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Traveling Waves in a Band Domain with Quasi-Periodically Undulating Boundaries

1. Introduction

We discuss traveling waves for a curvature-driven motion of plane curves in a band domain $\Omega_\epsilon$, where $\epsilon > 0$ is a certain small parameter and the boundaries of $\Omega_\epsilon$ undulate quasi-periodically as specified below. The law of motion of the curve is given by the equation

$$ V = \kappa + A, \quad (1) $$

where $V$ denotes the normal velocity of the curve, $\kappa$ denotes the curvature and $A$ is a positive constant representing a constant driving force. The domain $\Omega_\epsilon$ is defined as follows: Let $g_1(y)$ and $g_2(y)$ be smooth quasi-periodic functions satisfying

$$ g_1(y) \geq 0, \quad \inf_y g_1(y) = 0, \quad \sup_y g_1'(y) = \tan \alpha_i, \quad \inf_y g_1'(y) = -\tan \beta_i \quad (i = 1, 2), $$

for some $\alpha_i, \beta_i \in (0, \frac{\pi}{2})$ and $\alpha_i + \beta_i < \frac{\pi}{2} \ (i = 1, 2)$. $\Omega_\epsilon$ is defined by

$$ \Omega_\epsilon := \{(x, y) \in \mathbb{R}^2 \mid -g_{1\epsilon}(y) < x < g_{2\epsilon}(y), \ y \in (-\infty, \infty)\} $$

with $g_{i\epsilon}(y) := 1 + \epsilon g_i(y) \ (i = 1, 2)$ (see Figure 1).

In this paper, by a solution of (1) we mean a time-dependent simple curve $\Gamma_t$ in $\Omega_\epsilon$ which satisfies (1) and contacts the each boundary of $\Omega_\epsilon$ vertically. To avoid sign confusion, the
normal to the curve $\Gamma_t$ will always be chosen toward the direction of the right-hand side region, and the sign of the normal velocity $V$ and the curvature $\kappa$ will be understood in accordance with this choice of the direction of the normal. Consequently, $\kappa$ is negative at those points where the curve is concave (see Figure 1).

![Figure 1: Domain and Curves](image)

We will only consider the case where the curves are expressed as a graph of a certain function $y = y(x, t)$, so (1) is equivalent to

$$y_t = \frac{y_{xx}}{1 + y_x^2} + A\sqrt{1 + y_x^2}, \quad t > 0,$$

with boundary conditions

$$y_x(x, t)|_{(-g_1(y), y)} = g_1'(y/\epsilon), \quad y_x(x, t)|_{(g_2(y), y)} = -g_2'(y/\epsilon),$$

and restrictions

$$-g_1(y) < x < g_2(y).$$

Let $\Omega_0 = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1\}$ be a straight band domain which is formally a limit of $\Omega_\epsilon$ as $\epsilon \to 0$. For $\Omega_0$ one can easily see that equation (1) has a traveling wave solution $\Gamma_t = \{(x, y_0 + At) \mid -1 < x < 1\}$ which moves at a constant speed $A$ remaining its shape (a line segment).

On the other hand, for $\Omega_\epsilon$, traveling wave solutions of (1) in the usual sense do not exist in general. For such undulating band domains, the notion of traveling waves has to be extended to the more general one in the same way as in [1].

**Case 1. Periodic traveling waves.** In the case where $g_1$ and $g_2$ are 1-periodic functions, a solution $\mathcal{Y}(x, t)$ of (2)-(4) is called a periodic traveling wave if

$$\mathcal{Y}(x, t + T_\epsilon) = \mathcal{Y}(x, t) + \epsilon$$
for some $T_\epsilon > 0$. Such a periodic traveling wave propagates in $y$-direction with average speed $c_\epsilon = \epsilon / T_\epsilon$, changing its profile periodically in time.

\[ \text{Fig.2 Periodic traveling wave} \]

**Case 2. Quasi-periodic traveling waves.** Roughly speaking, a quasi-periodic traveling wave for (1) is a curve which moves rightward changing its profile and speed quasi-periodically in time. To give a precise definition of quasi-periodic traveling waves, we introduce some notation and terminology. For any solution $y(x, t)$ of (2)-(4), we call

\[ \xi(t) := y(0, t) \text{ the current position;} \]

\[ \sigma_\xi(t)b = b(y + \xi(t)) \in \mathcal{H}_b \text{ the current landscape, where} \]

\[ b(y) = (g_{1\epsilon}(y), g_{2\epsilon}(y)) \text{ and } \mathcal{H}_b := \{\sigma_{r}b | r \in \mathbb{R}\}^{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \simeq \mathbb{T}^m \text{ for some } m \in \mathbb{N}; \]

\[ y(x, t) - \xi(t) \text{ the current profile.} \]

**Definition.** A solution $\mathcal{Y}(x, t)$ of (2)-(4) is called a quasi-periodic traveling wave if there exists $v(z, s) \in C(\mathcal{H}_b \times \mathbb{R}, \mathbb{R})$ such that

\[ \mathcal{Y}(x, t) - \xi^\epsilon(t) = v(\sigma_{\xi^\epsilon(t)}b, x), \]

where $\xi^\epsilon(t) = \mathcal{Y}(0, t)$ is the current position of $\mathcal{Y}(x, t)$. This means that the current profile depends continuously on the current landscape. A quasi-periodic traveling wave is called regular if \[ \inf_t \xi^\epsilon(t) > 0. \]

Note that this definition agrees with that of traveling waves for the homogeneous and the periodic cases. Moreover, we say that a quasi-periodic traveling wave has the **average speed** $c_\epsilon$. 
if
\[
\frac{\xi(t+T) - \xi(t)}{T} \to c_\epsilon \quad \text{as} \quad T \to \infty \quad \text{uniformly in} \ t.
\]

Recently, Matano has proved the existence of a regular quasi-periodic traveling wave having average speed for (2)-(4) on the assumption that \( A > (\sin \alpha_1 + \sin \alpha_2)/2 \). Moreover, one can discuss the uniqueness and stability of the traveling wave by using an argument similar to that in [3, 4].

The goal of this paper is to determine the homogenization limit of quasi-periodic traveling waves for (2)-(4). Our result is the following:

**Main Theorem.** Assume that \( A > (\sin \alpha_1 + \sin \alpha_2)/2 \) and let \( \mathcal{Y}_\epsilon(x,t) \) be the quasi-periodic traveling wave of (2)-(4), then

(i) The average speed \( c_\epsilon \) satisfies

\[
c_0 < c_\epsilon < c_0 + \mathcal{M}(\alpha_1, \alpha_2, A) \epsilon^{1/2}
\]

for small \( \epsilon \), where \( c_0 = c_0(\alpha_1, \alpha_2, A) < A \) is independent of \( \epsilon \), and is given by

\[
2 + F(\alpha_1, c_0) + F(\alpha_2, c_0) = 0,
\]

with

\[
F(\alpha, c) := \frac{\alpha}{c} - \frac{2A}{c \sqrt{A^2 - c^2}} \arctan \left( \frac{A + c}{A - c} \tan \frac{\alpha}{2} \right).
\]

(ii) \( \mathcal{Y}_\epsilon(x,t) \) converges (locally in \( C^{2,1} \)) to a homogenization limit \( \varphi(x; c_0) + \mathcal{Y}(x, c_0) \), where \( \varphi(x; c_0) \) is defined by

\[
\varphi(v; c_0) = -\frac{1}{c_0} \log \left| \frac{A - c_0 \cos(\arctan v)}{A - c_0} \right|,
\]

\[
x(v; c_0) = F(\arctan v, c_0) - 1 - F(\alpha_1, c_0),
\]

by a parameter \( v \in (-\tan \alpha_2, \tan \alpha_1) \).

**Remark.** (i) The above theorem implies that the effect of spatial inhomogeneity appears in the homogenization limit, although \( \Omega_\epsilon \) tends to \( \Omega_0 \) as \( \epsilon \to 0 \). Indeed, the homogenized traveling wave has non-planar profile \( \varphi \) and its propagation speed \( c_0 \) is less than \( A \).

(ii) The function \( \varphi(x; c_0) \) satisfies

\[
c_0 = \frac{\varphi_{xx}}{1 + \varphi_x^2} + A \sqrt{1 + \varphi_x^2}, \quad x \in (-\chi_1, \chi_2)
\]

for some \( \chi_1, \chi_2 > 1 \), and \( \varphi_x(-1) = \tan \alpha_1, \varphi_x(1) = -\tan \alpha_2 \) (cf. [2]).
2. Proof of Main Theorem

In this section, by constructing a lower solution and an upper solution we prove Main Theorem in the symmetric case: $g_1 = g_2$. The proof for the general case is similar and we omit it. In what follows, we write $g = g_1(= g_2)$, $\alpha = \alpha_1 (= \alpha_2)$ and $\varphi = \varphi(\cdot; c_0)$.

By Remark 1 (ii), we obtain

**Lemma 2.1** $\tilde{y}(x, t) := \varphi(x; c_0) + c_0 t$ is a lower solution of (2)-(4), and $c_0 < c_r$.

Let $\mathcal{Y}(x, t)$ be a periodic traveling wave of (2)-(4). We note that $\mathcal{Y}(\pm 1, t) = \varphi(\pm 1) = 0$, and $\mathcal{Y}(x_0, 0) = \varphi(x_0)$ for some $x_0 \in [-1,1]$. Take $L > \frac{4\sqrt{c_0 \varepsilon^2}}{\cos \alpha}$, and define

$$u(x, t) = L \varepsilon^\frac{1}{2} (1 - e^{-\varepsilon^2 t} \sin \frac{\pi(1+x)}{2}), \quad x \in [-1,1], \ t \geq 0.$$

**Lemma 2.2.** $\tilde{y}(x, t) := u(x, t) + \varphi(x) + c_0 t$ is an upper solution of (6) on $t \in [0,1]$, and hence

$$\overline{y}(x, t) \geq \mathcal{Y}(x, t), \quad x \in [-1,1], \ t \in [0,1].$$

**Sketch of the Proof.** To prove the Lemma, it suffices to show that

$$\tilde{y}_t \geq \frac{\tilde{y}_{xx}}{1 + \tilde{y}_x^2} + A \sqrt{1 + \tilde{y}_x^2} + A \sqrt{1 + \tilde{y}_x^2}, \quad x \in [-1,1], \ t \geq 0,$$

and

$$\mathcal{Y}(\pm 1, t) < \tilde{y}(\pm 1, t), \quad t \in [0,1].$$

The inequality (8) can be easily verified by our construction. Now we show that

$$\mathcal{Y}(\pm 1, t) < \tilde{y}(\pm 1, t), \quad t \in [0,1].$$

The other inequality in (9) can be treated similarly.

Suppose that $\tilde{t} < 1$ and

$$\mathcal{Y}(\tilde{t}, t) < \tilde{y}(\tilde{t}, t), \quad t \in [0, \tilde{t}].$$

Let $y_0 \in (0, 1)$ be such that $g'(y_0) = \tan \alpha$ and $g(y_0) = \frac{\theta}{\varepsilon} = O(1)$. Let $\zeta(x)$ be an arc with curvature $-A$ and satisfying $\zeta(-1 - \theta) = 0$, $\zeta'(-1 - \theta) = \tan \alpha$. Then we have

$$\zeta(x) = -\frac{1}{A} \cos \alpha + \frac{1}{A} \sqrt{\cos^2 \alpha + 2A \sin \alpha \cdot (1 + \theta + x) - A^2 (1 + \theta + x)^2}.$$
Since \[ \varphi(-1 + l\sqrt{\epsilon}) = \tan \alpha \cdot l\sqrt{\epsilon} + \left( \frac{c_0}{2 \cos^2 \alpha} - \frac{A}{2 \cos^3 \alpha} \right) l^2 \epsilon + \cdots, \]
for \( l = \frac{L \pi \cos \alpha e^{-\pi^2}}{6 c_0} \), we have
\[
\zeta(-1 + l\sqrt{\epsilon}) = \tan \alpha \cdot (l\sqrt{\epsilon} + \theta) - \frac{A}{2 \cos^3 \alpha} (l\sqrt{\epsilon} + \theta)^2 + \cdots \geq \tan \alpha \cdot \theta + \varphi(-1 + l\sqrt{\epsilon}) - M \epsilon
\]
for small \( \epsilon \), where \( M = \frac{\rho \cos \alpha}{\cos^3 \alpha} \).

Suppose that \( \zeta(x) + H(\tilde{t}) \) intersects \( \bar{y}(x, \tilde{t}) \) at \( x = -1 + l\sqrt{\epsilon} \) for some \( H(\tilde{t}) \), that is,
\[
\zeta(-1 + l\sqrt{\epsilon}) + H(\tilde{t}) = \bar{y}(-1 + l\sqrt{\epsilon}, \tilde{t}).
\]
Then we obtain
\[
H(\tilde{t}) = v(-1 + l\sqrt{\epsilon}, \tilde{t}) + \varphi(-1 + l\sqrt{\epsilon}) + c_0 \tilde{t} - \zeta(-1 + l\sqrt{\epsilon})
\leq v(-1 + l\sqrt{\epsilon}, \tilde{t}) - \tan \alpha \cdot \theta + M \epsilon + c_0 \tilde{t}
\leq L\sqrt{\epsilon} - L \frac{\pi l}{3} e^{-\pi^2} - \tan \alpha \cdot \theta + M \epsilon + c_0 \tilde{t}
= \bar{y}(-1, \tilde{t}) - L \frac{\pi l}{3} e^{-\pi^2} - \tan \alpha \cdot \theta + M \epsilon.
\]

On the other hand, there exists a \( \delta \in [0, \epsilon) \) such that the arc \( \zeta(x) + H(\tilde{t}) + \delta \) intersects \( \partial \Omega \) at some point \( (x^*, y^*) \), where
\[
x^* = -1 - \theta \quad \text{and} \quad g'(y^*) = \frac{g'(y^*)}{\epsilon} = \tan \alpha.
\]
This implies that the arc \( \zeta(x) + H(\tilde{t}) + \delta \) is a stationary curve of (2)-(4) on \([-1 - \theta, -1 + l\sqrt{\epsilon}] \).

Since
\[
Y^r(-1 + l\sqrt{\epsilon}, \tilde{t}) \leq \bar{y}(-1 + l\sqrt{\epsilon}, \tilde{t}) \leq \zeta(-1 + l\sqrt{\epsilon}) + H(\tilde{t}) + \delta,
\]
we have \( Y^r(x, \tilde{t}) \leq \zeta(x) + H(\tilde{t}) + \delta \) for \( x \in [-1 - \theta, -1 + l\sqrt{\epsilon}] \). Especially,
\[
Y^r(-1, \tilde{t}) \leq \zeta(-1) + H(\tilde{t}) + \delta \leq \tan \alpha \cdot \theta + H(\tilde{t}) + \epsilon
\leq \bar{y}(-1, \tilde{t}) + \left[ M + L \frac{\pi l}{3} e^{-\pi^2} \right] \cdot \epsilon \leq \bar{y}(-1, \tilde{t}) - 2 \epsilon
\]
by the choice of \( l \) and \( L \). Therefore we have
\[
\bar{y}(-1, \tilde{t} + t) \geq \bar{y}(-1, \tilde{t}) > Y^r(-1, \tilde{t}) + \epsilon \geq Y^r(-1, \tilde{t} + t), \quad t \in [0, T_{\epsilon}].
\]
This means that \( Y^r(-1, t) < \bar{y}(-1, t) \) on \( t \in [0, \tilde{t} + T_{\epsilon}] \).

Repeating the above discussion finite times, we get (10). This proves the Lemma.
Fig. 3 Upper Solution

Proof of Main Theorem. By Lemma 2.1 we only need the upper bound of $c_\epsilon$.
Denote by $\lfloor \chi \rfloor$ the integer part of $\chi > 0$. By Lemma 2.2 we have

$$\mathcal{Y}(x, 1) - \varphi(x) \leq \bar{y}(x, 1) - \varphi(x) = v(x, 1) + c_0 \leq \left[ \frac{L_\vee r^{\frac{1}{2}} + c_0}{\vee c} + 1 \right] \cdot \epsilon.$$  

On the other hand,

$$\mathcal{Y}(x, 1) - \varphi(x) \leq \bar{y}(x, 1) - \varphi(x) = v(x, 1) + c_0 \leq \left[ \frac{L_\vee r^{\frac{1}{2}} + c_0}{\vee c} + 1 \right] \cdot \epsilon,$$

and "equality" holds at some $x_0 \in [-1, 1]$. Therefore we obtain

$$1 \leq \left[ \frac{L_\vee r^{\frac{1}{2}} + c_0}{\vee c} + 1 \right] \cdot T_\epsilon \leq \left( \frac{L_\vee r^{\frac{1}{2}} + c_0}{\vee c} + 1 \right) \cdot T_\epsilon,$$

and hence

$$c_\epsilon = \frac{\epsilon}{T_\epsilon} \leq c_0 + L_\vee r^{\frac{1}{2}} + \epsilon$$

This proves (5).

Statement (ii) follows from the comparison theorem, standard parabolic estimates and (5).

Remark 2. To give a scent to the readers for the relation between $c_0$ and $A$, as well as that between $c_0$ and $\alpha$, we consider the problem in a band domain with ratchet boundaries (see Figure 4).
We divide the traveling wave into two parts: the part near the boundaries (we call it boundary solution), and the part away from the boundaries (we call it interior solution). Via a rather intriguing asymptotic expansion approach, we find that the interior solution is approximately a traveling wave with constant speed and profile, while the behavior of the boundary solution is complex. In one period the motion of the boundary solution consists of three stages (see Figure 5).

Stage 1 – Contact points (where the solution curves contact with the boundary) are on $PQ$. In this stage the profile of the solution is like $\varphi$ and the propagation speed is of order $O(1)$.

Stage 2 – Contact points are on $QR$. In this stage, the contact point $\gamma(t)$ moves rapidly from $Q$ to $R$ in a short time $O(\varepsilon^2)$, while the interior solution almost remains stationary.

Stage 3 – Contact points stay at $R$. In this stage the propagation speed of the boundary solution varies from of order $O(\frac{1}{\varepsilon})$ to of order $O(1)$.
References

1. H. Matano. Traveling waves in spatially inhomogeneous diffusive media with bistable non-linearity I, submitted to *Discrete and Continuous Dynamical Systems*.

