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# Some remarks on Hayato Chiba's theory about Kuramoto conjecture

By

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**Abstract**

In §1 we review the celebrated result of H. Chiba about Kuramoto conjecture, and give some remarks on his proof. In §2, after introducing some notation and the related results due to Chiba, we explain our remarks more explicitly, for example, we give a rigorous analysis of the resonance poles of the resolvent of some important unbounded operator, which is not stated in his paper, but theoretically important.

## § 1. A short review of H. Chiba's theory

Y. Kuramoto [1] introduced in 1975 a mathematical model describing synchronization phenomena between coupled harmonic oscillators moving over a unit circle :

$$\frac{d\theta_i(t)}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)), \quad i = 1, \dots, N, \tag{1.1}$$

where $\theta_i(t) \in \mathbb{R}/(2\pi\mathbb{Z})$ is the phase of the $i$-th oscillator with a fixed frequency $\omega_i \in \mathbb{R}$ $(i = 1, \dots, N)$, and $K$ is the coupling constant. In this model, he used the center of gravity of the oscillators

$$\eta(t) = \frac{1}{N}\sum_{j=1}^{N} e^{\sqrt{-1}\theta_j(t)} \in \{z \in \mathbb{C}; |z| \leq 1\}$$

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as the order parameter of this system. Then, Kuramoto's conjecture is the following ([2, 3]):

*Let $N$ be sufficiently large. Then, as $t \to +\infty$, $|\eta(t)| \to 0$ or $|\eta(t)|$ converges to a positive value. More precisely: $\exists K_c > 0$: independent of $N$ such that*

$$\lim_{t\to\infty} |\eta(t)| \to 0 \quad \text{for } 0 < K < K_c \ \text{(non synchronization)},$$

$$\liminf_{t\to\infty} |\eta(t)| > 0 \quad \text{for } K > K_c \quad \text{(synchronization)}.$$

H. Chiba considered the continuous version of this problem as follows: Let us consider the density function of a discrete model:

$$\rho(t,\theta,\omega) := \frac{1}{N}\sum_{i=1}^{N} \delta(\theta - \theta_i(t))\delta(\omega - \omega_i).$$

Then we have the following equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}\left\{\left(\omega + K\int_{-\infty}^{\infty} d\omega' \int_0^{2\pi} \rho(t,\theta',\omega')\sin(\theta'-\theta)d\theta'\right)\rho\right\} = 0.$$

This is because we have

$$\begin{aligned}\partial_t \rho &= \frac{1}{N}\sum_{i=1}^{N}\left\{-\omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta_i(t))\right\}\delta'(\theta-\theta_i(t))\delta(\omega-\omega_i)\\ &= \partial_\theta\left[\left\{-\omega - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta)\right\}\rho\right].\end{aligned}$$

Further we consider the following problem for $\rho(t,\theta,\omega)$ in $[0,\infty)_t \times (\mathbb{R}/2\pi\mathbb{Z})_\theta \times \mathbb{R}_\omega$:

$$\text{(1.2)} \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial \theta}\left\{\left(\omega + K\displaystyle\int_{-\infty}^{\infty} d\omega' \int_0^{2\pi} \rho(t,\theta',\omega')\sin(\theta'-\theta)d\theta'\right)\rho\right\} = 0, \\ \rho(0,\theta,\omega) = g(\omega)h(\theta), \end{cases}$$

where the initial distribution is assumed to have a form $g(\omega)h(\theta)$; $g(\omega)$ is an analytic distribution on $\mathbb{R}$, for example, Gauss or Cauchy distributions, and $h(\theta)$ is any distribution on $S^1$. The order parameter $\eta(t)$ is written as

$$\eta(t) := \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} e^{\sqrt{-1}\theta}\rho(t,\theta,\omega)d\theta.$$

By considering $\rho' := \rho(t,\theta,\omega)/g(\omega)$ instead of $\rho$, we have

$$\begin{cases} \dfrac{\partial \rho'}{\partial t} + \dfrac{\partial}{\partial \theta}\left\{\rho'\left(\omega + K\displaystyle\int_{-\infty}^{\infty} g(\omega')d\omega' \int_0^{2\pi} \rho'(t,\theta',\omega')\sin(\theta'-\theta)d\theta'\right)\right\} = 0, \\ \rho'(0,\theta,\omega) = h(\theta). \end{cases}$$

This form of initial value problems is argued in H. Chiba's paper [4]. Hereafter we use the notation $\rho$ instead of $\rho'$. Then, his main results in [4] are the following:

**Theorem 1.1.** *Let $g(\omega)$ be a Gaussian distribution with mean $m$, deviation $\sigma > 0$, and let $K_c := 2/(\pi g(m))$. Suppose $0 < K < K_c$. Then, $\exists\delta > 0$ such that if $h(\theta)$ satisfies*

$$\left| \int_0^{2\pi} e^{\sqrt{-1}j\theta} h(\theta) d\theta \right| \leq \delta, \quad \forall j = 1, 2, \ldots, \tag{1.3}$$

*$|\eta(t)|$ decreases exponentially to 0 as $t \to +\infty$.*

**Theorem 1.2.** *$g, m, \sigma, K_c$ be as above. Then, $\exists\epsilon_0, \exists\delta > 0$ such that if $K_c < K < K_c + \epsilon_0$, and $h(\theta)$ satisfies*

$$\left| \int_0^{2\pi} e^{\sqrt{-1}j\theta} h(\theta) d\theta \right| \leq \delta, \quad \forall j = 1, 2, \ldots, \tag{1.4}$$

*we have*

$$\lim_{t \to +\infty} |\eta(t)| = \sqrt{\frac{-16(K - K_c)}{\pi K_c^4 g''(m)}} + O(K - K_c).$$

The following is a rough sketch of his proof.

(1) Expand $\rho(t, \theta, \omega) = \sum_{j \in \mathbb{Z}} Z_j(t, \omega) e^{\sqrt{-1}j\theta}$. Then derive a system of equations for $Z_j(t, \omega)$ $(j \in \mathbb{Z})$.
(2) Consider the linearization of the system of equations for infinitely small initial data $h(\theta)$. It is considered as a time-evolution equation in the Hilbert space $H = L^2(\mathbb{R}; g(\omega)d\omega)$. The analysis of this linearized system is used to approximate the solution of the original equation.
(3) The generator $T$ of the linearized evolution equation is unbounded. So, following Hille-Yosida theory, we must consider the resolvent $(\lambda - T)^{-1}$. Chiba's idea is the analytic continuation in $\lambda$ of the inner product $((\lambda - T)^{-1}\varphi, \psi)$, where $\varphi, \psi$ are taken in suitable dense subspaces $X_\pm$ of $H$. Indeed, $(\lambda - T)^{-1}$ is defined only in $\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0, \lambda \notin \{\text{eigenvalues}\}\}$ with continuous spectrum on $\{\operatorname{Re}\lambda = 0\}$, but he found that $((\lambda - T)^{-1}\varphi, \psi)$ extends to $\{\operatorname{Re}\lambda \leq 0\}$ as a meromorphic function in $\lambda$.
(4) For a Gaussian distribution $g(\omega)$, we can get detailed properties of $((\lambda - T)^{-1}\varphi, \psi)$; poles and residues. Indeed, the poles are all of first order, and Chiba called these poles generalized eigenvalues of $T$ (or resonance poles). Further for each generalized eigenvalue we can write down the eigen-vectors in $X'_\pm$ explicitly, which are generalized functions on $\mathbb{R}$. By using these eigen-vectors, he obtained an expansion theorem of the resolvent.

(5) According to Chiba, $X_\pm$ are introduced as subspaces of $H = L^2(\mathbb{R}; g(\omega)d\omega)$, and they have DFS-space structures; topological vector spaces with uncountable semi-norm systems. But their duals $X'_\pm$ are FS-spaces, in particular complete metric spaces. He used this metric in the analysis of the original non-linear dynamical system.

(6) In the same way as the usual arguments in dynamical systems in Banach spaces, he could employ the center manifold theory. In this case, the center manifold is 1-dimensional, and so the argument reduces to the bifurcation theory for some non-linear ordinary differential equation with parameter $K$.

## Our remarks concerning his proof:

(a) We can choose dense subspaces $X_\pm$ in $H = L^2(\mathbb{R}; g(\omega)d\omega)$, which have Banach space structures. So, the arguments become much simpler.

(b) For a Gaussian distribution $g(\omega)$, we gave a rigorous proof for the existence and the asymptotic behavior about the resonance poles. Further, our method covers some class of analytic distributions including Gaussian distributions, and $e^{-a\omega^4}/M$ ($a, M$ are positive constant).

## § 2. The key operator and the resonance poles of its resolvent

Firstly, we consider our equation by using Fourier coefficients of the solution. Since the distribution function $\rho(t,\theta,\omega)$ is periodic in $\theta$, we can define the Fourier coefficients:

$$Z_j(t,\omega) := \int_0^{2\pi} e^{\sqrt{-1}j\theta}\rho(t,\theta,\omega)d\theta.$$

Then we have

$$\eta(t) = \int_{-\infty}^{\infty} Z_1(t,\omega)g(\omega)d\omega, \tag{2.1}$$

$$\frac{dZ_j}{dt} = j\sqrt{-1}\omega Z_j + \frac{jK}{2}(\eta(t)Z_{j-1} - \overline{\eta(t)}Z_{j+1}), \quad \forall j \in \mathbb{Z}. \tag{2.2}$$

In particular, $Z_0$ does not depend on $t$, and so $Z_0(t,\omega) = Z_0(0,\omega) = \int_0^{2\pi} h(\theta)d\theta = 1$. Thus, the equations on $Z_j$ ($j \geq 0$) are closed among them. Further the order parameter $\eta(t)$ is determined only by $Z_1$.

**Definition 2.1. (Key operator $T$)**
Consider the infinitesimal variation $\psi_j := \delta Z_j$ ($j \in \mathbb{Z}$) of $Z_j$. Then we have the following linear equations for $\psi_j$'s:

$$\frac{d\psi_j}{dt} = \sqrt{-1}j\omega\psi_j + \frac{jK}{2}\delta_{j,0}\int_{-\infty}^{\infty}\psi_1(t,\omega)g(\omega)d\omega. \tag{2.3}$$

Put $j = 1$, then we have our key operator $T$:

(2.4) $$\frac{d\psi_1}{dt} = T\psi_1, \quad T := \sqrt{-1}M + \frac{K}{2}P,$$

where $M : q(\omega) \mapsto \omega q(\omega)$, and $Pq = (q, 1) \cdot 1$ with the inner product on $L^2(\mathbb{R}, g(\omega)d\omega)$ defined by

(2.5) $$(q_1, q_2) = \int_{\mathbb{R}} q_1(\omega)\overline{q_2(\omega)}g(\omega)d\omega.$$

**Definition 2.2. (Eigen-vectors and the resolvent $(\lambda - T)^{-1}$ of $T$)**
$T$ is not a bounded operator because $Mq(\omega) = \omega q(\omega)$ is not bounded. On the other hand, we can easily find the eigen-vectors for $\lambda \in \mathbb{C}$ because

$$Tq = \lambda q \iff \lambda q(\omega) = \sqrt{-1}\omega q(\omega) + K(q, 1)/2,$$

that is, $q(\omega) = K(q, 1)/\{2(\lambda - \sqrt{-1}\omega)\}$ $(\forall\omega)$. Further the condition for an eigenvalue $\lambda(\in \mathbb{C})$ is that

$$1 - \frac{K}{2}(\frac{1}{\lambda - \sqrt{-1}\omega}, 1) = 0.$$

In the same way, as for the resolvent $(\lambda - T)^{-1}$, we have

$$\{(\lambda - T)^{-1}q\}(\omega) \equiv q'(\omega) = \frac{c + q(\omega)}{\lambda - \sqrt{-1}\omega},$$

where $c = (K/2)(q', 1)$ is a constant, and so

$$c = \frac{K}{2} \cdot (\frac{q}{\lambda - \sqrt{-1}\omega}, 1) \cdot \left\{1 - \frac{K}{2}(\frac{1}{\lambda - \sqrt{-1}\omega}, 1)\right\}^{-1}.$$

Therefore, $(\lambda - T)^{-1}$ is a bounded operator if $\mathrm{Re}\,\lambda > 0$ and the denominator above is not equal to zero; that is, $\lambda$ is not any eigenvalue.

**Proposition 2.3.** *([4]). For $m \in \mathbb{R}, \sigma > 0$, we set $\hat{g}(x) := \sigma \cdot g(\sigma x + m)$. Suppose that $\hat{g}(x)$ is an even function of $x$, and is monotonously decreasing in $x > 0$. Then, putting $K_c = 2\sigma/(\pi\hat{g}(0))$, $T$ has no eigenvalue if $K < K_c$. Further, if $K > K_c$, $T$ has a unique eigenvalue $\lambda_+$, and the eigen-vector is $(\lambda_+ - \sqrt{-1}\omega)^{-1}$, where the condition of $\lambda_+ - \sqrt{-1}m = u > 0$ is the following:*

$$\sigma/K = \int_0^\infty \frac{\hat{g}((u/\sigma)y)}{1 + y^2}dy.$$

**Definition 2.4. (Resolvent functions $F(\lambda; \varphi, \psi), F_0(\lambda; \varphi, \psi)$).**
We define a holomorphic function $F(\lambda; \varphi, \psi)$ of $\lambda \in \mathbb{C}$ by

$$F(\lambda; \varphi, \psi) := ((\lambda - T)^{-1}\varphi, \psi).$$

By the arguments above, we know that $F$ is holomorphic in $\{\mathrm{Re}\,\lambda > 0, \lambda \neq \lambda_+\}$, and that $\lambda_+$ is a simple pole. Further, we have

$$F(\lambda;\varphi,\psi) = c(\frac{1}{\lambda-\sqrt{-1}\omega},\psi) + (\frac{\varphi}{\lambda-\sqrt{-1}\omega},\psi)$$

with

$$c = \frac{K}{2}\cdot(\frac{\varphi}{\lambda-\sqrt{-1}\omega},1)\cdot\left\{1-\frac{K}{2}(\frac{1}{\lambda-\sqrt{-1}\omega},1)\right\}^{-1}.$$

Since

$$(\frac{\varphi}{\lambda-\sqrt{-1}\omega},\psi) = \sqrt{-1}\int_{-\infty}^{\infty}\frac{\varphi(\omega)\overline{\psi(\omega)}g(\omega)}{\omega-(-\sqrt{-1}\lambda)}d\omega,$$

this term extends analytically to $\mathrm{Re}\,\lambda \leq 0$ across $\mathrm{Re}\,\lambda = 0$ if $\varphi(\omega)\overline{\psi(\omega)}g(\omega)$ extends analytically to $\mathrm{Im}\,\omega > 0$. Hence, suppose that $g$ is real analytic in $\mathbb{R}$, $\varphi$ is a boundary value of a holomorphic function from $\mathrm{Im}\,\omega > 0$, and that $\psi$ is a boundary value of a holomorphic function from $\mathrm{Im}\,\omega < 0$. Then, $F(\lambda;\varphi,\psi)$ extends analytically to $\mathrm{Re}\,\lambda \leq 0$ across $\mathrm{Re}\,\lambda = 0$ (except for the zero point of $1-K(1/(\lambda-\sqrt{-1}\omega),1)/2$ ). Hence, hereafter we suppose that

(1) $g(\omega)$ is real analytic in $\mathbb{R}$.
(2) $\varphi(\omega)$ and $\psi^*(\omega)$ are boundary values of holomorphic functions from $\mathrm{Im}\,\omega > 0$, where $\psi^*(\omega) := \overline{\psi(\bar{\omega})}$.

Further, by introducing the following change of the variable and the notation:

$$x := \frac{\omega-m}{\sigma}, \quad \hat{T} := \frac{T-m}{\sigma},$$

with the fact $g(\omega)d\omega = \hat{g}(x)dx$, we have only to consider the case $m=0, \sigma=1$. Hereafter, we suppose that $m=0, \sigma=1$, and use the variable $x$ instead of $\omega$. Thus,

$$F(\lambda;\varphi,\psi) = \int_{\mathbb{R}}\big((\lambda-T)^{-1}\varphi\big)(x)\cdot\psi^*(x)g(x)dx.$$

Further we define

$$F_0(\lambda;\varphi,\psi) = \int_{\mathbb{R}}\frac{\varphi(x)}{\lambda-\sqrt{-1}x}\psi^*(x)g(x)dx \tag{2.6}$$

for $\mathrm{Re}\,\lambda \neq 0$.

**Proposition 2.5.** *([4]).*
*(i) For* $\mathrm{Re}\,\lambda > 0$, *we have*

$$F(\lambda;\varphi,\psi) = F_0(\lambda;\varphi,\psi) + \frac{(K/2)F_0(\lambda;\varphi,1)F_0(\lambda;1,\psi)}{1-(K/2)F_0(\lambda;1,1)}. \tag{2.7}$$

*(ii) We denote by* $F_0^-(\lambda;\varphi,\psi), F^-(\lambda;\varphi,\psi)$ *the analytic continuation to* $\operatorname{Re}\lambda < 0$ *from* $\operatorname{Re}\lambda > 0$ *of* $F_0(\lambda;\varphi,\psi), F(\lambda;\varphi,\psi)$*, respectively. Then, for* $0 < -\operatorname{Re}\lambda \ll 1$*, we have*

$$F_0^-(\lambda;\varphi,\psi) = F_0(\lambda;\varphi,\psi) + 2\pi\varphi(-i\lambda)\psi^*(-i\lambda)g(-i\lambda), \tag{2.8}$$

$$F^-(\lambda;\varphi,\psi) = F_0^-(\lambda;\varphi,\psi) + \frac{(K/2)F_0^-(\lambda;\varphi,1)F_0^-(\lambda;1,\psi)}{1-(K/2)F_0^-(\lambda;1,1)}. \tag{2.9}$$

**Definition 2.6.** $\lambda_0(\in\mathbb{C},\operatorname{Re}\lambda_0\le 0)$ is called a resonance pole of the resolvent $T$ if the analytic extension of $F$ to $\{\operatorname{Re}\lambda\le 0\}$ has a pole at $\lambda_0$.

Chiba's assumption on $\varphi,\psi$ for the following proposition is stronger, that is, analyticity in a neighborhood of $\mathbb{R}$. But, it is obvious that the half-analyticity is sufficient.

**Proposition 2.7.** *Suppose that* $g(x)$ *is an entire function, and that* $\varphi(x),\psi^*(x)$ *extend holomorphically to the upper half plane* $\{z\in\mathbb{C};\operatorname{Im} z>0\}$*. Then,* $\lambda_0(\in\mathbb{C},\operatorname{Re}\lambda_0\le 0)$ *is a resonance pole if and only if* $1-(K/2)F_0^-(\lambda;1,1)=0$*; that is,*

$$2/K = \int_{\mathbb{R}} \frac{g(x)}{\lambda - ix}dx + 2\pi g(-i\lambda). \tag{2.10}$$

The following condition for a resonance pole of the resolvent for the Gaussian distribution $g(x)=e^{-x^2/2}/\sqrt{2\pi}$ is more useful than Chiba's condition in [4].

**Lemma 2.8.** $\lambda_0$ *is a resonance pole iff* $\operatorname{Re}\lambda\le 0$*, and*

$$1-(K/2)\left(\sqrt{2\pi}e^{\lambda^2/2} - \int_0^\infty e^{-s^2/2+\lambda s}ds\right) = 0. \tag{2.11}$$

*Further, we can calculate the asymptotic expansion of the above integral as* $|\lambda|\to\infty$ *in the following way:*

$$\begin{aligned}\int_0^\infty e^{\lambda s - s^2/2}ds &= \lambda^{-1}\int_0^\infty \partial_s(e^{\lambda s})\cdot e^{-s^2/2}ds \\ &= -\lambda^{-1} + \lambda^{-2}\int_0^\infty \partial_s(e^{\lambda s})\cdot s e^{-s^2/2}ds = \cdots\end{aligned}$$

*Proof.*

$$G(\lambda) := \int_{\mathbb{R}} \frac{g(x)}{\lambda - ix}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-x^2/2}}{\lambda - ix}\,dx \quad (\operatorname{Re}\lambda < 0).$$

Then,

$$\begin{aligned}G'(\lambda) &= -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-x^2/2}}{(\lambda-ix)^2}\,dx = \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d}{dx}\left(\frac{1}{\lambda-ix}\right)\cdot e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{ix-\lambda+\lambda}{\lambda-ix}\cdot e^{-x^2/2}\,dx = -1+\lambda G(\lambda).\end{aligned}$$

Therefore,

$$G(\lambda) = -e^{\lambda^2/2}\int_{-\infty}^{\lambda} e^{-s^2/2}ds + Ce^{\lambda^2/2}.$$

Considering $G(-\infty) = 0$, we have $C = 0$, and so

$$\int_{\mathbb{R}} \frac{g(x)}{\lambda - ix}dx = G(\lambda) = -e^{\lambda^2/2}\int_{-\infty}^{\lambda} e^{-s^2/2}ds. \tag{2.12}$$

Hence,

$$F_0^-(\lambda; 1, 1) = G(\lambda) + 2\pi g(-i\lambda) = e^{\lambda^2/2}\left(-\int_{-\infty}^{\lambda} e^{-s^2/2}ds + \sqrt{2\pi}\right). \tag{2.13}$$

Put $s = \lambda - s'$, and finally we have

$$G(\lambda) + 2\pi g(-i\lambda) \equiv \sqrt{2\pi}e^{\lambda^2/2} - \int_0^{\infty} e^{-s^2/2+\lambda s}ds. \tag{2.14}$$

□

**Theorem 2.9.** *(i) Any resonance pole is simple. (ii) If there is a resonance pole* $\lambda_0$ *on* $\mathrm{Re}\,\lambda = 0$*, then* $\lambda_0 = 0$ *and* $K = K_c$*. (iii) There exists a large constant* $C > 0$ *depending only on* $K$ *such that for any resonance pole satisfying* $|\lambda| > C$*, there exists an integer* $n \neq 0$ *satisfying at least anyone of inequalities:*

$$\left|\lambda - 2\sqrt{|n|\pi}\, e^{3\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right| < \frac{C}{|n|\sqrt{|n|}} \quad (n < 0),$$
$$\left|\lambda - 2\sqrt{|n|\pi}\, e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right| < \frac{C}{|n|\sqrt{|n|}} \quad (n > 0).$$

*Further, there is a large integer* $N > 0$ *such that for any integer* $n$ *satisfying* $|n| > N$ *there exists just one resonance pole in each disc above.*

*Proof.* We give only the proof of (iii). Let $\mathrm{Re}\,\lambda < 0$. Then, the condition for a resonance pole is

$$\sqrt{2\pi} - \frac{2e^{-\lambda^2/2}}{K} = e^{-\lambda^2/2}\int_0^{\infty} e^{\lambda s - s^2/2}ds.$$

Hence, under $\mathrm{Re}\,\lambda \leq 0, \lambda \neq 0$ we have

$$I(\lambda) := \lambda\int_0^{\infty} e^{\lambda s - s^2/2}ds = \int_0^{\infty} \partial_s(e^{\lambda s})\cdot e^{-s^2/2}ds = -1 + \int_0^{\infty} se^{\lambda s}e^{-s^2/2}ds.$$

Therefore,

$$|I(\lambda)| \leq 1 + \int_0^{\infty} se^{-s^2/2}ds = 1 + 1 = 2 \quad (\mathrm{Re}\,\lambda \leq 0). \tag{2.15}$$

Note that

$$1-(K/2)F_0^-(\lambda;1,1)\equiv 1-(K/2)\left(\sqrt{2\pi}e^{\lambda^2/2}-I(\lambda)/\lambda\right).$$

Hence we have the following condition for $\lambda$:

$$e^{\lambda^2/2}=\frac{\sqrt{2}}{\sqrt{\pi}K}\left(1+(K/2)I(\lambda)/\lambda\right).$$

So we have some integer $n$ satisfying

$$\lambda^2/2=2n\pi i-\log(\sqrt{\pi/2}K)+\log\left(1+(K/2)I(\lambda)/\lambda\right).$$

Suppose $|\lambda|>2K$, then

$$|(K/2)I(\lambda)/\lambda|<1/2.$$

Hence,

$$|\log\left(1+(K/2)I(\lambda)/\lambda\right)|\le\log 2+\pi/6.$$

Further we suppose

$$|\lambda|^2/2>3(|\log(\sqrt{\pi/2}K)|+\log 2+\pi/6),$$

then

$$|2n\pi|>2(|\log(\sqrt{\pi/2}K)|+\log 2+\pi/6).$$

So, $n\ne 0$, and

$$\begin{aligned}&2n\pi i-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)\\&=2n\pi i\left(1+\frac{-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)}{2n\pi i}\right),\end{aligned}$$

where we have the following estimate:

$$\left|\arg\left(1+\frac{-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)}{2n\pi i}\right)\right|\le\pi/6.$$

Since

$$\lambda=\pm\sqrt{2\big(2n\pi i-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)\big)}$$

and $\operatorname{Re}\lambda<0$, we can conclude $\lambda=$

$$2\sqrt{|n|\pi}\,e^{\frac{5\pi i}{4}}\left(1+\frac{-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)}{2n\pi i}\right)^{1/2}\quad(n>0),$$

$$2\sqrt{|n|\pi}\,e^{\frac{3\pi i}{4}}\left(1+\frac{-\log(\sqrt{\pi/2}K)+\log(1+(K/2)I(\lambda)/\lambda)}{2n\pi i}\right)^{1/2}\quad(n<0).$$

Put

$$H(\lambda) := \left(1 + \frac{-\log(\sqrt{\frac{\pi}{2}}K) + \log(1 + (K/2)I(\lambda)/\lambda)}{2n\pi i}\right)^{1/2}.$$

Then, since $\operatorname{Re} H(\lambda) \geq 0$,

$$\left|H(\lambda) - \left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right| \leq \frac{|H(\lambda)^2 - \left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)^2|}{\operatorname{Re} H(\lambda) + 1}$$
$$\leq \frac{|\log(1 + (K/2)I(\lambda)/\lambda)| + \frac{(\log(\sqrt{\pi/2}K))^2}{8|n|\pi}}{2|n|\pi}.$$

Because

$$|\log(1+z)/z| \leq 2\log(3/2) \quad (\forall z; |z| \leq 1/2),$$

we have

$$\left|H(\lambda) - \left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right|$$
$$\leq \frac{2\log(3/2)K|\lambda|^{-1} + \frac{(\log(\sqrt{\pi/2}K))^2}{8|n|\pi}}{2|n|\pi} \leq \frac{16\pi\log(3/2)K|n||\lambda|^{-1} + (\log(\sqrt{\pi/2}K))^2}{16n^2\pi^2}.$$

On the other hand, since

$$|\lambda| = 2\sqrt{|n|\pi}\,|H(\lambda)| \geq 2\sqrt{|n|\pi}\sqrt{\operatorname{Re}\left(1 + \frac{-\log(\sqrt{\pi/2}K) + \log(1 + (K/2)I(\lambda)/\lambda)}{2n\pi i}\right)}$$
$$\geq \sqrt{4|n|\pi\left(1 - \frac{\pi/6}{2|n|\pi}\right)} \geq \sqrt{11|n|\pi/3},$$

we obtain

$$|\lambda|^{-1} \leq \sqrt{3/(11\pi)}|n|^{-1/2} \leq |n|^{-1/2}.$$

Hence,

$$\left|H(\lambda) - \left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right| \leq \frac{16\pi\log(3/2)K|n||\lambda|^{-1} + (\log(\sqrt{\pi/2}K))^2}{16|n|^2\pi^2}$$
$$\leq \frac{16\pi\log(3/2)K + (\log(\sqrt{\pi/2}K))^2/\sqrt{|n|}}{16\pi^2|n|^{3/2}} \leq \frac{16\pi\log(3/2)K + (\log(\sqrt{\pi/2}K))^2}{16\pi^2|n|^{3/2}}.$$

Thus we can take the large constant $C$ in the theorem as

$$C = \max\left\{2K, \sqrt{6(|\log(\sqrt{\pi/2}K)| + \log 2 + \pi/6)}, \frac{16\pi\log(3/2)K + (\log(\sqrt{\pi/2}K))^2}{16\pi^2}\right\}.$$

As for the existence of the resonance poles, we use Rouché's theorem. We put

$$f(\lambda) := \lambda^2/2 - 2n\pi i + \log(\sqrt{\pi/2}K) - \log(1 + (K/2)I(\lambda)/\lambda),$$
$$h(\lambda) := \lambda^2/2 - 2n\pi i + \log(\sqrt{\pi/2}K),$$

where $n \neq 0$ is any integer satisfying $|n| > \log(\sqrt{\pi/2}K)$. Since the argument goes in the same way, we consider only the case $n > 0$. Easily to see, the zero point of $h$ in $\mathrm{Re}\,\lambda \leq 0$ is given by

$$\lambda_n = \sqrt{4n\pi}e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{2n\pi i}\right)^{1/2}.$$

For some $c$ satisfying $n|\lambda_n|/2 > c > 0$, which will be determined after this argument, we put

$$D_n := \{\lambda \in \mathbb{C}; |\lambda - \lambda_n| < c/n\}.$$

We compare the numbers of zero points in $D_n$ for $f, h$. Since $c/n < |\lambda_n|$, it is clear that the number of zero points in $D_n$ of $h$ is just 1. So, we will show the following inequality:

$$\sup\{|f(\lambda) - h(\lambda)|; \lambda \in \partial D_n\} < \inf\{|h(\lambda)|; \lambda \in \partial D_n\}.$$

Indeed,

$$\begin{aligned}
&\sup\{|f(\lambda) - h(\lambda)|; \lambda \in \partial D_n\} = \sup\{|\log(1 + (K/2)I(\lambda)/\lambda)|; \lambda \in \partial D_n\} \\
&\leq \sup\{2\log(3/2)|(K/2)I(\lambda)/\lambda)|; \lambda \in \partial D_n\} \leq \sup\{2K\log(3/2)/|\lambda|; \lambda \in \partial D_n\} \\
&\leq 2K\log(3/2)/(|\lambda_n| - c/n) \leq 4K\log(3/2)/|\lambda_n| \leq 4K\log(3/2)/\sqrt{4n\pi}.
\end{aligned}$$

On the other hand, since

$$h(\lambda_n + w) = \lambda_n w + w^2/2 = \lambda_n w(1 + w/(2\lambda_n)),$$

we have

$$\inf\{|h(\lambda)|; \lambda \in \partial D_n\} \geq |\lambda_n|(c/n)(1 - (c/n)/|2\lambda_n|) \geq \sqrt{4n\pi}(c/n)(1 - 1/2)) \geq c\sqrt{\pi/n}.$$

Therefore we have only to take $c$ as

$$c\sqrt{\pi/n} > 4K\log(3/2)/\sqrt{4n\pi} \quad (\text{or equivalently,} \quad 2K\log(3/2)/\pi < c).$$

In fact, we can choose such $c$ if

$$2K\log(3/2)/\pi < n|\lambda_n|/2 = \sqrt{n^3\pi}\left|1 - \frac{\log(\sqrt{\pi/2}K)}{2n\pi i}\right|^{1/2}.$$

Therefore, it is sufficient if $n > \max\{\log(\sqrt{\pi/2}K), \{2K\log(3/2)\}^{2/3}/\pi\}$. Hence take

$$N = \max\{\log(\sqrt{\pi/2}K), \{2K\log(3/2)\}^{2/3}/\pi\} + 1.$$

By the following lemma 2.10, the resonance pole which we obtained is just the one in $(iii)$. This completes the proof. □

**Lemma 2.10.** *Put*

$$E_n := \left\{ \left| \lambda - 2\sqrt{|n|\pi}\, e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)\right| < \frac{C}{|n|\sqrt{|n|}} \right\}.$$

*Then, for large integers $n, m$, we have*

$$D_n \cap E_m = \emptyset \ (n \neq m).$$

*Proof.*

$$d_n := \left| 2\sqrt{|n|\pi}\, e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right) - \sqrt{4n\pi} e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{2n\pi i}\right)^{1/2} \right|$$

$$= 2\sqrt{|n|\pi} \left| \frac{\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right)^2 - \left(1 - \frac{\log(\sqrt{\pi/2}K)}{2n\pi i}\right)}{\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right) + \left(1 - \frac{\log(\sqrt{\pi/2}K)}{2n\pi i}\right)^{1/2}} \right| < \frac{2\sqrt{|n|\pi}(\log(\sqrt{\pi/2}K))^2}{16n^2\pi^2}.$$

Therefore, suppose that $n$ satisfies

$$\frac{2\sqrt{|n|\pi}(\log(\sqrt{\pi/2}K))^2}{16n^2\pi^2} < \frac{c}{n}, \quad \left(\text{or equivalently } n > \left(\frac{2\sqrt{\pi}(\log(\sqrt{\pi/2}K))^2}{16\pi^2 c}\right)^2\right).$$

Then, since $d_n < c/n$, the center of $E_n$ belongs to $D_n$. As for the distance between the centers of $E_n, E_m$ $(n \neq m)$ we have

$$\begin{aligned} d(n,m) &= \left| 2\sqrt{|n|\pi}\, e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right) - 2\sqrt{|m|\pi}\, e^{5\pi i/4}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4m\pi i}\right)\right| \\ &= 2\sqrt{\pi}\left| \sqrt{|n|}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4n\pi i}\right) - \sqrt{|m|}\left(1 - \frac{\log(\sqrt{\pi/2}K)}{4m\pi i}\right)\right| \\ &> 2\sqrt{\pi}\left(\left|\sqrt{|n|} - \sqrt{|m|}\right|\right) \end{aligned}$$

(see the real part and the imaginary part). Therefore, in order to prove

$$D_n \cap E_m = \emptyset$$

for $n \neq m, n, m > N$ with a sufficiently large $N$, we have only to see

$$d(n,m) > d_n + \frac{c}{n} + \frac{C}{m^{3/2}}.$$

Further, by the arguments above, we have only to prove the following inequality:

$$2\sqrt{\pi}\left(\left|\sqrt{|n|} - \sqrt{|m|}\right|\right) > \frac{c}{n} + \frac{c}{n} + \frac{C}{m\sqrt{m}} = 2c/n + C/(m\sqrt{m}).$$

Put

$$f(n) := 2\sqrt{\pi}\left(\left|\sqrt{|n|} - \sqrt{|m|}\right|\right) - 2c/n - C/(m\sqrt{m}).$$

Then, when $n > m$, $f = 2\sqrt{\pi}(\sqrt{n} - \sqrt{m}) - 2c/n - C/(m\sqrt{m})$ and so $f'(n) > 0$. Hence,

$$f(n) \geq f(m+1) = 2\sqrt{\pi}/(\sqrt{m+1} + \sqrt{m}) - 2c/(m+1) - C/(m\sqrt{m}).$$

It is clear that the right side above is positive for any sufficiently large $m$. On the other hand, when $n < m$, $f = 2\sqrt{\pi}(\sqrt{m} - \sqrt{n}) - 2c/n - C/(m\sqrt{m})$ and so

$$f'(n) = -\sqrt{\pi/n} + 2c/n^2 = \frac{2c - \sqrt{\pi n^3}}{n^2}$$

is negative for any sufficiently large $n$. Therefore, since

$$f(n) \geq f(m-1) = 2\sqrt{\pi}/(\sqrt{m} + \sqrt{m-1}) - 2c/(m-1) - C/(m\sqrt{m}),$$

$f(m-1)$ and so $f(n)$ is positive for any sufficiently large $m$. Consequently we have a large $N > 0$ such that for any $n, m > N (n \neq m)$ we have $f(n) > 0$. □

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