数理解析研究所講究録 第2107巻 2019年 46-68

Analysis of non-stationary Navier-Stokes equations approximated by the pressure stabilization method

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1 Introduction

The mathematical description of fluid flow is given by the Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi = f \qquad \nabla\cdot u = 0 & t\in(0,\infty), x\in\Omega, \\ u(0,x) = a & x\in\Omega, \\ u(t,x) = 0 & x\in\partial\Omega, \end{cases} \tag{NS}$$

where the fluid vector fields $u = u(t,x) = (u_1(t,x),\dots,u_n(t,x))$ and the pressure $\pi = \pi(t,x)$ are unknown function, the external force $f = f(t,x)$ is a given vector function, the initial data a is a given solenoidal function and Ω is some bounded domain (see section 2 for detail). It is well-known that analysis of Navier-Stokes equations (NS) is very important in view of both mathematical analysis and engineering, however the problem concerning existence and regularity of solution to (NS) is unsolved for a long time. One of the difficulty of analysis for (NS) is the pressure term $\nabla\pi$ and incompressible condition $\nabla\cdot u = 0$.

In order to overcome this difficulty, we often use Helmholtz decomposition. The Helmholtz decomposition means that for $1 < p < \infty$, the following relation holds:

$$L_p(\Omega)^n = L_{p,\sigma}(\Omega) \oplus G_p(\Omega),$$

where $L_{p,\sigma}(\Omega) = \overline{\{u \mid u_j \in C_0^\infty(\Omega), \nabla\cdot u = 0\}}^{\|\cdot\|_{L_p}}$ and $G_p(\Omega) = \{\nabla\pi \in L_p(\Omega)^n \mid \pi \in L_{p,\mathrm{loc}}(\Omega)\}$. We remark that whether the Helmholtz decomposition holds depends on the shape of the region in the case where $p \neq 2$ (see Galdi [6] for detail).

On the other hand, in numerical analysis, some penalty methods (quasi-compressibility methods) are employed as the method to overcome this difficulty. They are methods that eliminate the pressure by using approximated incompressible condition. For example, setting $\alpha > 0$ as a perturbation parameter, we use $\nabla\cdot u = -\pi/\alpha$ in the penalty method, $\nabla\cdot u = \Delta\pi/\alpha$ in the pressure stabilization method and $\nabla\cdot u = -\partial_t\pi/\alpha$ in the pseudo-compressible method. In this paper, we consider the Navier-Stokes equations with incompressible condition approximated by pressure stabilization method. Namely we consider

In this note, we reorganize and summarize the paper [8] by the author and R. Matsui. More information and detail proofs can be found in [8]

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the following equations:

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla \pi_\alpha = f & t \in (0,\infty), x \in \Omega, \\ \nabla \cdot u_\alpha = \Delta \pi_\alpha / \alpha & t \in (0,\infty), x \in \Omega, \\ u_\alpha(0,x) = a_\alpha & x \in \Omega, \\ u_\alpha(t,x) = 0, \qquad \partial_n \pi_\alpha(t,x) = 0 & x \in \partial\Omega. \end{cases} \tag{NSa}$$

(NSa) may be considered as a singular perturbation of (NS). As $\alpha \to \infty$, (NSa) tends to (NS) formally and we cancel the Neumann boundary condition for the pressure.

There are many results concerning the stationary Stokes equations and Navier-Stokes equations by using the pressure stabilization method (for example [1],[7]). However there are few results concerning the nonstationary Stokes equations and Navier-Stokes equations. As far as the authors know, only the result due to Prohl [9] is known as the results concerning the nonstationary problem. In [9], Prohl considered the sharp a priori estimate for the pressure stabilization method under some assumptions and showed the following error estimates:

$$\begin{aligned} &\|u_\alpha - u\|_{L^\infty([0,T],L_2(\Omega))} + \|\tau(\pi_\alpha - \pi)\|_{L^\infty([0,T],W_2^{-1}(\Omega))} \leq C\alpha^{-1}, \\ &\|u_\alpha - u\|_{L^\infty([0,T],W_2^1(\Omega))} + \|\sqrt{\tau}(\pi_\alpha - \pi)\|_{L^\infty([0,T],L_2(\Omega))} \leq C\alpha^{-1/2}, \end{aligned} \tag{1.1}$$

where $\tau = \tau(t) = \min(t,1)$. He proved a priori error estimate by using energy method. In other words, he proved that if we can prove the existence of the local in time solution to (NSa), the solution to (NSa) satisfies (1.1). So goal of this paper is to show the existence theorem for (NSa) and the error estimates. In this paper, we shall use the maximal regularity theorem in order to prove the local in time existence theorem and the error estimate in the L_p in time and the L_q in space framework with $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Here, the maximal regularity theorem means that each term in the abstract Cauchy problem is well-defined and has the same regularity. To be precisely, when we consider the Cauchy problem

$$\partial_t u(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0, \tag{1.2}$$

where X be a Banach space, A be closed linear unbounded operator in X with dense domain $D(A)$ and $f : \mathbb{R}_+ \to X$ is a given function has the maximal regularity, the maximal regularity theorem means for each $f \in L_p(\mathbb{R}_+, X)$ there exists a unique solution u to (1.2) almost everywhere and satisfying $\partial_t u, Au \in L_p(\mathbb{R}_+, X)$. However it is difficult to analyze equations (NSa) as it is by using the maximal regularity theorem because the regularity of solution to the first equation is different from the one of the second equations in (NSa). For this purpose, in order to adjust the regularity of the solution to their equations, we consider the following equations instead of approximated incompressible conditions in (NSa):

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega \qquad (\forall \varphi \in \widehat{W}^1_{q'}(\Omega)) \tag{1.3}$$

for $1 < q < \infty$. We notice that (1.3) is a weak form of the approximated incompressible condition $\nabla \cdot u_\alpha = \alpha^{-1}\Delta\pi_\alpha$ and $\partial_n \pi_\alpha = 0$ on $\partial\Omega$. We call (1.3) approximated weak incompressible condition in this paper. Therefore we consider

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla \pi_\alpha = f & t \in (0,\infty), x \in \Omega, \\ u_\alpha(0,x) = a_\alpha & x \in \Omega, \\ u_\alpha(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{1.4}$$

under the approximated weak incompressible condition (1.3) in L^q-framework.

2 Main results

Before we describe main theorem, we shall introduce some functional spaces and notations throughout this paper. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a, b, \ldots$. The values of constants C and $C_{a,b,\ldots}$ may change from line to line. For $1 < q < \infty$, let $q' = q/(q-1)$. For any two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y and we write $\mathcal{L}(X) = \mathcal{L}(X,X)$ for short. $\mathrm{Hol}(U,X)$ denotes the set of all X-valued holomorphic functions defined on a complex domain U. As the complex domain where a resolvent parameter belongs, we use $\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$ and $\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0\}$ for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. For any domain D, Banach space X and $1 \leq q \leq \infty$, $L_q(D,X)$ denotes the usual Lebesgue space of X-valued functions defined on D and $\|\cdot\|_{L_q(D,X)}$ denotes its norm. We use the notation $L_q(D) = L_q(D,\mathbb{R})$, $\|\cdot\|_{L_q(D)} = \|\cdot\|_{L_q(D,\mathbb{R})}$ and for $a, b, \ldots, c \in L_q(D)$, $\|(a,b,\ldots,c)\|_{L_q(D)} = \|a\|_{L_q(D)} + \|b\|_{L_q(D)} + \cdots + \|c\|_{L_q(D)}$. In a similar way, for $1 \leq q \leq \infty$ and a positive integer m, $W_q^m(D,X)$ denotes the Sobolev spaces of X-valued functions of defined on D. We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion. For $1 \leq p, q \leq \infty$, $B_{q,p}^{2(1-1/p)}(D)$ denotes the real interpolation space defined by $B_{q,p}^{2(1-1/p)}(D) = (L_q(D), W_q^2(D))_{1-1/p,p}$. For a Banach space X and some $\gamma_0 \in \mathbb{R}$, we set

$$\begin{aligned} L_{p,\gamma_0}(\mathbb{R},X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R},X) \mid \|e^{-\gamma t} f\|_{L_p(\mathbb{R},X)} < \infty,\ (\gamma \geq \gamma_0)\}, \\ L_{p,\gamma_0,(0)}(\mathbb{R},X) &= \{f(t) \in L_{p,\gamma_0}(\mathbb{R},X) \mid f(t) = 0\ (t < 0)\}, \\ W^1_{p,\gamma_0,(0)}(\mathbb{R},X) &= \{f(t) \in L_{p,\gamma_0,(0)}(\mathbb{R},X) \mid f'(t) \in L_{p,\gamma_0}(\mathbb{R},X)\}. \end{aligned}$$

In order to deal with the pressure term, we use the following functional spaces:

$$\begin{aligned} L_{q,\mathrm{loc}}(D) &= \{f \mid f|_K \in L_q(K),\ K \text{ is any compact set in } D\}, \\ \widehat{W}_q^1(D) &= \{\theta \in L_{q,\mathrm{loc}}(D) \mid \nabla\theta \in L_q(D)^n\}. \end{aligned}$$

Since our proof is based on Fourier analysis, we next introduce the Fourier transform and the Laplace transform. We define the Fourier transform, its inverse Fourier transform, the Laplace transform and its inverse Laplace transform by

$$\begin{aligned} \hat{f}(\xi) &= \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, & \mathcal{F}_\xi^{-1}[f](x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)d\xi, \\ \mathcal{L}_t[f](\lambda) &= \mathcal{F}_t[e^{-\gamma t} f(t)](\tau), & \mathcal{L}_\tau^{-1}[f](t) &= e^{\gamma t}\mathcal{F}_\tau^{-1}[f](t), \end{aligned}$$

respectively, where $x, \xi \in \mathbb{R}^n$, $\lambda = \gamma + i\tau \in \mathbb{C}$ and $x \cdot \xi$ is usual inner product: $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$. Furthermore, we define the Fourier-Laplace transform by

$$\mathcal{L}_t[\mathcal{F}_x[v(t,x)]](\lambda, \xi) = \mathcal{F}_{t,x}[e^{-\gamma t} v(t,x)](\lambda, \xi) = \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} e^{-(\lambda t + i x \cdot \xi)} v(t,x) dx \right) dt.$$

By using Fourier transform and Laplace transform, we define $H^s_{p,\gamma_0}(\mathbb{R}, X)$ for a Banach space X. For $\lambda = \gamma + i\tau$, we define the operator Λ^s_γ as

$$(\Lambda^s_\gamma f)(t) = \mathcal{L}^{-1}_\tau [|\lambda|^s \mathcal{L}_t[f](\lambda)](t) = e^{\gamma t} \mathcal{F}^{-1}_\tau [(\tau^2 + \gamma^2)^{s/2} \mathcal{F}_t[e^{-\gamma t} f(t)](\tau)](t).$$

For $0 < s < 1$ and $\gamma_0 > 0$, we define the space $H^s_{p,\gamma_0}(\mathbb{R}, X)$ as

$$H^s_{p,\gamma_0}(\mathbb{R}, X) = \{ f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid \|e^{-\gamma t} \Lambda^s_\gamma f\|_{L_p(\mathbb{R},X)} < \infty (\forall \gamma \geq \gamma_0) \}.$$

In this paper, we assume next assumption for our domain Ω.

Assumption 2.1. *Let $n/2 < q < \infty$ and $n < r < \infty$. Let Ω be a uniform $W^{2-1/r}_r$ domain introduced in [5] and $L_q(\Omega)$ has the Helmholtz decomposition.*

Here the assumption on a uniformly $W^{2-1/r}_r$ domain is used when we reduce the problem on the bounded domain to one on the bent half-space and on the whole space. According to Galdi [6], that "$L_q(\Omega)$ has the Helmholtz decomposition" is equivalent that the following weak Neumann problem is uniquely solvable: for $f \in L_q(\Omega)$,

$$(\nabla \theta, \nabla \varphi) = (f, \nabla \varphi) \qquad (\forall \varphi \in \widehat{W}^1_{q'}(\Omega)).$$

The map P_Ω and Q_Ω are defined by $Q_\Omega f = \theta$, where θ is the solution to the above weak Neumann problem and $P_\Omega f = f - \nabla Q_\Omega f$. P_Ω is called the Helmholtz projection. We remark that if $q = 2$, $L_2(\Omega)$ has the Helmholtz decomposition for any Ω (see Galdi [6]).

First main result is concerned with the local in time existence theorem for (1.4) with approximated weak incompressible condition (1.3).

Theorem 2.1. *Let $n \geq 2$, $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Let $\alpha > 0$ and $T_0 \in (0, \infty)$. For any $M > 0$, assume that the initial data $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega)$ and the external force $f \in L_p((0,T_0), L_q(\Omega)^n)$ satisfy*

$$\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p((0,T_0), L_q(\Omega)^n)} \leq M. \tag{2.1}$$

Then, there exists $T^* \in (0, T_0)$ depending on only M such that (1.4) under (1.3) has a unique solution (u_α, π_α) of the following class:

$$u_\alpha \in W^1_p((0,T^*), L_q(\Omega)^n) \cap L_p((0,T^*), W^2_q(\Omega)^n), \qquad \pi_\alpha \in L_p((0,T^*), \widehat{W}^1_q(\Omega)).$$

Moreover the following estimate holds:

$$\|u_\alpha\|_{L_\infty((0,T^*),L_q(\Omega))} + \|(\partial_t u_\alpha, \nabla^2 u_\alpha, \nabla \pi_\alpha)\|_{L_p((0,T^*),L_q(\Omega))} + \|\nabla u_\alpha\|_{L_r((0,T^*),L_q(\Omega))} \leq C_{n,p,q,T^*}$$

for $1/p - 1/r \leq 1/2$.

Here we state the outline of the proof of main theorem (Theorem 2.1). We show Theorem 2.1 by using the contraction mapping principle with two type maximal regularity theorems (Theorem 2.2 and Theorem 2.9). In order to prove Theorem 2.2, we use the Weis' operator valued Fourier multiplier theorem. For this purpose, we have to show the existence of $\mathcal{R}$-bouned solution operator to the generalized resolvent problem of (1.4) (see Theorem 2.7 for detail). In order to prove Theorem 2.9, we need the some estimate of semigroup $T_\alpha(t)$ for linearlized problem of (1.4). For this purpose, we have to show the resolvent estimate (Corollary 2.8), which is a corollary of Theorem 2.7. Therefore our main task is to show Theorem 2.7.

We shall explain the proof of Theorem 2.1 in more detail. In order to prove Theorem 2.1, we use the contraction mapping principle and maximal L_p-L_q regularity theorem for the following linearized problems corresponding to (1.4):

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & t>0, x \in \Omega, \\ u_\alpha(t,x) = 0 & x \in \partial\Omega, \\ u_\alpha(0,x) = a_\alpha & x \in \Omega \end{cases} \tag{2.2}$$

under the approximated weak incompressible condition

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega). \tag{2.3}$$

First result is concerned with the maximal L_p-L_q regularity theorem for (2.2) under (2.3) with $a_\alpha = 0$.

Theorem 2.2. *Let $1 < p, q < \infty$ and $\alpha > 0$. Then there exists a positive number γ_0 such that the following assertion holds: for any $f, g \in L_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega))$, (2.2) under (2.3) with $a_\alpha = 0$ has a unique solution:*

$$u_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, W^2_q(\Omega)) \cap W^1_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega)), \qquad \pi_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, \widehat{W}^1_q(\Omega)).$$

Moreover, the following estimate holds:

$$\begin{aligned} &\|e^{-\gamma t}(\partial_t u_\alpha, \gamma u_\alpha, \Lambda_\gamma^{\frac{1}{2}}\nabla u_\alpha, \Lambda^{1/2}_{\gamma+\alpha}(\nabla\cdot u_\alpha), \nabla^2 u_\alpha, \nabla\pi_\alpha)\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\qquad \le C_{n,p,q}\|e^{-\gamma t}(f, \alpha g)\|_{L_p(\mathbb{R},L_q(\Omega))} \end{aligned}$$

for any $\gamma \ge \gamma_0$.

Remark 2.3. *By the property of Helmholtz decomposition, we can solve (2.3) for $u_\alpha, g \in L_q(\Omega)$ and we see $\pi_\alpha = \alpha Q_\Omega(u_\alpha - g)$.*

In order to prove Theorem 2.2, we use the operator valued Fourier multiplier theorem due to Weis [13]. This theorem needs $\mathcal{R}$-boundedness of solution operator. To this end, we first introduce the definition of $\mathcal{R}$-boundedness.

Definition 2.4. *The family of the operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ $(j = 1,\dots,N)$ and for all sequences $\{\gamma_j(u)\}_{j=1}^N$ of independent, symmetric, $\{-1,1\}$-valued random variables on [0,1], there holds the inequality:*

$$\int_0^1 \|\sum_{j=1}^N \gamma_j(u)T_jf_j\|_Y^p du \le C\int_0^1 \|\sum_{j=1}^N \gamma_j(u)f_j\|_X^p du.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}(\mathcal{T})$.

Remark 2.5. *According to [3], the following properties concerning $\mathcal{R}$-boundedness is known. From Definition 2.4, $\mathcal{R}$-boundedness of the family of operators implies uniform boundedness.*

$$\|T\|^p_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|T(x)\|^p_Y \leq \mathcal{R}(\mathcal{T}).$$

Moreover it is well-known that $\mathcal{R}$-bounds behave like norms. Namely, the following properties hold.

(i) *Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X,Y)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S} = \{T+S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S})$.*

(ii) *Let X, Y, Z be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X,Y)$ and $\mathcal{S} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}(\mathcal{S})\mathcal{R}(\mathcal{T})$.*

The following theorem is the operator valued Fourier multiplier theorem proved by Weis [5] for $X = Y = L_q(\Omega)$.

Theorem 2.6. *Let $1 < p, q < \infty$ and $M(\tau) \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X,Y))$ be satisfy*

$$\mathcal{R}(\{M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) = c_0 < \infty, \quad \mathcal{R}(\{|\tau|\partial_\tau M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) = c_1 < \infty.$$

Then, T_M defined by $[T_M f](t) = \mathcal{F}^{-1}_\xi[M(\tau)\mathcal{F}_x[f](\tau)](t) (f \in \mathcal{S}(\mathbb{R}, X))$ is the bounded operator from $L_p(\mathbb{R}, X)$ to $L_p(\mathbb{R}, Y)$. Moreover, the following estimate holds:

$$\|T_M f\|_{L_p(\mathbb{R},Y)} \leq C(c_0 + c_1)\|f\|_{L_p(\mathbb{R},X)} \qquad (f \in L_p(\mathbb{R}, X)),$$

where C is a positive constant depending on p, X.

In order to prove the maximal L_p-L_q regularity theorem with the help of Theorem 2.6, we need the $\mathcal{R}$-boundedness for solution operator to the following generalized resolvent problem

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & \text{in } \Omega, \\ u_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

under the approximated weak incompressible condition (2.3), where the resolvent parameter λ varies in $\Sigma_{\varepsilon,\lambda_0}$ $(0 < \varepsilon < \pi/2, \lambda_0 > 0)$.

We can show the existence of the $\mathcal{R}$-boundedness operator to (2.4) under (2.3) as follows:

Theorem 2.7. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\Omega) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\Omega)\}$, then there exist a $\lambda_0 > 0$ and operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in Hol(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), W^2_q(\Omega)^n)), \qquad \mathcal{P}(\lambda) \in Hol(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), \widehat{W}^1_q(\Omega)))$$

such that for any $f, g \in L_q(\Omega)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to (2.4) under (2.3) and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^n)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda+\alpha)^{1/2}(\nabla \cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

By Remark 2.5, we can prove the resolvent estimate for (2.4) under (2.3).

Corollary 2.8. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let $\lambda_0 > 0$ be a number obtained in Theorem 2.7. For $f, g \in L_q(\Omega)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, there exists a unique solution (u_α, π_α) to (2.4) under (2.3) which satisfies the following inequality:*

$$\|(\lambda u_\alpha, \lambda^{1/2}\nabla u_\alpha, \nabla^2 u_\alpha, (\lambda+\alpha)^{1/2}(\nabla\cdot u_\alpha), \nabla\pi_\alpha)\|_{L_q(\Omega)} \le C\|(f, \alpha g)\|_{L_q(\Omega)}.$$

Let $\mathcal{A}_\alpha$ be the linear operator defined by $\mathcal{A}_\alpha u_\alpha = \Delta u_\alpha - \alpha\nabla Q_\Omega u_\alpha$ and $\mathcal{D}(\mathcal{A}_\alpha) = \{u \in W_q^2(\Omega)^n \mid u|_{\partial\Omega} = 0\}$. By Corollary 2.8 with $g = 0$, we see that $\mathcal{A}_\alpha$ generates the semigroup $\{T_\alpha(t)\}_{t\ge 0}$ on $L_q(\Omega)^n$. Moreover there exists a positive constant $C > 0$ such that for any $a_\alpha \in L_q(\Omega)^n$, $u_\alpha(t) = T_\alpha(t)a_\alpha$ satisfies

$$\|(u_\alpha, t^{1/2}\nabla u_\alpha, t\nabla^2 u_\alpha, t\partial_t u_\alpha)\|_{L_q(\Omega)} \le Ce^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)} \qquad (t > 0).$$

By the equations (2.2), we have

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} \le \|\partial_t u_\alpha\|_{L_q(\Omega)} + \|-\Delta u_\alpha\|_{L_q(\Omega)} \le Ct^{-1}e^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)}, \tag{2.5}$$

which means that we can not estimate the pressure term $\nabla\pi_\alpha$ near $t = 0$. On the other hands, since $\pi_\alpha = \alpha Q_\Omega u_\alpha$ is the pressure associated with $u_\alpha = T_\alpha(t)a_\alpha$ and $\nabla\pi_\alpha = \alpha(u_\alpha - P_\Omega u_\alpha)$, (u_α, π_α) enjoys (2.2) under (2.3) and $\nabla\pi_\alpha$ satisfies the following estimate:

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} = \alpha\|u_\alpha - P_\Omega u_\alpha\|_{L_q(\Omega)} \le \alpha\|u_\alpha\|_{L_q(\Omega)} \le C\alpha e^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)},$$

which implies the boundedness of $\nabla\pi_\alpha$ near $t = 0$ and $\|\nabla\pi_\alpha\|_{L_\infty((0,T),L_q(\Omega))} \le C\alpha e^{\lambda_0 T}\|a_\alpha\|_{L_q(\Omega)}$. This is the effect of the pressure stabilization method.

By real interpolation, we can see the following maximal L_p-L_q regularity theorem for (2.2) with $f = g = 0$.

Theorem 2.9. *Let $\alpha > 0$ and $1 < p, q < \infty$. Let λ_0 be a number obtained in Theorem 2.7. For $a_\alpha \in B_{q,p}^{2(1-1/p)}(\Omega)$, $u_\alpha = T_\alpha(t)a_\alpha$ satisfy*

$$\begin{aligned}
\|e^{-\lambda_0 t}(\partial_t u_\alpha, \nabla^2 u_\alpha)\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B_{q,p}^{2(1-1/p)}(\Omega)},\\
(\gamma-\lambda_0)^{1/p}\|e^{-\gamma t}u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{L_q(\Omega)},\\
(\gamma-\lambda_0)^{1/(2p)}\|e^{-\gamma t}\nabla u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B_{q,p}^{2(1-1/p)}(\Omega)}
\end{aligned}$$

for any $\gamma > \lambda_0$. Moreover $\pi_\alpha = \alpha Q_\alpha u_\alpha$ satisfy

$$\begin{aligned}
\|e^{-\lambda_0 t}\nabla\pi_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B_{q,p}^{2(1-1/p)}(\Omega)},\\
\|\nabla\pi_\alpha\|_{L_\infty(0,T),L_q(\Omega))} &\le C_{n,p,q}\alpha e^{\lambda_0 T}\|a_\alpha\|_{L_q(\Omega)}
\end{aligned}$$

for any $T > 0$.

Next we consider the error estimate between the solution (u, π) to (NS) under the weak incompressible condition $(u, \nabla\varphi)_\Omega = 0$ for $\varphi \in \widehat{W}_{q'}^1(\Omega)$ and solution (u_α, π_α) to (1.4) under (1.3). To this end, setting $u_E = u - u_\alpha$ and $\pi_E = \pi - \pi_\alpha$, we see that (u_E, π_E) enjoys that

$$\begin{cases} \partial_t u_E - \Delta u_E + \nabla \pi_E + N(u_E, u_\alpha) = 0, & t \in (0,\infty), x \in \Omega, \\ u_E(0,x) = a_E, & x \in \Omega, \\ u_E(t,x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

where $N(u_E, u_\alpha) = (u_E \cdot \nabla)u_E + (u_E \cdot \nabla)u_\alpha + (u_\alpha \cdot \nabla)u_E$ and $a_E = a - a_\alpha$ under the approximated weak incompressible condition

$$(u_E, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_E, \nabla\varphi)_\Omega + \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega) \tag{2.7}$$

for $1 < q < \infty$. In a similar way to Theorem 2.1, we consider (2.2) under (2.7) for $a_\alpha = a_E$. By Theorem 2.2 with $f = 0$, $g = \alpha^{-1}\nabla\pi$ and Theorem 2.9, we obtain the following theorems:

Theorem 2.10. *Let $1 < p, q < \infty$ and $\alpha > 0$. Let γ_0 be a positive number obtained in Theorem 2.7. If usual Stokes equations under the weak incompressible condition has a unique solution (u, π) in $(L_{p,\gamma_E,(0)}(\mathbb{R}, W^2_q(\Omega)^n) \cap W^1_{p,\gamma_E,(0)}(\mathbb{R}, L_q(\Omega)^n)) \times L_{p,\gamma_E,(0)}(\mathbb{R}, \widehat{W}^1_q(\Omega))$, (2.2) under (2.7) with $a_E = 0$ has a unique solution:*

$$u_E \in L_{p,\gamma_E,(0)}(\mathbb{R}, W^2_q(\Omega)^n) \cap W^1_{p,\gamma_E,(0)}(\mathbb{R}, L_q(\Omega)^n), \qquad \pi_E \in L_{p,\gamma_E,(0)}(\mathbb{R}, \widehat{W}^1_q(\Omega)).$$

Moreover, the following estimate holds.

$$\begin{aligned} \|e^{-\gamma t}(\partial_t u_E, \alpha u_E, \Lambda_\gamma^{\frac{1}{2}} \nabla u_E, \nabla^2 u_E, \Lambda^{1/2}_{\gamma+\alpha}(\nabla \cdot u_E), \nabla \pi_E)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ \leq C_{n,p,q} \|e^{-\gamma t} \nabla \pi\|_{L_p(\mathbb{R}, L_q(\Omega))} \end{aligned}$$

for any $\gamma \geq \gamma_E$.

Theorem 2.11. *Let $1 < p, q < \infty$ and $\alpha > 0$. Let λ_0 be a number obtained in Theorem 2.7. For $a_E \in B^{2(1-1/p)}_{q,p}(\Omega)$, $u_E = T_\alpha(t)a_E$ and $\pi_E = \alpha Q_\Omega u_E - \pi$ satisfy*

$$\begin{aligned} \|e^{-\lambda_0 t}(\partial_t u_E, \nabla^2 u_E, \nabla \pi_E)\|_{L_p((0,\infty), L_q(\Omega))} &\leq C_{n,p,q} \|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)}, \\ (\gamma - \lambda_0)^{1/p} \|e^{-\gamma t} u_E\|_{L_p((0,\infty), L_q(\Omega))} &\leq C_{n,p,q} \|a_E\|_{L_q(\Omega)}, \\ (\gamma - \lambda_0)^{1/(2p)} \|e^{-\gamma t} \nabla u_E\|_{L_p((0,\infty), L_q(\Omega))} &\leq C_{n,p,q} \|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \end{aligned}$$

for any $\gamma > \lambda_0$. If $\pi \in L_\infty((0,\infty), \widehat{W}^1_q(\Omega))$, π_E satisfies

$$\|e^{-\lambda_0 t} \nabla \pi_E\|_{L_\infty((0,T), L_q(\Omega))} \leq C\alpha \|a_E\|_{L_q(\Omega)} + \|\nabla \pi\|_{L_\infty((0,\infty), L_q(\Omega))}$$

for any $T > 0$.

By above two theorems, we can obtain the following theorem concerned with the error estimates.

Theorem 2.12. *Let $n \geq 2$, $n/2 < q < \infty$, $\max\{1, n/q\} < p < \infty$ and $\alpha > 0$. Let T^* be a positive constant obtained in Theorem 2.1 and (u_α, π_α) be a solution obtained in Theorem 2.1. For any $M > 0$, assume that $a_E \in B^{2(1-1/p)}_{q,p}(\Omega)$ satisfies*

$$\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq M\alpha^{-1}. \tag{2.8}$$

Then there exists $T^\flat \in (0, T^*)$ *such that* (2.6) *has a unique solution* (u_E, π_E) *which satisfies*

$$\begin{aligned}&\|u_E\|_{L_\infty((0,T^\flat),L_q(\Omega))} + \|\nabla u_E\|_{L_r((0,T^\flat),L_q(\Omega))}\\ &\qquad + \|(\nabla^2 u_E, \partial_t u_E, \nabla \pi_E)\|_{L_p((0,T^\flat),L_q(\Omega))} \le C_{n,p,q,T^\flat}\alpha^{-1}\end{aligned} \tag{2.9}$$

for $1/p - 1/r \le 1/2$.

Remark 2.13. (2.9) *means the following error estimates for the Navier-Stokes equations:*

$$\begin{aligned}&\|u - u_\alpha\|_{L_\infty((0,T^\flat),L_q(\Omega))} \le C\alpha^{-1},\\ &\|(\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha))\|_{L_p((0,T^\flat),L_q(\Omega))} \le C\alpha^{-1},\end{aligned}$$

In comparison with the result due to Prohl [9], we can extend L_2 *framework to* L_q *framework with respect to the error estimate.*

3 Preliminary

In this section, we shall introduce some lemmas and definitions, which plays important role for our proof. Before we describe some propositions and lemmas, we introduce the notation of symbols. Set

$$\begin{aligned}&r = |\xi'|, \qquad \omega_\lambda = \sqrt{\lambda + r^2}, \qquad \omega = \sqrt{\lambda + \alpha + r^2},\\ &\mathcal{E}(z) = e^{-z(x_n + y_n)}, \qquad \mathcal{M}(a, b, x_n) = \frac{e^{-ax_n} - e^{-bx_n}}{a - b},\end{aligned} \tag{3.1}$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Here $\mathcal{E}(\omega_\lambda)$ is the symbol corresponding to heat equation and $\mathcal{M}(\omega_\lambda, r, x_n)$ is the symbol corresponding to Stokes equations.

We next introduce some lemmas. In order to apply the operator-valued Fourier multiplier theorem proved by Weis [13], we need the $\mathcal{R}$-boundedness of solution operator to (2.2). However since it is difficult to prove $\mathcal{R}$-boundedness directly from its definition, we first introduce the following sufficient condition for showing $\mathcal{R}$-boundedness of solution operator given in Theorem 3.3 in Enomoto and Shibata [4].

Theorem 3.1. *Let* $1 < q < \infty$ *and* $0 < \varepsilon < \pi/2$. *Let* $m(\lambda, \xi)$ *be a function defined on* $\Sigma_\varepsilon \times (\mathbb{R}^n \backslash \{0\})$ *such that for any multi-index* $\beta \in \mathbb{N}_0^n (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ *there exists a constant* C_β *depending on* β *and* λ *such that*

$$|\partial_\xi^\beta m(\lambda, \xi)| \le C_\beta |\xi|^{-|\beta|}$$

for any $(\lambda, \xi) \in \Sigma_\varepsilon \times (\mathbb{R}^n \backslash \{0\})$. *Let* K_λ *be an operator defined by*

$$[K_\lambda f](x) = \mathcal{F}_\xi^{-1}[m(\lambda, \xi)\mathcal{F}_x[f](\xi)](x).$$

Then the set $\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}$ *is* $\mathcal{R}$*-bounded on* $\mathcal{L}(L_q(\mathbb{R}^n))$ *and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n))}(\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}) \le C \max_{|\beta| \le n+2} C_\beta$$

with some constant C *that depends solely on* q *and* n.

To prove the $\mathcal{R}$-boundedness of the solution operator in $\mathbb{R}^n_+$, we use the following lemma proved by Shibata and Shimizu [12] (see Lemma 5.4 in [12]).

Lemma 3.2. *Let $0<\varepsilon<\pi/2$, $1<q<\infty$. Let $m(\lambda,\xi')$ be a function defined on Σ_ε such that for any multi-index $\delta'\in\mathbb{N}_0^{n-1}$ there exists a constant $C_{\delta'}$ depending on δ', ε and N such that*

$$|\partial_{\xi'}^{\delta'}m(\lambda,\xi')|\le C_{\delta'}r^{-|\delta'|}.$$

Let $K_j(\lambda,m)$ $(j=1,\dots,5)$ be the operators defined by

$$\begin{aligned}
[K_1(\lambda,m)g](x)&=\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\lambda,\xi')r\mathcal{E}(\omega_\lambda)\widetilde{g}(\xi',y_n)\right](x')dy_n,\\
[K_2(\lambda,m)g](x)&=\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\lambda,\xi')r^2\mathcal{M}(\omega_\lambda,r,x_n+y_n)\widetilde{g}(\xi',y_n)\right](x')dy_n,\\
[K_3(\lambda,m)g](x)&=\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\lambda,\xi')|\lambda|^{1/2}r\mathcal{M}(\omega_\lambda,r,x_n+y_n)\widetilde{g}(\xi',y_n)\right](x')dy_n,\\
[K_4(\lambda,m)g](x)&=\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\lambda,\xi')\omega r\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)\widetilde{g}(\xi',y_n)\right](x')dy_n,\\
[K_5(\lambda,m)g](x)&=\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m(\lambda,\xi')|\lambda|^{1/2}r\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)\widetilde{g}(\xi',y_n)\right](x')dy_n.
\end{aligned}$$

Then, the sets $\{(\tau\partial_\tau)^\ell K_j(\lambda,m)\mid\lambda\in\Sigma_\varepsilon\}$ $(j=1,\dots,5,\ell=0,1)$ are $\mathcal{R}$-bounded families in $\mathcal{L}(L_q(\mathbb{R}^n_+))$. Moreover, there exists a constant $C_{n,q,\varepsilon}$ such that

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n_+))}(\{(\tau\partial_\tau)^\ell K_j(\lambda,m)\mid\lambda\in\Sigma_\varepsilon\})\le C_{n,q,\varepsilon}\qquad(j=1,\dots,5,\ell=0,1).$$

This lemma is proved in a similar way to Lemma 5.4 in [12] with the following lemma.

Lemma 3.3. *For $0<\varepsilon<\pi/2$, let $\lambda\in\Sigma_\varepsilon$.*

(i) *There exist positive constants C_1,C_2 and C_3 depending on ε such that the following inequalities hold:*

$$|\omega_\lambda|\ge C_1(|\lambda|^{1/2}+r),\quad C_2(\alpha^{1/2}+|\lambda|^{1/2}+r)\le Re\ \omega\le C_3(\alpha^{1/2}+|\lambda|^{1/2}+r).\tag{3.2}$$

(ii) *There exist positive constants C such that the following inequalities hold:*

$$\begin{aligned}
&|D^{\delta'}_{\xi'}r^s|\le Cr^{s-|\delta'|},\\
&|D^{\delta'}_{\xi'}\omega_\lambda^s|\le C(|\lambda|^{1/2}+r)^{s-|\delta'|},\\
&|D^{\delta'}_{\xi'}\omega^s|\le C(\alpha^{1/2}+|\lambda|^{1/2}+r)^{s-|\delta'|},\\
&|D^{\delta'}_{\xi'}(r+\omega_\lambda)^s|\le C(|\lambda|^{1/2}+r)^s r^{-|\delta'|},\\
&|D^{\delta'}_{\xi'}(r+\omega)^s|\le C(|\lambda|^{1/2}+\alpha^{1/2}+r)^s r^{-|\delta'|},\\
&|D^{\delta'}_{\xi'}(\omega+\omega_\lambda)^s|\le C(|\lambda|^{1/2}+\alpha^{1/2}+r)^s(|\lambda|^{1/2}+r)^{-|\delta'|}
\end{aligned}\tag{3.3}$$

for any $s\in\mathbb{R}$ and multi-index δ.

(iii) *There exist positive constants C such that the following inequalities hold:*

$$
\begin{aligned}
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-rx_n}\}| &\le Cr^{-|\delta'|}e^{-(1/2)rx_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-\omega_\lambda x_n}\}| &\le C(|\lambda|^{1/2}+r)^{-|\delta'|}e^{-d(|\lambda|^{1/2}+r)x_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-\omega x_n}\}| &\le C(\alpha^{1/2}+|\lambda|^{1/2}+r)^{-|\delta'|}e^{-d(\alpha^{1/2}+|\lambda|^{1/2}+r)x_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell \mathcal{M}(\omega_\lambda,r,x_n)\}| &\le C(x_n \text{ or } |\lambda|^{-1/2})e^{-drx_n}r^{-|\delta'|},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell \mathcal{M}(\omega_\lambda,\omega,x_n)\}| &\le C(x_n \text{ or } \alpha^{-1/2})e^{-d(|\lambda|^{1/2}+r)x_n}(|\lambda|^{1/2}+r)^{-|\delta'|} \qquad (3.4)
\end{aligned}
$$

for $\ell=0,1$ and any multi-index δ' and $(\xi',x_n)\in(\mathbb{R}^{n-1}\backslash\{0\})\times(0,\infty)$, where d is a positive constant independent of ε and δ'.

Proof. (i) (3.2) are proved by elementary calculation.
(ii) Let $f(t)=t^{s/2}$. By Bell formula, we see

$$
D_\xi^\delta r^s=\sum_{\ell=1}^{|\delta|}f^{(\ell)}(r^2)\sum_{\delta_1+\cdots+\delta_\ell=\delta,|\delta_i|\ge1}\Gamma^\ell_{\delta_1,\dots,\delta_\ell}(D_\xi^{\delta_1}r^2)\cdots(D_\xi^{\delta_\ell}r^2),
$$

where $\Gamma^\ell_{\alpha_1,\dots,\alpha_\ell}$ is some constant and $f^{(\ell)}(t)=d^\ell f(t)/dt^\ell$. Since $|D_\xi^{\delta_j}r^2|\le 2r^{2-|\delta_j|}$, we can obtain the first estimate. We can prove the other estimates in a similar way to the first estimate taking the elementary estimate: $|\lambda+|\xi|^2|\ge(\sin\varepsilon)(|\lambda|+|\xi|^2)$ $(0<\varepsilon<\pi/2$, $\xi\in\mathbb{R}^n)$ into account.

(iii) It is sufficient to prove the last estimate with $\ell=0$ in (3.4), since we can prove the other estimates similarly.
Since $\mathcal{M}(\omega_\lambda,\omega,x_n)=-x_n\int_0^1 e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}d\theta$, by Bell formula, we have

$$
\begin{aligned}
|D_{\xi'}^{\delta'}e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}| \le C_{\delta'}\sum_{\ell=1}^{|\delta'|}&x_n^\ell e^{-(c_1(1-\theta)(|\lambda|^{1/2}+r)+c_2\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}\\
&\times((1-\theta)(|\lambda|^{1/2}+r)^{1-|\delta'_1|}+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r)^{1-|\delta'_1|})\\
&\times\cdots\times((1-\theta)(|\lambda|^{1/2}+r)^{1-|\delta'_\ell|}+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r)^{1-|\delta'_\ell|}),
\end{aligned}
$$

where we used $|e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}|=e^{-((1-\theta)\mathrm{Re}\omega_\lambda+\theta\mathrm{Re}\omega)x_n}$. Setting $c=\min(c_1,c_2)$, we see

$$
|D_{\xi'}^{\delta'}e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}|\le C_{\delta'}e^{-(c/2)((1-\theta)(|\lambda|^{1/2}+r)+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}(|\lambda|^{1/2}+r)^{-|\delta'|},
$$

which implies

$$
\begin{aligned}
|D_{\xi'}^{\delta'}\mathcal{M}(\omega_\lambda,\omega,x_n)|&\le C_{\delta'}\int_0^1 e^{-(c/2)((1-\theta)(|\lambda|^{1/2}+r)+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}d\theta x_n(|\lambda|^{1/2}+r)^{-|\delta'|}\\
&=C_{\delta'}\int_0^1 e^{-(c/2)(|\lambda|^{1/2}+r)x_n}e^{-\theta(c/2)\alpha^{1/2}x_n}d\theta x_n(|\lambda|^{1/2}+r)^{-|\delta'|}.
\end{aligned}
$$

By integrating this right hand side, we have

$$
|D_{\xi'}^{\delta'}\mathcal{M}(\omega_\lambda,\omega,x_n)|\le C_{\delta'}(c/2)^{-1}\alpha^{-1/2}e^{-(c/2)(|\lambda|^{1/2}+r)x_n}(|\lambda|^{1/2}+r)^{-|\delta'|}. \qquad (3.5)
$$

On the other hands, by $e^{-\theta(c/2)\alpha^{1/2}x_n} \leq 1$, we have

$$|D^{\delta'}_{\xi'}\mathcal{M}(\omega_\lambda, \omega, x_n)| \leq C_{\delta'} x_n e^{-(c/2)(|\lambda|^{1/2}+r)x_n}(|\lambda|^{1/2} + r)^{-|\delta'|}. \tag{3.6}$$

Therefore, we obtain the last estimate with $\ell = 0$ in (3.4). □

4 $\mathcal{R}$-boundedness of the solution operator to resolvent problem

Goal of this section is to prove the $\mathcal{R}$-boundedness of the solution operator to the following resolvent problem (2.4) in Ω:

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & \text{in } \Omega, \\ u_\alpha = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\lambda \in \Sigma_{\varepsilon,\lambda_0}(0 < \varepsilon < \pi/2, \lambda_0 > 0)$ under the approximated weak incompressible condition (2.3). Our method is based on cut-off technique. For this purpose, we shall first prove the whole space case. Secondly we shall prove the half-space case by using the result for the whole space case and some lemma introduced in section 3. Next we shall prove the bent half-space case by reducing to the result for the half-space case with the change of variable. Finally we shall prove the bounded domain case by using the result for the whole space and the bent half-space case with cut-off technique. In this paper, we focus the whole space case and the half-space case (see [8] for the bent half-space case and the bounded domain case).

4.1 Problem in the whole space

In this subsection, we shall prove the following theorem:

Theorem 4.1. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\mathbb{R}^n) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n)\}$. Then, there exist operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in \mathit{Hol}(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n), W^2_q(\mathbb{R}^n)^n)), \qquad \mathcal{P}(\lambda) \in \mathit{Hol}(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n), \widehat{W}^1_q(\mathbb{R}^n)))$$

such that for any $f, g \in L_q(\mathbb{R}^n)^n$ and $\lambda \in \Sigma_\varepsilon$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to (2.4) under (2.3) for the case $\Omega = \mathbb{R}^n$ and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^n)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda + \alpha)^{1/2}(\nabla \cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

Proof. In order to prove the $\mathcal{R}$-boundedness of solution operator by using Theorem 3.1, we shall obtain the solution formula to (2.4) under (2.3) by using Fourier transform. By the property of Helmholtz projection, we know $\nabla\pi_\alpha = \alpha\nabla Q_{\mathbb{R}^n}(u_\alpha - g)$ and $\mathcal{F}[\nabla Q_{\mathbb{R}^n} v] = |\xi|^{-2}\xi(\xi \cdot \widehat{v})$. Applying the Fourier transform to (2.4), we obtain the following solution formula : $u_{\alpha,j}(x) = u_j(x) + u^E_{\alpha,j}(x)$ and $\pi_\alpha(x) = \pi(x) + \pi^E_\alpha(x)$, where (u, π) is the solution to Stokes equations given by

$$u_j(x) = \mathcal{F}^{-1}_\xi \left[\frac{1}{\lambda + |\xi|^2} \widehat{f}_j(\xi) \right](x) - \sum_{k=1}^n \mathcal{F}^{-1}_\xi \left[\frac{\xi_j \xi_k}{(\lambda + |\xi|^2)|\xi|^2} \widehat{f}_k(\xi) \right](x), \tag{4.1}$$

$$\pi(x) = -i \sum_{k=1}^n \mathcal{F}^{-1}_\xi \left[\frac{\xi_k}{|\xi|^2} \hat{f}_k(\xi) \right](x) \tag{4.2}$$

for $j = 1, \ldots, n$ and the error term $(u^E_\alpha, \pi^E_\alpha)$ given by

$$u^E_{\alpha,j} = \sum_{k=1}^n \mathcal{F}^{-1}_\xi \left[\frac{\xi_j \xi_k (\hat{f}_k(\xi) - \alpha \hat{g}_k)}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right](x),$$
$$\pi^E_\alpha = i \sum_{k=1}^n \mathcal{F}^{-1}_\xi \left[\frac{\xi_k (\lambda + |\xi|^2)(\hat{f}_k(\xi) - \alpha \hat{g}_k(\xi))}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right](x) \tag{4.3}$$

for $j = 1, \ldots, n$. Since in the whole space case, it is well-known that the solution operator to Stokes equations is $\mathcal{R}$-bounded ([12] for detail), we consider the only error term $(u^E_\alpha, \pi^E_\alpha)$. By Leibniz rule, for $\ell = 0, 1$, we obtain

$$\begin{aligned} &\left| (\tau \partial_\tau)^\ell D^\delta_\xi \frac{(\lambda + \alpha)\xi_j \xi_k}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right| \le C_{\varepsilon,\delta} |\xi|^{-|\delta|}, \quad \left| (\tau \partial_\tau)^\ell D^\delta_\xi \frac{(\lambda + \alpha)^{1/2} \xi_m \xi_j \xi_k}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right| \le C_{\varepsilon,\delta} |\xi|^{-|\delta|}, \\ &\left| (\tau \partial_\tau)^\ell D^\delta_\xi \frac{\xi_m \xi_n \xi_j \xi_k}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right| \le C_{\varepsilon,\delta} |\xi|^{-|\delta|}, \quad \left| (\tau \partial_\tau)^\ell D^\delta_\xi \frac{\xi_j \xi_k (\lambda + |\xi|^2)}{|\xi|^2(\lambda + \alpha + |\xi|^2)} \right| \le C_{\varepsilon,\delta} |\xi|^{-|\delta|}, \end{aligned} \tag{4.4}$$

which implies from Theorem 3.1

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^{\tilde{N}})}(\{(\tau \partial_\tau)^\ell (G_{\lambda,\alpha} \mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^n)}(\{(\tau \partial_\tau)^\ell (\nabla \mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1).$$

This completes the proof of Theorem 4.1. □

Remark 4.2. *By Theorem 4.1, we see that the existence of the solution* (u_α, π_α) *to the resolvent problem* (2.4)*. Moreover by Theorem 2.6 and Remark 2.5,* (u_α, π_α) *satisfies the following resolvent estimate:*

$$\|(\lambda u_\alpha, \lambda^{1/2} \nabla u_\alpha, \nabla^2 u_\alpha, (\lambda + \alpha)^{1/2}(\nabla \cdot u_\alpha), \nabla \pi_\alpha)\|_{L_q(\mathbb{R}^n)} \le C_{n,q,\varepsilon} \|(f, \alpha g)\|_{L_q(\mathbb{R}^n)}.$$

4.2 Problem in the half-space

In this section we shall prove the following theorem:

Theorem 4.3. *Let* $\alpha > 0$, $1 < q < \infty$ *and* $0 < \varepsilon < \pi/2$. *Set* $X_q(\mathbb{R}^n_+) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n_+)\}$. *Then, there exist operator families* $\mathcal{U}(\lambda)$ *and* $\mathcal{P}(\lambda)$ *with*

$$\mathcal{U}(\lambda) \in Hol(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n_+), W^2_q(\mathbb{R}^n_+)^n), \qquad \mathcal{P}(\lambda) \in Hol(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n_+), \widehat{W}^1_q(\mathbb{R}^n_+)),$$

such that for any $f, g \in L_q(\mathbb{R}^n_+)^n$ *and* $\lambda \in \Sigma_\varepsilon$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, *where* $F = (f, \alpha g)$, *is a unique solution to* (2.4) *under* (2.3) *and* $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ *satisfies the following estimates:*

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+)^{\tilde N})}(\{(\tau\partial_\tau)^\ell (G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+)^n)}(\{(\tau\partial_\tau)^\ell (\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda+\alpha)^{1/2}(\nabla \cdot u))$ *and* $\widetilde{N} = 1 + n + n^2 + n^3$.

In order to prove Theorem 4.3 by Lemma 3.2, we shall obtain the solution formula to (2.4) under (2.3). By density argument, we may let $f, g \in C_0^\infty(\mathbb{R}^n_+)$. In this case, equation (2.4) under (2.3) is equivalent to the following equations:

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f, & \nabla \cdot u_\alpha - \alpha^{-1}\Delta \pi_\alpha = \nabla \cdot g \quad \text{in } \mathbb{R}^n_+, \\ u|_{\partial\mathbb{R}^n_+} = 0, & \partial_n \pi_\alpha|_{\partial\mathbb{R}^n_+} = 0. \end{cases} \tag{4.5}$$

We shall obtain the solution formula to (4.5). For this purpose, we extend the external force f and g to the whole space. For $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$, let $F = (f_1^e, \ldots, f_{n-1}^e, f_n^o)$ and $G = (g_1^e, \ldots, g_{n-1}^e, g_n^o)$, where

$$f_j^e(x) = \begin{cases} f_j(x', x_n) & (x_n > 0) \\ f_j(x', -x_n) & (x_n < 0) \end{cases}, \quad f_n^o(x) = \begin{cases} f_n(x', x_n) & (x_n > 0) \\ -f_n(x', -x_n) & (x_n < 0) \end{cases},$$

where $x' = (x_1, \ldots, x_{n-1})$. We consider the resolvent problem with F and G:

$$\lambda U_\alpha - \Delta U_\alpha + \nabla \Pi_\alpha = F, \qquad \nabla \cdot U_\alpha = \alpha^{-1}\Delta \Pi_\alpha + \nabla \cdot G \qquad \text{in } \mathbb{R}^n. \tag{4.6}$$

Here we remark that from the definition of our extension, (U_α, Π_α) enjoys the boundary condition

$$U_{\alpha,n}(x', 0) = 0, \qquad \partial_n \Pi_\alpha(x', 0) = 0. \tag{4.7}$$

By the result for the whole space and the definition of our extension, the following estimates hold:

$$\begin{aligned} \|(\lambda U_\alpha, \lambda^{1/2}\nabla U_\alpha, \nabla^2 U_\alpha, (\lambda+\alpha)^{1/2}(\nabla \cdot U_\alpha), \nabla \Pi_\alpha)\|_{L_q(\mathbb{R}^n)} &\leq C\|(F, \alpha G)\|_{L_q(\mathbb{R}^n)} \\ &\leq C\|(f, \alpha g)\|_{L_q(\mathbb{R}^n_+)}. \end{aligned} \tag{4.8}$$

Setting $u_\alpha = w_\alpha + U_\alpha$ and $\pi_\alpha = \rho_\alpha + \Pi_\alpha$, we see that to solve (4.5) is equivalent to solve

$$\begin{cases} \lambda w_\alpha - \Delta w_\alpha + \nabla \rho_\alpha = 0, & \nabla \cdot w_\alpha = \Delta \rho_\alpha / \alpha \quad \text{in } \mathbb{R}^n_+, \\ (w_\alpha)_j|_{x_n=0} = h_j|_{x_n=0}, & \partial_n \rho_\alpha|_{x_n=0} = 0, \end{cases} \tag{4.9}$$

where $h_j = -(U_\alpha)_j$ for $j = 1, \ldots, n-1$ and $h_n = 0$. Applying div and $(\lambda + \alpha - \Delta)\Delta$ to the first equation in (4.9), we obtain

$$(\lambda + \alpha - \Delta)\Delta \rho_\alpha = 0, \qquad (\lambda + \alpha - \Delta)(\lambda - \Delta)\Delta w_\alpha = 0. \tag{4.10}$$

By applying the partial Fourier transform defined by

$$\widetilde{g}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} g(x', x_n)dx'$$

to (4.9) and (4.10) , we have

$$\begin{aligned}
\lambda(\widetilde{w_\alpha})_j + r^2(\widetilde{w_\alpha})_j - \partial_n^2(\widetilde{w_\alpha})_j + (i\xi_j)\widetilde{\rho_\alpha} = 0,\\
\lambda(\widetilde{w_\alpha})_n + r^2(\widetilde{w_\alpha})_n - \partial_n^2(\widetilde{w_\alpha})_n + \partial_n\widetilde{\rho_\alpha} = 0,\\
i\xi'\cdot\widetilde{w_\alpha}' + \partial_n(\widetilde{w_\alpha})_n = \alpha^{-1}(-r^2\widetilde{\rho_\alpha} + \partial_n^2\widetilde{\rho_\alpha}),\\
(\widetilde{w_\alpha})_j(\xi', 0) = \widetilde{h}_j(\xi', 0), \qquad (\widetilde{w_\alpha})_n(\xi', 0) = 0, \qquad \partial_n\widetilde{\rho_\alpha}(\xi', 0) = 0
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&(\lambda + \alpha + r^2 - D_n^2)(r^2 - D_n^2)\widetilde{\rho_\alpha} = 0,\\
&(\lambda + \alpha + r^2 - D_n^2)(\lambda + r^2 - D_n^2)(r^2 - D_n^2)\widetilde{w_\alpha} = 0,
\end{aligned} \tag{4.12}$$

where $i\xi'\cdot\widetilde{w_\alpha}' = \sum_{j=1}^{n-1}(i\xi_j)(\widetilde{w_\alpha})_j$. Since from (4.12), we see the solution $(\widetilde{w_\alpha}, \widetilde{\rho_\alpha})$ can be expressed by

$$\widetilde{\rho_\alpha} = pe^{-rx_n} + qe^{-\omega x_n}, \qquad (\widetilde{w_\alpha})_j = a_j e^{-rx_n} + b_j e^{-\omega_\lambda x_n} + c_j e^{-\omega x_n} \tag{4.13}$$

for $j = 1, \dots, n$, we shall find the solution to (4.11) having the form (4.13). By substituting (4.13) to (4.11), we see

$$\begin{cases}
\lambda a_j + (i\xi_j)p = 0, \quad -\alpha c_j + (i\xi_j)q = 0,\\
\lambda a_n - rp = 0, \qquad -\alpha c_n - \omega q = 0,\\
i\xi'\cdot a' - ra_n = 0, \quad i\xi'\cdot b' - \omega_\lambda b_n = 0, \quad i\xi'\cdot c' - \omega c_n = \alpha^{-1}(\alpha+\lambda)q,\\
a_j + b_j + c_j = \widetilde{h}_j, \quad a_n + b_n + c_n = 0, \qquad -rp - \omega q = 0
\end{cases}$$

for $j = 1, \dots, n-1$. Setting $\mathcal{A} = \lambda(\omega_\lambda\omega - r^2)$ and $\mathcal{B} = \alpha\omega(\omega_\lambda - r)$, we see

$$\begin{aligned}
&p = -\frac{\alpha\lambda\omega i}{r(\mathcal{A}+\mathcal{B})}\xi'\cdot\widetilde{h}', && q = -\frac{r}{\omega}p, &&\\
&a_j = -\frac{i\xi_j}{\lambda}p, && b_j = \widetilde{h}_j + \frac{i\xi_j}{\lambda}p + \frac{i\xi_j r}{\alpha\omega}p, && c_j = -\frac{i\xi_j r}{\alpha\omega}p,\\
&a_n = \frac{r}{\lambda}p, && b_n = -\frac{r}{\lambda}p - \frac{r}{\alpha}p, && c_n = \frac{r}{\alpha}p.
\end{aligned}$$

Therefore, we obtain the solution formula $(\widetilde{w_\alpha})_j = \widetilde{w}_j + \widetilde{w_\alpha}_j^E$ and $\widetilde{\rho_\alpha} = \widetilde{\rho} + \widetilde{\rho_\alpha}^E$, where $(\widetilde{w}, \widetilde{w_\alpha}^E, \widetilde{\rho}, \widetilde{\rho_\alpha}^E)$ is given

$$\begin{aligned}
\widetilde{w}_j &= \widetilde{h}_j e^{-\omega_\lambda x_n} + \frac{\xi_j}{r}\xi'\cdot\widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n),\\
\widetilde{w_\alpha}_j^E &= -\frac{\xi_j}{r}\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\xi'\cdot\widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n) - \frac{\xi_j}{\omega_\lambda + r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\xi'\cdot\widetilde{h}'\mathcal{M}(\omega, \omega_\lambda, x_n),\\
\widetilde{w}_n &= i\xi'\cdot\widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n),
\end{aligned}$$

$$\widetilde{w_{\alpha n}}^E = \frac{\mathcal{B}}{\mathcal{A}+\mathcal{B}} i\xi' \cdot \widetilde{h'}\mathcal{M}(\omega_\lambda, r, x_n) - \frac{\alpha\omega\lambda}{(\omega+\omega_\lambda)(\mathcal{A}+\mathcal{B})} i\xi' \cdot \widetilde{h'}\mathcal{M}(\omega, \omega_\lambda, x_n),$$
$$\widetilde{\rho} = -\frac{\omega_\lambda + r}{r} i\xi' \cdot \widetilde{h'} e^{-rx_n},$$
$$\widetilde{\rho}^E = \frac{\omega_\lambda + r}{r} \frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}} i\xi' \cdot \widetilde{h'} e^{-rx_n} + \frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}} i\xi' \cdot \widetilde{h'} e^{-\omega x_n}.$$

Since the symbol $\mathcal{M}(a, b, x_n)$ defined by (3.1) has the following properties:

$$\partial_n \mathcal{M}(a, b, x_n) = -e^{-ax_n} - b\mathcal{M}(a, b, x_n),$$
$$\partial_n^2 \mathcal{M}(a, b, x_n) = (a+b)e^{-ax_n} + b^2\mathcal{M}(a, b, x_n)$$

and by $g(0) = -\int_0^\infty \partial_n g(y_n) dy_n$, we have

$$\widetilde{h}(\xi', 0)e^{-ax_n} = \int_0^\infty \mathcal{E}(a)(a - D_n)\tilde{h}(\xi', y_n) dy_n,$$
$$\tilde{h}(\xi', 0)\mathcal{M}(a, b, x_n) = \int_0^\infty \{\mathcal{E}(a)\tilde{h}(y_n) + \mathcal{M}(a, b, x_n + y_n))(b - D_n)\tilde{h}(\xi', y_n)\} dy_n,$$

where $\mathcal{E}(z)$ is defined by (3.1). Therefore, setting $\bar{\xi_j} = \xi_j/r$, we obtain

$$\begin{aligned}
w_j(x) = &\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\mathcal{E}(\omega_\lambda)(\omega_\lambda - D_n)\widetilde{h_j}(\xi', y_n)](x') dy_n \\
&+ \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_j}\bar{\xi_k}(\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n) \\
&\qquad + \mathcal{M}(\omega_\lambda, r, x_n + y_n)(r - D_n) r\widetilde{h_k}(\xi', y_n))](x') dy_n,
\end{aligned}$$

$$\begin{aligned}
(w_\alpha)_j^E(x) = &-\sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_j}\bar{\xi_k} \frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}} (\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n) \\
&\qquad + \mathcal{M}(\omega_\lambda, r, x_n + y_n)(r - D_n) r\widetilde{h_k}(\xi', y_n))](x') dy_n \\
&+ \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{r\bar{\xi_j}\bar{\xi_k}}{\omega_\lambda + r} \frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}} (\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n) \\
&\qquad + \mathcal{M}(\omega_\lambda, \omega, x_n + y_n)(\omega - D_n) r\widetilde{h_k}(\xi', y_n))](x') dy_n,
\end{aligned}$$

$$\begin{aligned}
w_n(x) = &\sum_{k=1}^{n-1} i \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k}(\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n) \\
&\qquad + \mathcal{M}(\omega_\lambda, r, x_n + y_n)(r - D_n) r\widetilde{h_k}(\xi', y_n))](x') dy_n,
\end{aligned}$$

$$\begin{aligned}
(w_\alpha)_n^E(x) = &\sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k} \frac{i\mathcal{B}}{\mathcal{A}+\mathcal{B}} (\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n) \\
&\qquad + \mathcal{M}(\omega_\lambda, r, x_n + y_n)(r - D_n) r\widetilde{h_k}(\xi', y_n))](x') dy_n \\
&+ \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k} \frac{\omega i}{\omega_\lambda + \omega} \frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}} (\mathcal{E}(\omega_\lambda) r\widetilde{h_k}(\xi', y_n)
\end{aligned}$$

$$+ \mathcal{M}(\omega_\lambda, \omega, x_n + y_n)(\omega - D_n) r \widetilde{h}_k(\xi', y_n))](x') dy_n,$$

$$\rho(x) = -\sum_{k=1}^{n-1} i \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{\omega_\lambda + r}{r} \mathcal{E}(r)(r - D_n) r \bar{\xi}_k \widetilde{h}_k(\xi', y_n)](x') dy_n,$$

$$(\rho_\alpha)^E(x) = \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi}_k \frac{\omega_\lambda + r}{r} \frac{\mathcal{A}}{\mathcal{A} + \mathcal{B}} i \mathcal{E}(r)(r - D_n) r \widetilde{h}_k(\xi', y_n)](x') dy_n$$

$$+ \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi}_k \frac{\alpha\lambda}{\mathcal{A} + \mathcal{B}} i \mathcal{E}(\omega)(\omega - D_n) r \widetilde{h}_k(\xi', y_n)] dy_n. \tag{4.14}$$

We remark that (w, ρ) is the solution to the usual Stokes equations and (w^E, ρ^E) is the error between the solution to Stokes equations and Stokes equations approximated by pressure stabilization. Since Shibata and Shimizu [12] proved $\mathcal{R}$-boundedness of solution operator to Stokes equations, it is sufficient to consider $(w_\alpha^E, \rho_\alpha^E)$ only. For this purpose, we prepare the following lemma.

Lemma 4.4. *Let $0 < \varepsilon < \pi/2$ and $\alpha > 0$. For any multi-index δ' and $(\lambda, \xi', x_n) \in \Sigma_\varepsilon \times (\mathbb{R}^{n-1} \backslash \{0\}) \times (0, \infty)$, $m(\lambda, \xi') = r(\omega_\lambda + r)^{-1}, \omega(\omega_\lambda + \omega)^{-1}, \mathcal{A}(\mathcal{A} + \mathcal{B})^{-1}, \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A} + \mathcal{B})^{-1}$ enjoy*

$$|\partial_{\xi'}^{\delta'} m(\lambda, \xi')| \le C r^{-|\delta'|}, \tag{4.15}$$

where C is a positive constant which is dependent of ε and δ'.

Proof. We first show that $m(\lambda, \xi') = r(\omega_\lambda + r)^{-1}$ and $\omega(\omega_\lambda + \omega)^{-1}$ enjoy (4.15). By Leibniz rule with (3.3), we see

$$\left| D_{\xi'}^{\delta'} \frac{r}{\omega_\lambda + r} \right| \le C \sum_{\delta' = \delta_1' + \delta_2'} r^{1 - |\delta_1'|} \frac{r^{-|\delta_2'|}}{|\lambda|^{1/2} + r} \le C r^{-|\delta'|},$$

$$\left| D_{\xi'}^{\delta'} \frac{\omega}{\omega_\lambda + \omega} \right| \le C \sum_{\delta' = \delta_1' + \delta_2'} (|\lambda|^{1/2} + \alpha^{1/2} + r) r^{-|\delta_1'|} \frac{r^{-|\delta_2'|}}{(|\lambda|^{1/2} + \alpha^{1/2} + r)} \le C r^{-|\delta'|}.$$

In order to prove $m(\lambda, \xi') = \mathcal{A}(\mathcal{A} + \mathcal{B})^{-1}, \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A} + \mathcal{B})^{-1}$, we shall consider $D_{\xi'}^{\delta'}(\mathcal{A} + \mathcal{B})$. Since

$$\mathcal{A} + \mathcal{B} = (\lambda + \alpha)\omega(\omega_\lambda - r) + \lambda r(\omega - r) = \frac{\lambda(\lambda + \alpha)\omega}{\omega_\lambda + r} + \frac{\lambda(\lambda + \alpha) r}{\omega + r},$$

we have

$$\left| D_{\xi'}^{\delta'}(\mathcal{A} + \mathcal{B}) \right| \le C|\lambda|(|\lambda| + \alpha) \left\{ \frac{|\lambda|^{1/2} + \alpha^{1/2} + r}{|\lambda|^{1/2} + r} + \frac{r}{|\lambda|^{1/2} + \alpha^{1/2} + r} \right\} r^{-|\delta'|}$$

$$\le C|\lambda|(|\lambda|^{1/2} + \alpha^{1/2})^2 (|\lambda|^{1/2} + \alpha^{1/2} + r)(|\lambda|^{1/2} + r)^{-1} r^{-|\delta'|}. \tag{4.16}$$

Since $|\arg[\omega(\omega+r)/r(\omega_\lambda+r)]| < \pi-\varepsilon$, we know $\omega r^{-1}(\omega+r)(\omega_\lambda+r)^{-1} \in \Sigma_\varepsilon$, which implies that

$$\begin{aligned}|\mathcal{A}+\mathcal{B}| &= |\lambda+\alpha||\lambda|\left|\frac{r}{\omega+r}\right|\left|\frac{\omega}{\omega_\lambda+r}\cdot\frac{\omega+r}{r}+1\right| \\ &\geq C(|\lambda|^{1/2}+\alpha^{1/2})^2|\lambda|r(|\lambda|^{1/2}+\alpha^{1/2}+r)^{-1}\left(\left|\frac{\omega}{\omega_\lambda+r}\cdot\frac{\omega+r}{r}\right|+1\right) \\ &\geq C(|\lambda|^{1/2}+\alpha^{1/2})^2|\lambda|(|\lambda|^{1/2}+\alpha^{1/2}+r)(|\lambda|^{1/2}+r)^{-1}.\end{aligned}$$

By Bell's formula with (4.16), we obtain

$$\left|D_{\xi'}^{\delta'}(\mathcal{A}+\mathcal{B})^{-1}\right| \leq C|\lambda|^{-1}(|\lambda|^{1/2}+\alpha^{1/2})^{-2}(|\lambda|^{1/2}+\alpha^{1/2}+r)^{-1}(|\lambda|^{1/2}+r)r^{-|\delta'|},$$

which implies (4.15) for $m(\lambda,\xi') = \mathcal{A}(\mathcal{A}+\mathcal{B})^{-1}, \mathcal{B}(\mathcal{A}+\mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A}+\mathcal{B})^{-1}$. □

Proof of Theorem 4.3. We shall prove Theorem 4.3 by Lemma 3.2 with Lemma 4.4. Set $(w_\alpha)^E_{j,k,\ell}(x)(k=1,\dots,n-1, \ell=1,\dots,6)$ as follows

$$\begin{aligned}(w_\alpha)^E_{j,k,1}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar\xi_j\bar\xi_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\right](x')dy_n, \\ (w_\alpha)^E_{j,k,2}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar\xi_j\bar\xi_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,r,x_n+y_n)r^2\widetilde{h_k}(\xi',y_n)\right](x')dy_n, \\ (w_\alpha)^E_{j,k,3}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar\xi_j\bar\xi_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,r,x_n+y_n)rD_n\widetilde{h_k}(\xi',y_n)\right](x')dy_n, \\ (w_\alpha)^E_{j,k,4}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar\xi_j\bar\xi_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\right](x')dy_n, \\ (w_\alpha)^E_{j,k,5}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar\xi_j\bar\xi_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)\omega r\widetilde{h_k}(\xi',y_n)\right](x')dy_n, \\ (w_\alpha)^E_{j,k,6}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar\xi_j\bar\xi_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)rD_n\widetilde{h_k}(\xi',y_n)\right](x')dy_n.\end{aligned}$$

Setting $K_{\alpha,\ell,j}(h_k) = (w_\alpha)^E_{j,k,\ell}(x)$ for $\ell=1,2,4,5$, by Lemma 3.2, Lemma 4.4 and (4.8), we see that $K_{\alpha,\ell,j}$ is $\mathcal{R}$-bounded. Since $h_k = -(U_\alpha)_k$, $U_\alpha = \mathcal{U}_{\mathbb{R}^n}(\lambda)F$, where $\mathcal{U}_{\mathbb{R}^n}(\lambda)$ is the solution operator in $\mathbb{R}^n$ and $F=(f,\alpha g)$, setting $\mathcal{V}_{j,k,\ell}(\lambda)F = K_{\alpha,j,\ell}((\mathcal{U}_{\mathbb{R}^n}(\lambda)F)_k)$, we see that $G_{\lambda,\alpha}\mathcal{V}_{j,k,\ell}(\lambda)F = K_{\alpha,\ell,j}(G_{\lambda,\alpha}(\mathcal{U}_{\mathbb{R}^n}((\lambda)F)$ is $\mathcal{R}$-bounded by Remark 2.5.

Since Lemma 3.2 and Lemma 4.4 and the relation:

$$\begin{aligned}\lambda(w_\alpha)^E_{j,k,3}(x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}[\bar\xi_j\bar\xi_k&\frac{\mathcal{A}}{(\mathcal{A}+\mathcal{B})}\mathcal{M}(\omega_\lambda,r,x_n+y_n) \\ &\times r|\lambda|^{1/2}\frac{\lambda}{|\lambda|}(|\lambda|^{1/2}D_n\widetilde{h_k}(\xi',y_n))](x')dy_n,\end{aligned}$$

we see there exists a $\mathcal{R}$-bouned operator $K_{\alpha,3,j}$ such that $K_{\alpha,3,j}(|\lambda|^{1/2}D_nh_k) = \lambda(w_\alpha)^E_{j,k,3}(x)$. Setting $\lambda\mathcal{V}_{j,k,3}(\lambda)F = K_{\alpha,3,j}(|\lambda|^{1/2}D_n(\mathcal{U}_{\mathbb{R}^n}F)_k)$, we see $\lambda\mathcal{V}_{j,k,3}(\lambda)F$ is $\mathcal{R}$-bounded. In a similar way, we can show that $G_{\lambda,\alpha}\mathcal{V}_{j,k,\ell}(\lambda)F(\ell=3,6)$ is $\mathcal{R}$-bounded. Summing up, setting

$(\mathcal{U}(\lambda)F)_j = \sum_{k,\ell} \mathcal{V}_{j,k,\ell}(\lambda)F$ and $\mathcal{U}(\lambda)F = ((\mathcal{U}(\lambda)F)_j)_{j=1,\dots,n}$, we see $\mathcal{U}(\lambda)F$ is the solution operator in $\mathbb{R}^n_+$ and $G_{\lambda,\alpha}\mathcal{U}(\lambda)F$ is $\mathcal{R}$-bounded.

In the same way, we obtain the results for $(w_\alpha)^E_n(x)$ from the results for $(w_\alpha)^E_j(x)$ and the results for $(\rho_\alpha)^E(x)$ from the equations (2.4) and the results for $(w_\alpha)^E_j(x)$ and $(w_\alpha)^E_n(x)$.

□

5 Application to the approximated Navier-Stokes equations

In this section, we shall prove the local in time existence theorem for (NSa) and (2.6) (Theorem 2.1 and Theorem 2.12) by the method due to Shibata-Kubo [11]. Before we prove these theorems, we shall describe some facts shown by using maximal L_p-L_q regularity theorem (Theorem 2.2).

Let $(w, \tau) = M_T(f)$ be the solution to

$$\begin{cases} \partial_t w - \Delta w + \nabla\tau = f & x \in \Omega, t \in (0,T), \\ w(0,x) = 0 & x \in \Omega, \\ w(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{5.1}$$

under the approximated weak incompressible condition (1.3)

For $f \in L_p((0,T), L_q(\Omega))$, let $f_0(t) = f(t)$ $(0 < t < T)$ and $f_0(t) = 0$ $(t \notin (0,T))$. Then, letting (w,τ) be the solution to Stokes equation for $f = f_0$ on $t \in (0,\infty)$, (w,τ) can define on $t \in \mathbb{R}$. Moreover, this solution satisfies $w(t) = \tau(t) = 0$ $(t \leq 0)$ and (5.1) on $t \in (0,T)$. Furthermore, by Theorem 2.2, the following estimate holds: for $0 < S \leq T$, we have

$$\|\partial_t w\|_{L_p((0,S),L_q(\Omega))} \leq e^{\gamma S}\|e^{-\gamma t}\partial_t w\|_{L_p((0,T),L_q(\Omega))} \leq C_{n,p,q}e^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))}. \tag{5.2}$$

Similarly we have

$$\|\nabla^2 w\|_{L_p((0,S),L_q(\Omega))} + \|\nabla\tau\|_{L_p((0,S),L_q(\Omega))} \leq C_{n,p,q}e^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))}. \tag{5.3}$$

Moreover taking into account the fact about Bessel potential space:

$$\|e^{-\gamma t}u\|_{L_q(\mathbb{R},X)} \leq C\|e^{-\gamma t}\Lambda^\alpha_\gamma u\|_{L_p(\mathbb{R},X)} \leq C\gamma^{-(\beta-\alpha)}\|e^{-\gamma t}\Lambda^\beta_\gamma u\|_{L_p(\mathbb{R},X)} \tag{5.4}$$

for Banach space X, $1 < p < q < \infty$, $\alpha = 1/p - 1/q$, $\alpha < \beta < \infty$ and $\gamma \geq 0$ and the estimate:

$$\|e^{-\gamma t}u\|_{L_\infty(\mathbb{R},X)} \leq C\|e^{-\gamma t}\Lambda^\alpha_\gamma u\|_{L_p(\mathbb{R},X)}$$

for $0 < \alpha - 1/p < 1$ and $1 < p < \infty$ (see [2]), by Theorem 2.2 we obtain

$$\begin{aligned} &\|\nabla w\|_{L_r((0,S),L_q(\Omega))} + \|w\|_{L_\infty((0,S),L_q(\Omega))} \\ &\quad\leq Ce^{\gamma S}\|e^{-\gamma t}\Lambda^\alpha_1\nabla w\|_{L_q(\mathbb{R},L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma t}\Lambda^1_1 w\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\quad\leq Ce^{\gamma S}\|e^{-\gamma t}\Lambda^{1/2}_1\nabla w\|_{L_p(\mathbb{R},L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma t}\Lambda^1_1 w\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\quad\leq Ce^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))}, \end{aligned} \tag{5.5}$$

where $1/p - 1/r \le 1/2$.

Letting $\beta = n/(2q)$ and $\ell_k (k = 1, 2, 3)$ are the positive constants satisfying

$$0 < \frac{1}{p} - \frac{1}{\beta p \ell_1} \le \frac{1}{2}, \qquad 0 < \frac{1}{p} - \frac{1}{(1-\beta)p\ell_2} \le \frac{1}{2}, \qquad \beta + \frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} = 1$$

and setting

$$\gamma = 1/(\ell_3 p), \qquad r_1 = \beta p \ell_1, \qquad r_2 = (1-\beta)p\ell_2, \tag{5.6}$$

by Sobolev embedding theorem and Holder's inequality, we obtain

$$\begin{aligned} &\|(v\cdot\nabla)w\|_{L_p((0,S),L_q(\Omega))} \\ &\le S^\gamma \|v\|^{1-\beta}_{L_\infty((0,S),L_q(\Omega))} \|\nabla v\|^{\beta}_{L_{r_1}((0,S),L_q(\Omega))} \|\nabla w\|^{1-\beta}_{L_{r_2}((0,S),L_q(\Omega))} \|\nabla^2 w\|^{\beta}_{L_p((0,S),L_q(\Omega))} \end{aligned} \tag{5.7}$$

for any $v, w \in W^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), W^2_q(\Omega))$ and $0 < S \le T$.

Proof of Theorem 2.1. Setting $u^* = T_\alpha(t)a_\alpha$ and $\pi^* = \alpha Q_\Omega u_\alpha$, by Theorem 2.9 and (2.5), (u^*, π^*) is the solution to (2.2) under (2.3) and satisfies

$$\|e^{-\lambda_0 t}(\partial_t u^*, \nabla^2 u^*, \nabla \pi^*)\|_{L_p((0,\infty),L_q(\Omega))} \le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le CM, \tag{5.8}$$

where $1 < p, q < \infty$ and λ_0 is a positive number obtained in Theorem 2.7. Setting $v_\alpha = u_\alpha - u^*$, and $\rho_\alpha = \pi_\alpha - \pi^*$, we see that what (u_α, π_α) is the solution to (1.4) under (2.3) is equivalent to what (v_α, ρ_α) is the solution to

$$\begin{cases} \partial_t v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = f - N_1(v_\alpha) - N_2(u^*) & t \in (0,T), x \in \Omega, \\ v_\alpha(0,x) = 0 & x \in \Omega, \\ v_\alpha(t,x) = 0 & t \in (0,T),\ x \in \partial\Omega \end{cases} \tag{5.9}$$

under the approximated weak incompressible condition (1.3), where

$$N_1(v_\alpha, u^*) = (v_\alpha\cdot\nabla)v_\alpha + (u^*\cdot\nabla)v_\alpha + (v_\alpha\cdot\nabla)u^*, \qquad N_2(u^*) = (u^*\cdot\nabla)u^*.$$

In order to prove Theorem 2.1, we consider (5.9) under (1.3). For this purpose, we set

$$\begin{aligned} \langle (w,\tau)\rangle_T = &\|\partial_t w\|_{L_p((0,T),L_q(\Omega))} + \|\nabla^2 w\|_{L_p((0,T),L_q(\Omega))} + \|\nabla\tau\|_{L_p((0,T),L_q(\Omega))} \\ &+ \|w\|_{L_\infty((0,T),L_q(\Omega))} + \|\nabla w\|_{L_{r_1}((0,T),L_q(\Omega))} + \|\nabla w\|_{L_{r_2}((0,T),L_q(\Omega))} \end{aligned} \tag{5.10}$$

with r_1, r_2 is defined by (5.6). By (2.1), (5.2), (5.3) and (5.5), we have

$$\langle M_{T^*}(f))\rangle_{T^*} \le C_{n,p,q}e^{\lambda_0 T^*}\|f\|_{L_p((0,T^*),L_q(\Omega))} \le C_{n,p,q}e^{\lambda_0 T^*}M. \tag{5.11}$$

Set $L = C_{n,p,q}e^{\lambda_0 T^*}M$. To prove Theorem 2.1 by contraction mapping principle, we shall define the underlying space $X_{T,L}$ as follows:

$$\begin{aligned} X_{T,L} = \{&(w,\tau) \in W^1_p((0,T), L_q(\Omega)^n) \cap L_p((0,T), W^2_q(\Omega)^n)) \\ &\times L_p((0,T), \widehat{W}^1_q(\Omega)) \mid w|_{t=0} = 0,\ \langle (w,\tau)\rangle_T \le 2L\}. \end{aligned} \tag{5.12}$$

Here the constant T is determined later as the sufficiently small constant. We define the map Φ as

$$\Phi(w,\theta) = M_T(f) - M_T(N_1(v_\alpha, u^*)) - M_T(N_2(u^*)),$$

where M_T is the solution operator to (5.1) under (1.3). We shall prove that Φ is the contraction mapping on $X_{T,L}$. By (5.7) and (5.8) we have

$$\|N_2(u^*)\|_{L_p((0,S),L_q(\Omega))} \le \|(u^*\cdot\nabla)u^*\|_{L_p((0,S),L_q(\Omega))} \le CS^\gamma e^{2\lambda_0 S}M^2$$

for $1 < p \le \infty$ and $n/2 < q < \infty$. By (5.2) the following inequality holds:

$$\langle M_{T^*}(N_2(u^*))\rangle_{T^*} \le C_{n,p,q}e^{2\lambda_0 T^*}\|N_2(u^*)\|_{L_p((0,T^*),L_q(\Omega))} \le C_{n,p,q}(T^*)^\gamma e^{2\lambda_0 T^*}M^2 \qquad (5.13)$$

for $0 < T^* \le T_0$. In a similar way, for $(v_\alpha,\rho_\alpha) \in X_{T^*,L}$ we obtain

$$\|N_1(v_\alpha,u^*)\|_{L_p((0,S),L_q(\Omega))} \le Ce^{\lambda_0 T^*}S^\gamma ML,$$

which implies

$$\langle M_{T^*}(N_1(v_\alpha,u^*))\rangle_{T^*} \le C_{n,p,q}\|N_1(v_\alpha,u^*)\|_{L_p((0,T^*),L_q(\Omega))} \le C(T^*)^\gamma e^{\lambda_0 T^*}ML. \qquad (5.14)$$

Therefore there exists a constant $C = C_{n,p,q,T_0}$ such that

$$\langle\Phi(v_\alpha,\rho_\alpha)\rangle_{T^*} \le L + C(T^*)^\gamma\left(e^{2\lambda_0 T^*}M^2 + e^{\lambda_0 T^*}ML\right)$$

for $(v_\alpha,\rho_\alpha) \in X_{T^*}$. Taking the time $T^*(\le T_0)$ sufficiently small such that $C(T^*)^\gamma e^{\lambda_0 T^*}M \le 1/2$ and $C(T^*)^\gamma e^{2\lambda_0 T^*}M^2 \le L/2$, we have $\langle\Phi(w,\tau)\rangle_{T^*} \le 2L$. Therefore, Φ is the mapping on $X_{T^*,L}$. Moreover taking into account the facts:

$$\Phi(w_1,\tau_1) - \Phi(w_2,\tau_2) = M_{T^*}(N_1(w_2,u^*) - N_1(w_1,u^*))$$

and

$$N_1(w_2,u^*) - N_1(w_1,u^*) = ((w_2-w_1)\cdot\nabla)u^* + (u^*\cdot\nabla)(w_2-w_1)$$

for $(w_i,\tau_i) \in X_{T^*,L}$ $(i=1,2)$, by (5.7), (5.8) and (5.12), we can show the following inequality holds:

$$\|N_1(w_2) - N_1(w_1)\|_{L_p((0,T^*),L_q)} \le C_{n,p,q,T_0}(T^*)^\gamma e^{\lambda_0 T^*}M\langle(w_2,\tau_2)-(w_1,\tau_1)\rangle_{T^*},$$

which implies

$$\langle\Phi(w_1,\tau_1) - \Phi(w_2,\tau_2)\rangle_{T^*} \le C_{n,p,q,T_0}(T^*)^\gamma e^{\lambda_0 T^*}M\langle(w_2,\tau_2)-(w_1,\tau_1)\rangle_{T^*}.$$

Taking T^* sufficiently small such that $C(T^*)^\gamma e^{\lambda_0 T^*}M \le 1/2$ if necessary, we obtain

$$\langle\Phi(w_1,\tau_1) - \Phi(w_2,\tau_2)\rangle_{T^*} \le (1/2)\langle(w_1,\tau_1)-(w_2,\tau_2)\rangle_{T^*}.$$

Therefore, we see that Φ is the contraction mapping on X_{T^*}. By the contraction mapping principle, we see that Φ has fixed point (v_α,ρ_α). Satisfying $\Phi(v_\alpha,\rho_\alpha) = (v_\alpha,\rho_\alpha)$, by (5.13), we see that $(u_\alpha,\pi_\alpha) = (u^*+v_\alpha, \pi^*+\rho_\alpha)$ is the unique solution for (1.4) under (1.3). Therefore we obtain Theorem 2.1. □

Proof of Theorem 2.12. Let (u^*, π^*) be a solution to (2.2) with $f = g = 0$ and $a_\alpha = a_E$. By Theorem 2.9, the following estimates hold.

$$\|e^{-\lambda_0 t}(\partial_t u^*, \nabla^2 u^*, \nabla \pi^*)\|_{L_p((0,\infty),L_q(\Omega))} \le C_{n,p,q}\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le CM\alpha^{-1}, \tag{5.15}$$

where $1 < p, q < \infty$. In order to look for the solution (v_α, ρ_α) of (2.6) as $v_\alpha = u_E - u^*$ and $\rho_\alpha = \pi_E - \pi^*$, we shall obtain the solution to

$$\begin{cases} \partial_t v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = -N_1(v_\alpha, u^*) - N_2(u^*, u_\alpha) & t \in (0,\infty), x \in \Omega, \\ v_\alpha(0,x) = 0 & x \in \Omega, \\ v_\alpha(t,x) = 0, & x \in \partial\Omega, \end{cases} \tag{5.16}$$

under the approximated weak incompressible condition (2.7), where

$$\begin{aligned} N_1(v_\alpha, u^*) &= (v_\alpha \cdot \nabla)v_\alpha + ((u^* + u_\alpha)\cdot\nabla)v_\alpha + (v_\alpha\cdot\nabla)(u^* + u_\alpha), \\ N_2(u^*, u_\alpha) &= (u^*\cdot\nabla)(u^* + u_\alpha) + (u_\alpha\cdot\nabla)u^*. \end{aligned}$$

In a similar way to Theorem 2.1, we shall define underlying space X_{T,L_E} as follows:

$$\begin{aligned} X_{T,L_E} = \{(w,\tau) \in (&W^1_p((0,T), L_q(\Omega)^n) \cap L_p((0,T), W^2_q(\Omega)^n)) \\ &\times L_p((0,T), \widehat{W}^1_q(\Omega)) \mid w|_{t=0} = 0,\ \alpha\langle (w,\tau)\rangle_T \le L_E\}, \end{aligned} \tag{5.17}$$

where $\langle (w,\tau)\rangle_T$ is defined in (5.10). Setting the map Φ defined by

$$\Phi(w,\theta) = -M_{T^*}(N_1(v_\alpha, u^*)) - M_{T^*}(N_2(u^*, u_\alpha)),$$

where $M_T(f)$ is a solution operator to (5.1) under (2.7), we shall estimate $N_1(v_\alpha, u^*)$ and $N_2(u^*, u_\alpha)$ in a similar way to Theorem 2.1. Setting $\beta, \ell_k (k = 1,2,3), \gamma, r_i (i = 1,2)$ as the same positive constant in proof of Theorem 2.1, we see

$$\|N_1(v_\alpha, u^*)\|_{L_p((0,S),L_q(\Omega))} \le \frac{CS^\gamma}{\alpha}\left(\frac{1}{\alpha}L_E^2 + \frac{1}{\alpha}e^{\lambda_0 T^*}ML_E + LL_E\right)$$

and

$$\|N_2(u^*, u_\alpha)\|_{L_p((0,S),L_q(\Omega))} \le C\frac{S^\gamma}{\alpha}\left(\frac{1}{\alpha}e^{2\lambda_0 T^*}M^2 + e^{\lambda_0 T^*}ML\right)$$

for $1 < p < \infty$, by (2.8), (5.2) for $0 < T^\flat \le T^*$, the following inequality holds:

$$\alpha\langle M_{T^\flat}(N_1(v_\alpha, u^*) + N_2(u^*, u_\alpha))\rangle_{T^\flat} \le C_{n,p,q,M,L,L_E}(T^\flat)^\gamma.$$

In a similar way to Theorem 2.1, taking $T^\flat$ sufficiently small if necessary, we can prove that Φ is the contraction mapping on $X_{T^\flat,L_E}$. Therefore we obtain Theorem 2.12. □

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