enough and time, would find all Borcherds products in $S_k(K(N))$.

## 2.7 Examples of paramodular newforms.

Both Borcherds products and Gritsenko lifts have Jacobi forms as their source data. An excellent source of Jacobi forms is given by the theory of theta blocks, which is due to Gritsenko, Skoruppa, and Zagier. Let $\epsilon$ be the muliplier of the Dedekind Eta function and view the Dedekind Eta function as a Jacobi cusp form of index zero and weight one-half,

$$\eta \in J_{1/2,0}^{\text{cusp}}(\epsilon); \qquad \eta(\tau) = q^{1/24} \prod_{n\in\mathbb{N}} (1 - q^n).$$

Let $\vartheta \in J_{1/2,1/2}^{\text{cusp}}(\epsilon^3 v_H)$ be the odd Jacobi theta function

$$\vartheta(\tau, z) = q^{1/8}\left(\zeta^{1/2} - \zeta^{-1/2}\right) \prod_{n\in\mathbb{N}} (1 - q^n)(1 - q^n\zeta)(1 - q^n\zeta^{-1}).$$

The simple theta blocks we will use are all of the following form

$$\mathrm{TB}_k[d_1, d_2, \ldots, d_\ell](\tau, z) = \eta(\tau)^{2k-\ell} \prod_{j=1}^{\ell} \vartheta(\tau, d_j z) \in J_{k,m}^{\mathrm{wh}}(\epsilon^{2k+2\ell}),$$

where $2m = d_1^2 + d_2^2 + \cdots + d_\ell^2$ and $d_i \in \mathbb{N}$.

**Example** $N = 277$**.** We have $\dim S_2(K(277)) = 11$, whereas the dimension of Gritsenko lifts in $S_2(K(277))$ is $\dim J_{2,277}^{\mathrm{cusp}} = 10$. Therefore, there is one nonlift eigenform in $S_2(K(277))$, necessarily a new form, because 277 is prime. Let $G_i = \mathrm{Grit}(\mathrm{TB}_2(\Sigma_i)) \in S_2(K(277))$ for $1 \le i \le 10$ be the lifts of the 10 theta blocks given by:

$\Sigma_i \in \{$ [2, 4, 4, 4, 5, 6, 8, 9, 10, 14], [2, 3, 4, 5, 5, 7, 7, 9, 10, 14], [2, 3, 4, 4, 5, 7, 8, 9, 11, 13], [2, 3, 3, 5, 6, 6, 8, 9, 11, 13], [2, 3, 3, 5, 5, 8, 8, 8, 11, 13], [2, 3, 3, 5, 5, 7, 8, 10, 10, 13], [2, 3, 3, 4, 5, 6, 7, 9, 10, 15], [2, 2, 4, 5, 6, 7, 7, 9, 11, 13], [2, 2, 4, 4, 6, 7, 8, 10, 11, 12], [ 2, 2, 3, 5, 6, 7, 9, 9, 11, 12] $\}$.

In [20] the rational function $f_{277} = Q/L$ in these Gritsenko lifts was proven to be *holomorphic* and an eigenform, where

$$\begin{aligned}
Q = {} & -14G_1^2 - 20G_8G_2 + 11G_9G_2 + 6G_2^2 - 30G_7G_{10} + 15G_9G_{10} + 15G_{10}G_1 \\
& - 30G_{10}G_2 - 30G_{10}G_3 + 5G_4G_5 + 6G_4G_6 + 17G_4G_7 - 3G_4G_8 - 5G_4G_9 \\
& - 5G_5G_6 + 20G_5G_7 - 5G_5G_8 - 10G_5G_9 - 3G_6^2 + 13G_6G_7 + 3G_6G_8 \\
& - 10G_6G_9 - 22G_7^2 + G_7G_8 + 15G_7G_9 + 6G_8^2 - 4G_8G_9 - 2G_9^2 + 20G_1G_2 \\
& - 28G_3G_2 + 23G_4G_2 + 7G_6G_2 - 31G_7G_2 + 15G_5G_2 + 45G_1G_3 - 10G_1G_5 \\
& - 2G_1G_4 - 13G_1G_6 - 7G_1G_8 + 39G_1G_7 - 16G_1G_9 - 34G_3^2 + 8G_3G_4 \\
& + 20G_3G_5 + 22G_3G_6 + 10G_3G_8 + 21G_3G_9 - 56G_3G_7 - 3G_4^2, \\
L = {} & -G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1.
\end{aligned}$$

In [6] the method of Faltings-Serre for proving the equality of Galois representations was generalized to GSp(4) and the modularity of $A_{277}$ was proven by demonstrating the equality of $L$-functions

$$L(A_{277}, s, \text{Hasse-Weil}) = L(f_{277}, s, \text{spin}).$$

This verification required calculating the $T(p)$-eigenvalues of $f_{277}$ up to $p \le 43$.

**Example** $N = 587^-$**.** In [12] a new eigenform $f_{587}^- \in S_2(K(587))^-$ was constructed by using a Borcherds product. Construct theta blocks $\phi \in J_{2,587}^{\mathrm{cusp}}$ and $\Xi \in J_{2,2\cdot 587}^{\mathrm{cusp}}$:

$$\begin{aligned}
\phi &= \mathrm{TB}_2[1,1,2,2,2,3,3,4,4,5,5,6,6,7,8,8,9,10,11,12,13,14], \\
\Xi &= \mathrm{TB}_2[1,10,2,2,18,3,3,4,4,15,5,6,6,7,8,16,9,10,22,12,13,14].
\end{aligned}$$

Construct a weakly holomorphic Jacobi form $\psi = (\phi\,|V_2 - \Xi)\,/\phi \in J_{0,587}^{\mathrm{wh}}(\mathbb{Z})$ and the corresponding Borcherds product $f_{587}^- = \mathrm{Borch}(\psi) \in S_2(K(587))^-$ by

$$\psi(\tau, z) = \sum_{n,r} c(n, r; \psi) q^n \zeta^r = 4 + \frac{1}{q} + \zeta^{-14} + \cdots + q^{134}\zeta^{561} + \cdots,$$

$$f_{587}^{-}\left[\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right] = q^2\zeta^{68}\xi^{587}\prod_{(n,m,r)\geq 0}\left(1-q^n\zeta^r\xi^{Nm}\right)^{c(nm,r;\psi)}.$$

In [6] the Faltings-Serre method for GSp(4) was also successful in proving

$$L(A_{587}^{-}, s, \text{Hasse-Weil}) = L(f_{587}^{-}, s, \text{spin}).$$

The $T(p)$ eigenvalues of $f_{587}^{-}$ were checked up to $p \leq 47$ with the Euler factor of $A_{587}^{-} = \text{Jac}(C)$, with $C$ given by $y^2 + (x^3+x+1)y = -x^2 - x$.

**Example** $N = 249$. The smallest known conductor of an abelian surface defined over $\mathbb{Q}$ with minimal endomorphisms is $N = 249$. It is proven [5] that this is the smallest possible *odd* conductor. Modularity has not been proven for this example but here is the candidate paramodular form $f_{249} = \text{Borch}(\psi_{249}) \in S_2(K(249))$, see [19].

$$\psi_{249}(\tau, z) = \frac{\vartheta(\tau,8z)}{\vartheta(\tau,z)}\frac{\vartheta(\tau,18z)}{\vartheta(\tau,6z)}\frac{\vartheta(\tau,14z)}{\vartheta(\tau,7z)} \in J_{0,249}^{\text{w.h.}}(\mathbb{Z});$$

$$\begin{aligned} f_{249}\left[\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right] &= 14\,q^2\zeta^{63}\xi^{498}\prod_{\substack{n,m,r\in\mathbb{Z}:\ m\geq 0\\ \text{if } m=0 \text{ then } n\geq 0\\ \text{if } m=n=0 \text{ then } r<0}}(1-q^n\zeta^r\xi^{mN})^{c(nm,r;\,\psi_{249})}\\ &- 6\ \text{Grit}(\text{TB}_2(2,3,3,4,5,6,7,9,10,13)) - 3\ \text{Grit}(\text{TB}_2(2,2,3,5,5,6,7,9,11,12))\\ &+ 3\ \text{Grit}(\text{TB}_2(1,3,3,5,6,6,6,9,11,12)) + 2\ \text{Grit}(\text{TB}_2(1,1,2,3,4,5,6,9,10,15))\\ &+ 7\ \text{Grit}(\text{TB}_2(1,2,3,3,4,5,6,9,11,14)).\end{aligned}$$

An abelian surface of conductor 249 is given by the Jacobian of the hyperelliptic curve given by $y^2 = x^6 + 4x^5 + 4x^4 + 2x^3 + 1$, see [5].

## 3 A counterexample and the modified Paramodular conjecture.

Frank Calegari has pointed out counterexamples to the Paramodular conjecture. We begin with some perspective on classification results. For isogeny classes of $E/\mathbb{Q}$, the Modularity Theorem gives a bijection with normalized $\mathbb{Q}$-newforms in $S_2(\Gamma_0(N))$. What is the situation for elliptic curves defined over quadratic extensions?

For real quadratic $K$, the bijection with isogeny classes of normalized Hilbert $\mathbb{Q}$-newforms in $S_2(\text{SL}(\mathcal{O}_K \oplus \mathfrak{a}))$ is believed but is not quite complete. The modularity of elliptic curves over real quadratic fields is proven, but the association of an $E/K$ to each appropriate Hilbert form is not finished, see the discussion in [10] for a reference. For imaginary quadratic $K$, it is believed that each $E/K$ has its modularity shown by some Bianchi $\mathbb{Q}$-newform in $S_2(\Gamma_0(\mathfrak{n}))$; however, there is a problem in going from Bianchi newforms to $E/K$.

Ciaran Schembri is my source for the following example, see [23]. Define a hyperelliptic curve $C_o/\mathbb{Q}(i)$ of genus two by $y^2 = x^6 + 4ix^5 - (6+2i)x^4 + (7-i)x^3 - (9-8i)x^2 - 10ix + (3+4i)$. Then $A_o = \text{Jac}(C_o)$ is an abelian surface over $\mathbb{Q}(i)$ of conductor $\mathfrak{p}_{5,1}^4\mathfrak{p}_{37,2}^4$ of norm $34225^2 = 185^4$. We are in the case of quaternionic multiplication because we have $\mathcal{O}_6 \hookrightarrow \text{End}_{\mathbb{Q}(i)}(A_o)$, where $\mathcal{O}_6$ is the maximal order of the rational quaternion algebra of discriminant 6. There is a Bianchi newform $f_o \in S_2(\Gamma_0(\mathfrak{p}_{5,1}^4\mathfrak{p}_{37,2}^4))$ with $\mathbb{Q}$-rational eigenvalues such that $L(A_o, s, \text{Hasse-Weil}) = L(f_o, s)^2$.

We now show that there can be no $E/\mathbb{Q}(i)$ with $L(E, s, \text{Hasse}) = L(f_o, s)$. By a theorem of Faltings, the Hasse-Weil $L$-function determines the isogeny class of an abelian variety. Note further that the ring of endomorphisms tensored with $\mathbb{Q}$ is an invariant of the isogeny class. Thus, if there were an $E/\mathbb{Q}(i)$ with $L(E, s, \text{Hasse}) = L(f_o, s)$, then $A_o$ and $E \oplus E$ would have the same $L$-function and hence would be in the same isogney class. This is impossible since $\text{End}_{\mathbb{Q}}(A_o) \otimes \mathbb{Q}$ and $\text{End}_{\mathbb{Q}}(E \oplus E) \otimes \mathbb{Q}$ differ. Thus, the pairing between $E/\mathbb{Q}(i)$ and Bianchi newforms is not perfect. This same mismatch can be transported by lifting and Weil restriction to the paramodular case.

By Weil restriction, $B = \text{WR}(A_o/\mathbb{Q}(i))$ is an abelian fourfold defined over $\mathbb{Q}$ with $\text{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q}$ an indefinite quaternion algebra. The lift of Berger, Dembélé, Pacetti, and Sengun gives $f = \text{lift}(f_o) \in S_2(K(N))$, see [1]. Therefore, we have $L(B, s, \text{H-W}) = L(f, s, \text{spin})^2$ and there can be no abelian surface $A/\mathbb{Q}$ with $L(A, s, \text{H-W}) = L(f, s, \text{spin})$ due to the different endomorphism rings of $\text{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q}$ and $\text{End}_{\mathbb{Q}}(A \oplus A) \otimes \mathbb{Q}$, using the same argument as above. Note that the conductor $N = (16 \cdot 185)^2 = 8\,761\,600$ is quite a bit higher than any computations done so far.

In light of this counterexample, Brumer and Kramer have modified the conjecture in the following way, see [4]. An abelian fourfold $B/\mathbb{Q}$ has *quaternionic multiplication* (QM) if $\text{End}_{\mathbb{Q}}(B)$ is an order in a non-split quaternion algebra over $\mathbb{Q}$. A cuspidal, nonlift Siegel paramodular newform $f \in S_2(K(N))$ with rational Hecke eigenvalues will be called a *suitable* paramodular form of level $N$.

**3.0.1 Paramodular Conjecture** 2.0 (Brumer–Kramer.) *Let $N \in \mathbb{N}$. Let $\mathcal{A}_N$ be the set of isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor $N$ with $\text{End}_{\mathbb{Q}} A = \mathbb{Z}$. Let $\mathcal{B}_N$ be the set of isogeny classes of QM abelian fourfolds $B/\mathbb{Q}$ of conductor $N^2$. Let $\mathcal{P}_N$ be the set of suitable paramodular forms of level $N$, up to nonzero scaling. There is a bijection $\mathcal{A}_N \cup \mathcal{B}_N \leftrightarrow \mathcal{P}_N$ such that*

$$L(X, s, \text{Hasse-Weil}) = \begin{cases} L(f, s, \text{spin}), & \text{if } X \in \mathcal{A}_N, \\ L(f, s, \text{spin})^2, & \text{if } X \in \mathcal{B}_N. \end{cases}$$

We should note that Brumer and Kramer have shown in [4] that QM implies $N = M^2 s$ with $s \mid \gcd(30, M)$. Thus the original conjecture is unaltered for squarefree $N$ and, in particular, for the examples discussed here.

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