数理解析研究所講究録 第2101巻 2019年 178-183

Movable Singularity and Blowup of Semi linear Wave Equation

By

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Abstract

In this paper we shall show the blowup of the self-similar radially symmetric solution of a semi linear wave equation. The solution satisfies the semi linear Heun equation which is called a profile equation. We construct a singular solution in terms of elliptic function and Birkhoff normal form theory.

§ 1. Introduction

Let $x = (x_1, \ldots, x_n)$, $n \geq 2$ be the variable in $\mathbb{R}^n$ and $t \in \mathbb{R}$. Consider the semi linear wave equation with focusing nonlinearity

$$U_{tt} - \Delta U - U^3 = 0,\ U = U(x,t),\ x \in \mathbb{R}^n, \tag{1.1}$$

where $U_{tt} = \partial^2 U/\partial t^2$, $\Delta U = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2)U$. When we consider the blowup of a solution, one often considers the self-similar solution $U \equiv u_\lambda(x,t) := \lambda u(\lambda x, \lambda t)$, $\lambda > 0$. In this paper, we shall consider the self-similar solution with radial symmetry

$$U := (T-t)^{-1} u(\frac{r}{T-t}) \tag{1.2}$$

where $T > 0$, $r^2 = x_1^2 + \cdots + x_n^2$, and u is a function of a single variable y, $u = u(y)$. One can easily verify that u satisfies the semi linear Heun equation

$$(1-y^2)\frac{d^2u}{dy^2} + \left(\frac{n-1}{y} - 4y\right)\frac{du}{dy} - 2u + u^3 = 0. \tag{1.3}$$

2010 Mathematics Subject Classification(s): Primary 35C10; Secondary 45E10.
Key Words: movable singularity, blowup, Birkhoff normal form
Partially supported by Grant-in-Aid for Scientific Research (No. 20540172), JSPS, Japan.
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The equation (1.3) is called a profile equation. It has four fixed regular singular points at $y = 0, \pm 1, \infty$. By the movable singularity we mean the singularity $y \neq 0, \pm 1, \infty$ which depends on the respective solution.

The blowup of the initial value problem of the semi linear wave equation has been studied by many authors. In contrast with these results, our object in this paper is to study the blowup phenomenon from the viewpoint of movable singularity of the profile equation. Namely, we construct a solution of the profile equation with a movable singularity at some point. This yields the existence of blowup solution of the semi linear wave equation with singularities on the characteristic cone.

The idea of proof of the main theorem is to transform the corresponding Hamiltonian system near the blowup point to a certain normal form by the similar argument as to Birkhoff normal form theory. The full proof of the main theorem will be published elsewhere.

This paper is organized as follows. In §2 we show the Birkhoff-type reduction theorem. In §3 we prove the existence of singular solution of the reduced profile equation satisfying the requirements of the reduction theorem in §2. Then we state our result concerning the existence of blowup solution of semi linear wave equation.

§ 2. Birkhoff reduction

Set $\tilde{A}(y) = (1-y^2)^{-1}(y^{-1}(n-1) - 4y)$ and write (1.3) in

$$\frac{d^2u}{dy^2} + \tilde{A}(y)\frac{du}{dy} - 2\frac{u}{1-y^2} + \frac{u^3}{1-y^2} = 0. \tag{2.1}$$

We shall eliminate the term containing the first derivative of u by introducing a new unknown function w with $u = \alpha w$, where

$$\alpha(y) = \exp\left(-\frac{1}{2}\int_{y_0}^{y} \tilde{A}(s)ds\right), \tag{2.2}$$

where $y_0 \neq 0, \pm 1, \infty$. The resultant equation is given by

$$w'' + A(y)w + \frac{\alpha^2}{1-y^2}w^3 = 0, \tag{2.3}$$

where

$$A(y) = \frac{1}{2}\left(-\tilde{A}' - \frac{\tilde{A}^2}{2} - \frac{4}{1-y^2}\right) = \frac{n-1}{2y^2(1-y^2)^2}\left(\frac{3-n}{2} + y^2\right). \tag{2.4}$$

By setting $w = q$, $w' = p$, $q_1 = y$ and

$$B(q_1) := \frac{\alpha^2}{4(1-q_1^2)}, \tag{2.5}$$

(2.3) is written in a Hamiltonian system with the Hamiltonian function $H(q_1)$

$$H(q_1) := \frac{1}{2}\left(p^2 + A(q_1)q^2\right) + B(q_1)q^4. \tag{2.6}$$

Let $z_0 \neq 0, \pm 1$ be such that $A(z_0)B(z_0) \neq 0$. Set $v = \frac{1}{2}\left(p^2 + A(z_0)q^2\right)$. Let q_2 and p_2 be the linear combinations of q and p such that $2v := p^2 + A(z_0)q^2 = q_2p_2$. Then we have $B(z_0)q^4 = B(z_0)(\beta q_2 + \gamma p_2)^4$ for some nonzero constants β and γ. We can write the right-hand side as the sum of $a(v) = cq_2^2p_2^2$ and the remaining ones uniquely, where c is a certain constant. We call $a(v)$ the resonance part. We have $H(q_1) = v + a(v) + \tilde{H}$ for some $\tilde{H}(q_1, q_2, p_2)$ which is a polynomial of q_2 and p_2.

Consider the autonomous Hamiltonian $p_1 + H(q_1)$. Let $\tilde{c}$ be a nonzero constant. We shall transform $p_1 + H(q_1)$ to $p_1 + v + a(v) + \tilde{c}(q_2^4 + p_2^4)/4$ formally. Indeed, we have

Theorem 2.1. *There exists a formal symplectic transformation which transforms $p_1 + H$ to $p_1 + v + a(v) + \tilde{c}(q_2^4 + p_2^4)/4$.*

The proof is essentially Birkhoff's reduction.

Next we shall give the meaning to the normal form given by Theorem 2.1. First we introduce the homology equation.

Let $x = (\tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \tilde{p}_2)$ and $y = (q_1, p_1, q_2, p_2)$ be the original and the transformed variables, respectively. For simplicity we sometimes write $y = (y_1, \ldots, y_4)$. We consider the transformation $x = u(y)$ for some $u = (u_1, \ldots, u_4)$. Define

$$R := \chi_{\tilde{H}}, \qquad S := \chi_{a(v)+\tilde{c}(q_2^4+p_2^4)/4}, \tag{2.7}$$

where χ_g denotes the Hamiltonian vector field with Hamiltonian g with respect to a standard symplectic structure. Write $R = r(x)\frac{\partial}{\partial x}$ and $S = s(y)\frac{\partial}{\partial y}$.

Define $\Lambda(y) = (1, 0, q_2/2, -p_2/2)$. Then we have

Lemma 2.2. *Suppose that u satisfy the homology equation*

$$\Lambda(y)\nabla u + s(y)\nabla u = r(u) + \Lambda(u). \tag{2.8}$$

Then, the transformation $x = u(y)$ maps the vector field $(\Lambda(x) + r(x))\frac{\partial}{\partial x}$ to $(\Lambda(y) + s(y))\frac{\partial}{\partial y}$.

Proof.

$$(\Lambda(x) + r(x))\frac{\partial}{\partial x} = (\Lambda(u) + r(u))(\nabla u)^{-1}\frac{\partial}{\partial y} = (\Lambda(y) + s(y))\frac{\partial}{\partial y}. \tag{2.9}$$

□

We shall solve (2.8). Define

$$q_2 = \alpha\zeta, \quad p_2 = \eta\zeta, \tag{2.10}$$

where ζ is a complex parameter. Assume that α and η satisfy

$$2c\alpha^2\eta^2(\eta - \alpha) + \tilde{c}(\eta^5 - \alpha^5) \neq 0, \qquad \eta + \alpha \neq 0. \tag{2.11}$$

Let $\eta_0 > 0$ be given. Set $\rho = q_2 + p_2$ and define

$$\Omega_0 := \{(\rho, q_1) | \, |q_1 - z_0| < \eta_0, \ |\rho| < \eta_0\}. \tag{2.12}$$

Then we have

Theorem 2.3. *Suppose that α and η satisfy (2.11). Then there exists an $\eta_0 > 0$ such that if p_1 is in some neighborhood of the origin and (q_1, q_2, p_2) is given by (2.10) with q_2 and p_2 replaced by q_2^{-1} and p_2^{-1}, respectively, and $\zeta = (\alpha + \eta)^{-1}\rho$, $(q_1, \rho) \in \Omega_0$, then the vector field $(\Lambda(x) + r(x))\frac{\partial}{\partial x}$ is transformed to $(\Lambda(y) + s(y))\frac{\partial}{\partial y}$ by an analytic change of coordinates.*

§ 3. Movable singularity and blowup solution

In this section we shall construct a solution of (1.3) with movable singularity and state our main result. In view of Theorem 2.3 we consider the Hamiltonian $p_1 + q_2p_2 + cq_2^2p_2^2 + \tilde{c}(q_2^4 + p_2^4)$, where $c \neq 0$ and $\tilde{c} \neq 0$ are constants. By setting $q_2 = q$ and $p_2 = p$ we consider the Hamiltonian

$$\tilde{H} := qp + \frac{\varepsilon}{2}q^2p^2 - \frac{\eta}{8}(q^2 - p^2)^2, \tag{3.1}$$

where ϵ and $\eta \neq 0$ are constants. Because $\tilde{c}$ can be chosen arbitrarily, we may assume $\epsilon \neq 0$, $\epsilon + \eta \neq 0$ without loss of generality. Suppose that (q, p) is the solution of the Hamiltonian system for $\tilde{H}$. Then there exists a constant C_2 such that $\tilde{H}(q, p) \equiv C_2$. Define

$$\zeta = \frac{q + p}{2}, \ \xi = \frac{q - p}{2i}. \tag{3.2}$$

Then we have

$$\begin{aligned} C_2 = \tilde{H} &= (\zeta^2 + \xi^2) + \frac{\epsilon}{2}(\zeta^2 + \xi^2)^2 + 2\eta\zeta^2\xi^2 \\ &= \frac{\epsilon + \eta}{2}(\zeta^2 + \xi^2 + \frac{1}{\epsilon + \eta})^2 - \frac{1}{2(\epsilon + \eta)} - \frac{\eta}{2}(\zeta^2 - \xi^2)^2. \end{aligned} \tag{3.3}$$

Hence we have

$$1 = \frac{(\epsilon+\eta)^2}{A}(\zeta^2+\xi^2+\frac{1}{\epsilon+\eta})^2 - \frac{\eta(\epsilon+\eta)}{A}(\zeta^2-\xi^2)^2 \tag{3.4}$$

where $A = 1+2C_2(\epsilon+\eta)$. We determine $\theta = \theta(z)$ such that

$$\sin^2\theta = \frac{(\epsilon+\eta)^2}{A}(\zeta^2+\xi^2+\frac{1}{\epsilon+\eta})^2, \quad \cos^2\theta = -\frac{\eta(\epsilon+\eta)}{A}(\zeta^2-\xi^2)^2. \tag{3.5}$$

Then, by (3.5) and simple computations we have

$$\zeta \equiv \zeta(z) = \sqrt{\frac{\sqrt{\frac{A(\epsilon+\eta)}{-\eta}}\cos\theta + \sqrt{A}\sin\theta - 1}{2(\epsilon+\eta)}}, \tag{3.6}$$

$$\xi \equiv \xi(z) = \sqrt{\frac{-\sqrt{\frac{A(\epsilon+\eta)}{-\eta}}\cos\theta + \sqrt{A}\sin\theta - 1}{2(\epsilon+\eta)}}. \tag{3.7}$$

Set $X(z) = \sin\theta(z) + \eta\epsilon^{-1}/\sqrt{A}$ and define

$$\mathcal{A} = \sqrt{\mathcal{E} + \frac{i}{2}\sqrt{\mathcal{F}}}, \quad \mathcal{B} = \sqrt{\mathcal{E} - \frac{i}{2}\sqrt{\mathcal{F}}}, \tag{3.8}$$

where

$$\mathcal{E} = \frac{1}{2(\epsilon+\eta)}\left(\sqrt{A}X(z) - \eta\epsilon^{-1} - 1\right), \tag{3.9}$$

$$\mathcal{F} = \frac{A}{\eta(\epsilon+\eta)}\left(1 - (X(z) - \eta\epsilon^{-1}/\sqrt{A})^2\right). \tag{3.10}$$

Then we see that $\zeta = \mathcal{A}$ and $\xi = \mathcal{B}$. Therefore, by (3.2) we obtain

$$q(z) = \mathcal{A} + i\mathcal{B}, \quad p(z) = \mathcal{A} - i\mathcal{B}. \tag{3.11}$$

Then we have

Lemma 3.1. *$X(z)$ is an elliptic function.*

In view of (3.11) we see that the solution has movable singularity given by the elliptic function. By virtue of the parametrization via the elliptic function, we shall construct the blowup solution of (1.1) with singularities on some characteristic cone. Let $a(v)$ be as in Theorem 2.1. Then we have

Theorem 3.2. *Let $T > 0$. Assume that $z_0 \neq 0, \pm 1$. Given a neighborhood Ω_0 of z_0. Then there exist $z_1 \in \Omega_0$ and a blowup solution U of (1.1) such that U blows up on the set $z_1(T-t) = r$, $r^2 = x_1^2 + \cdots + x_n^2$, $(t, x) \in \mathbb{R}^{n+1}$.*

Proof. We shall look for u in (1.2) such that u satisfies (1.3) and has a movable singularity. Indeed, we make the reduction as in Theorem 2.3 to (1.3) and we obtain the autonomous system. Indeed, the assumption (2.11) is the condition for q_2 and p_2, which can be satisfied in view of the parametrization of singular solution in the above by slight change of parameters.

Therefore we obtain a singular solution of (1.3) parametrized by the solution of the autonomous equation. Next one can easily show that there is a local diffeomorphic change of variables in some neighborhood of z_0 between the original variable $\tilde{q}_1$ and q_1. By expressing the singular solution in terms of the variable of (1.3) we obtain the singular solution of (1.3). The location of singularity is clear in view of the definition of a radially symmetric self-similar solution. □

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