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A finite element for Stokes with a commuting diagram

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1 Introduction

This paper provides a rather elementary and selfcontained presentation of a finite element space that can be used for Galerkin approximations of the Stokes equation. This finite element space was first defined in [8]. There, the language used was that of differential forms, which makes it possible to give a uniform treatment of all space dimensions, and to place the problem within the larger context of constructing complexes of differential forms. On the other hand this language has not been adopted by all finite element practitioners. Here, we use vectorfields in $\mathbb{R}^3$ and standard vector identities (involving for instance the vector product), and concentrate on the last two spaces in the complexes, which are the ones relevant to Stokes' equation. Compared with existing finite element methods, the advantage of this one is that the space comes equipped with degrees of freedom that are low order (don't involve derivatives) and produce interpolators that satisfy a commuting diagram. This makes the numerical analysis of the stability and convergence of the method very natural.

Hopefully this presentation can reach a larger audience than the rather technical paper [8]. We hope in particular it can provide the information necessary for implementing this method. It was also an honor and a pleasure to attend the RIMS workshop "Mathematical Analysis in Fluid and Gas Dynamics" (July 4 - 6, 2018) where these results were presented. Hopefully this paper can be useful to analysts who are curious about finite element methods. We notice that many theoretical existence proofs for Navier-Stokes equations are based on Galerkin methods. For instance, [4] uses this method with Galerkin spaces defined from eigenfunctions of the Stokes operator, but most of their analysis extends to the type of Galerkin spaces we define here. In this direction it can also be mentioned that some famous analysis tools have genuine extensions to a finite element setting [5][7].

The Stokes equation. The Stokes equation arises as a simplified model in fluid mechanics, and can be derived from the incompressible Navier-Stokes equations by looking for solutions that are constant in time and for which the non-linear term (convection) is negligible. It takes the following form. We fix a bounded connected domain S in $\mathbb{R}^3$ whose boundary is Lipschitz, in the sense that locally it is the graph of a scalar Lipshitz functions. Then we want to find a vector field u on S (modelling fluid velocities) and a scalar field p on S (modelling pressure) such that:

$$-\epsilon\Delta u - \operatorname{grad} p = f, \tag{1}$$

$$\operatorname{div} u = g. \tag{2}$$

The vector field f (modelling an external force) and the scalar field g (which would be 0 for an incompressible fluid) are supposed given. The most important boundary condition is the so-called no-slip boundary condition : all components of u on the boundary ∂S are zero : $u|_{\partial S} = 0$. For p there are no boundary conditions, but for uniqueness we impose that $\int p = 0$. There is a compatibility condition, namely $\int g = 0$.

The appropriate functional framework is to look for u in the Sobolev space $X = \mathrm{H}^1_0(S)^3$ of vector fields that are square integrable with first order partial derivatives that are also square integrable. The no-slip boundary condition is expressed in the subscript. The field p is looked for in $Y = \mathrm{L}^2_0(S)$, where the subscript indicates 0 integral. One supposes that $f \in X' = \mathrm{H}^{-1}(S)^3 = (\mathrm{H}^1_0(S)^3)'$ and $g \in Y' = Y = \mathrm{L}^2_0(S)$.

The variational formulation of the Stokes equation is then:

$$\left\{\begin{array}{l} u \in X \\ p \in Y \end{array}\right. \quad \left\{\begin{array}{lll} \forall u' \in X & \int \epsilon \operatorname{grad} u \cdot \operatorname{grad} u' + \int p \operatorname{div} u' & = \int f \cdot u' \\ \forall p' \in Y & \int p' \operatorname{div} u & = \int g p' \end{array}\right. \tag{3}$$

Given $(f, g) \in X' \times Y'$, there is a unique solution $(u, p) \in X \times Y$.

The main ingredient in the wellposedness of the Stokes equation is the surjectivity of $\operatorname{div} : \mathrm{H}^1_0(S)^3 \to \mathrm{L}^2_0(S)$. This is equivalent to the property that the adjoint operator, namely $\operatorname{grad} : \mathrm{L}^2_0(S) \to \mathrm{H}^{-1}(S)^3$, is injective and has closed range. This again is equivalent to the following inequality, due to Lions and Nečas: There exists $C > 0$ such that for all $u \in \mathrm{L}^2_0(S)$:

$$\|u\|_{\mathrm{L}^2(S)} \leq C \|\operatorname{grad} u\|_{\mathrm{H}^{-1}(S)^3}. \tag{4}$$

For proofs of this inequality, and its relation to the Stokes equation, we refer to Chapter 4 of [4] or Chapters 13 – 15 of [15].

It can also be remarked that on a domain S which is starshaped with respect to a ball, Bogovskiĭ has defined an integral operator that provides an explicit right inverse of $\operatorname{div} : \mathrm{H}^1_0(S)^3 \to \mathrm{L}^2_0(S)$. The Bogovskiĭ operator is the formal adjoint of the regularized Poincaré operator. On bounded Lipschitz domains, a partition of unity technique can be used to glue together regularized Poincaré operators, to provide a homotopy between the identity and a compact operator, and then Fredholm theory shows that $\operatorname{grad} : \mathrm{L}^2(S) \to \mathrm{H}^{-1}(S)^3$ has closed range, from which (4) follows. We refer to [11] for an exposition of these operators and their mapping properties, including the above mentioned gluing construction.

The role of Poincaré operators in the construction of certain finite element spaces (such as the Raviart-Thomas-Nédélec spaces defined below) has been highlighted in [12]. The finite elements we introduce for the Stokes equation are also based on such operators, but this is most apparent when the whole complex is considered (see the last section below).

Galerkin discretizations of the Stokes equation. A conforming discretization of the Stokes equation is to identify finite dimensional subspaces $X_h \subseteq X$ and $Y_h \subseteq Y$, depending on a small parameter h which is typically the maximal diameter of the tetrahedra in some mesh $\mathcal{T}_h$ of the domain S. One solves:

$$\left\{\begin{array}{l} u_h \in X_h \\ p_h \in Y_h \end{array}\right. \left\{\begin{array}{lll} \forall u'_h \in X_h & \int \epsilon \operatorname{grad} u_h \cdot \operatorname{grad} u'_h + \int p_h \operatorname{div} u'_h & = \int f_h \cdot u'_h \\ \forall p'_h \in Y_h & \int p'_h \operatorname{div} u_h & = \int g_h p'_h \end{array}\right. \tag{5}$$

One is interested in the convergence of u_h to u and p_h to p as h goes to 0. A *necessary* condition for this convergence to hold (for all $f \in X'$ and all $g \in Y'$) is the so-called Brezzi inf-sup condition: There exists a constant $C > 0$ such that for all h:

$$\inf_{q_h \in Y_h} \sup_{v_h \in X_h} \frac{\int q_h \operatorname{div} v_h}{\|q_h\|_Y \|v_h\|_X} \geq 1/C. \tag{6}$$

In the context of conforming finite element methods for the Stokes equation this condition is also *sufficient* for convergence (the inf-sup condition can be formulated in the more general context of mixed methods, where an additional condition must hold, that is trivially satisfied for the Stokes equation). It expresses that the pair (X_h, Y_h) must be compatible in a sense : for instance, the L^2-projection from Y_h to the range of div on X_h must be injective. Notice that if Y_h is much smaller than $\operatorname{div} X_h$ then, in the case $g = 0$ of incompressible flow, the numerical method does *not* in general produce divergence free vectorfields u_h. This *can* be a serious problem, even though the numerical method is convergent, as discussed in [13].

Ideally one would like to construct pairs (X_h, Y_h) such that $\operatorname{div} : X_h \to Y_h$ is surjective. Then the method produces divergence free vector fields when it should. However surjectivity is not enough ; when it holds, the Brezzi the inf-sup condition is equivalent to the following property:

> *Stable discrete surjectivity*: There is a constant $C > 0$ such that for each h, for each $q_h \in Y_h$ there is $v_h \in X_h$ such that $\operatorname{div} v_h = q_h$ and $\|v_h\|_X \leq C\|q_h\|_Y$.

This property should be compared with the main step to show that the continuous Stokes equation is well-posed, which we recall was to prove that $\operatorname{div} : X \to Y$ is surjective. The stable discrete surjectivity can be deduced from the continuous surjectivity if one knows how to construct surjective projections $P_h^X : X \to X_h$ and $P_h^Y : Y \to Y_h$ which are bounded in the operator norms of X and Y respectively, uniformly with respect to h, and such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{div}} & Y \\ \downarrow{\scriptstyle P_h^X} & & \downarrow{\scriptstyle P_h^Y} \\ X_h & \xrightarrow{\operatorname{div}} & Y_h \end{array} \tag{7}$$

In practice P_h^Y will often be the L^2-projection onto Y_h. The existence of a uniformly bounded P_h^X (called a Fortin operator) is equivalent to the stable discrete surjectivity, but in practice the way towards a proof of the inf-sup condition is to define a finite element for which a P_h^Y can be constructed a priori (see the discussion in §5.4 and §8.4 in [3]).

The construction of a Stokes pair (X_h, Y_h) equipped with natural operators P_h^Y and P_h^X with the above properties, is delicate. Many existing methods are justified by other means, in particular the macro-element technique of Stenberg (see §8.5 in [3]).

Notice that the first method one might think of, namely to use a tetrahedral mesh (with meshwidth h) and continuous piecewise affine velocities (for X_h) and piecewise constant pressures (for Y_h), is an *unstable* method. For an overview over finite element methods that have been proposed for the Stokes problem, including the various tricks that have been devised to prove the inf-sup condition, we refer to Chapter 8 of [3].

A related model problem. If we replace the viscosity term by a reactive term in the Stokes equation, we arrive at the following problem, for some $\lambda > 0$:

$$\lambda u - \operatorname{grad} p = f, \tag{8}$$

$$\operatorname{div} u = g \tag{9}$$

Now the relevant boundary condition for u is that the normal component on ∂S is 0 (so that the fluid velocity is parallel to the boundary) as in the Euler equation.

The appropriate function space for u is now $Z = \mathrm{H}^0_0(\operatorname{div}, S)$, the space of L^2 vectorfields with divergence in L^2, satisfying this boundary condition (indicated in the subscript). This space is strictly larger than X. The variational formulation of this equation is:

$$\left\{ \begin{array}{l} u \in Z \\ p \in Y \end{array} \right. \quad \left\{ \begin{array}{lll} \forall u' \in Z & \int \lambda u \cdot u' + \int p \operatorname{div} u' & = \int f \cdot u' \\ \forall p' \in Y & \int p' \operatorname{div} u & = \int g p' \end{array} \right. \tag{10}$$

Given $(f, g) \in Z' \times Y'$, there is a unique solution $(u, p) \in Z \times Y$.

Good discretizations are now obtained by defining subspaces $Z_h \subseteq Z$ and $Y_h \subseteq Y$ such that there exists a constant $C > 0$ such that for all h:

$$\inf_{q_h \in Y_h} \sup_{v_h \in Z_h} \frac{\int q_h \operatorname{div} v_h}{\|q_h\|_Y \|v_h\|_Z} \geq 1/C. \tag{11}$$

Good finite elements for this equations have been known for a long time, namely the Raviart-Thomas-Nédélec (RTN) spaces. They provide a pair (Z_h, Y_h) for which $\operatorname{div} : Z_h \to Y_h$ is stably surjective in the above sense. In the spaces Z_h, the vector fields are continuous only in the normal direction across two-dimensional faces, so that they would not provide conforming spaces for the Stokes equation. Recall that a vectorfield that is piecewise smooth with respect to a simplicial mesh, is in $\mathrm{H}^0(\operatorname{div}, S)$ if an only if it is continuous in the normal direction across two-dimensional faces.

We now provide the definition of the lowest order RTN spaces, since they provide a template for the Stokes element we will introduce later.

We suppose that the domain is subdivided into tetrahedra by a simplicial mesh $\mathcal{T}_h$, where h is the mesh-width. The elements of Z_h are the vectorfields in Z that are piecewise of the form:

$$v : x \mapsto ax + b, \tag{12}$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}^3$. On a given tetrahedron T, this is a 4-dimensional space, and an elements v is uniquely determined by the 4 numbers:

$$\int_F v \cdot n_F, \tag{13}$$

which are the integrals, for each triangular face F of T, of the normal component of v (n_F denotes the normal vector of F, for a choice of orientation). Since the normal component of a field of the form (12) is constant, globally determining a vectorfield by the numbers (13) guarantees $\mathrm{H}^0(\mathrm{div}, S)$ continuity. Incidentally this shows that the dimension of Z_h is the number of interior faces.

The degrees of freedom (13) determine a projection onto Z_h, the so-called associated interpolator. One can define it for instance on piecewise smooth vectorfields on S that are elements of Z, the space of which is denoted $\tilde{Z}_h$. The interpolator $I_h^Z : \tilde{Z}_h \to Z_h$ is defined by associating to vectorfield $v \in \tilde{Z}_h$, the element $I_h^Z v$ of Z_h with the same degrees of freedom.

For Y_h one takes the piecewise constant functions on S, with 0 integral. The degree of freedom associated with a tetrahedron T is the integral on T. The associated interpolator I_h^Y is then the L^2 projection on each tetrahedron. We let $\tilde{Y}_h$ denote the space of elements of Y that are piecewise smooth.

These spaces and operators have the property that the following diagram commutes:

$$\begin{array}{ccc} \tilde{Z}_h & \xrightarrow{\mathrm{div}} & \tilde{Y}_h \\ \downarrow I_h^Z & & \downarrow I_h^Y \\ Z_h & \xrightarrow{\mathrm{div}} & Y_h \end{array} \tag{14}$$

Indeed, given that I_h^Z and I_h^Y are projections, it is enough to check that if $I_h^Z u = 0$ then $I_h^Y \mathrm{div}\, u = 0$, and this follows from Stokes' theorem, written:

$$\int_T \mathrm{div}\, u = \sum_F o(T, F) \int_F u \cdot n_F, \tag{15}$$

where the sum is over the faces F of T, given their relative orientation $o(T, F) \in \{+1, -1\}$ (which is $+1$ iff n_F is outward pointing, compared with T).

Ideally, instead of diagram (14), one would like to have commuting projection operators onto Z_h that are defined on larger spaces, especially Z or even $\mathrm{L}^2(S)^3$. For this purpose one can precede the interpolation by a smoothing operator, where the smoothing kernel is adapted to the mesh. For quasi-uniform meshes and periodic domains one can use smoothing by convolution:

$$u \mapsto \phi_{\epsilon h} * u, \tag{16}$$

for a fixed small enough parameter $\epsilon > 0$. Here, $\phi_{\epsilon h}$ is the standard mollifier, scaled so that its support is in a ball of diameter ϵh. For details on how to modify the method to take into account boundary conditions or the possibility that the diameter of the cells in the mesh varies significantly across the domain (while the mesh remains shape-regular), we refer to [10] and [9]. The conclusion is that an interpolator such as the above, defined from degrees of freedom, can be modified by a smoothing step, so that one obtains commuting projections that are bounded in $\mathrm{L}^2(S)$, uniformly in h. From this the stable discrete surjectivity follows.

2 A finite element for Stokes

We now proceed to define a Stokes pair (X_h, Y_h), which, for its properties, mimicks as far as possible the RTN element, in particular in the sense that there are natural degrees of freedom, for which the associated interpolator satisfies a commuting diagram. Incidentally the space Y_h is the same as for the RTN element, as it consists of the piecewise constants, with 0 integral on S. Another source of inspiration for the Stokes element was the Clough-Tocher element for scalar fields of class $\mathrm{C}^1(S)$, in that we define vectorfields on a tetrahedron that are piecewise polynomial with respect to a subdivision of the tetrahedron. Finally it can be remarked that our degrees of freedom coincide with those of [2], though the vectorfields they define are quite different, in particular they do not have piecewise constant divergence.

The subdivision of a tetrahedron we use, as well as the degrees of freedom, are represented in Figure 1.

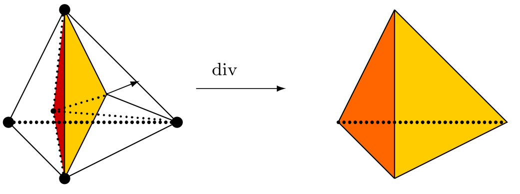

Figure 1: Stokes pair

The subdivision of the tetrahedron that we use is known as a Worsey-Farin split. A point W_T is chosen in the interior of the tetrahedron. Moreover, an interior point W_F is chosen in each face F. Then W_F is connected by edges to each vertex of F, defining a subdivision of the face F into three triangles. Finally W_T is connected by edges to each vertex of T, as well as to W_F for each face F of T. Interior faces and tetrahedra are added, so that we have, for each face F, three tetrahedra whose base are the three triangles in the face F already mentioned and whose top vertex is W_T. In total, this divides T into 12 smaller tetrahedra (3 for each of the 4 faces). We denote by $\mathcal{R}(T)$ this refinement of T.

As we shall see, and as is required in a proper Worsey-Farin split, when two tetrahedra T and T' share a two-dimensional face F, then W_F should lie on the line joining W_T and $W_{T'}$. This will be important for ensuring interelement continuity of the vectorfields. However the definition of the finite element does dot depend on this property and is local to each T.

Fix now a tetrahedron T. We define the following spaces of vectorfields on T:

$$K(T) = \{v \in \mathrm{C}^0\mathrm{P}^1(\mathcal{R}(T), \mathbb{R}^3) \ : \ \operatorname{div} v = 0\}, \tag{17}$$

$$X(T) = K(T) \oplus \{x \mapsto a(x - W_T) \ : \ a \in \mathbb{R}\} \tag{18}$$

In words, for $K(T)$ we consider the continuous vectorfields on T that are piecewise affine with respect to $\mathcal{R}(T)$, and that are moreover divergence free. In order to define $X(T)$ we add a one dimensional space of vectorfields with constant divergence, from the RTN space.

On the space $X(T)$ we consider the following degrees of freedom:

- evaluation of the vector field (3 components) at the 4 vertices.
- integration of the normal component on the 4 triangular faces.

This may be considered as a list of 16 linear forms, and one sees that they span a 16-dimensional space of linear forms on, say the continuous vectorfields on T.

Theorem 2.1. *The space $X(T)$ is 16-dimensional and the above degrees of freedom are unisolvent on $X(T)$, in the sense that given 16 real numbers, there is a unique element of $X(T)$ such that the degrees of freedom evaluate to these numbers.*

The proof of this theorem will occupy much of the remainder of this section, and will be decomposed into a succession of lemmas.

Lemma 2.2. *The space $K(T)$ has dimension at least 15.*

Proof. An element of $\mathrm{C}^0\mathrm{P}^1(\mathcal{R}(T), \mathbb{R}^3)$ is uniquely determined by its vertex values, at the vertices of $\mathcal{R}(T)$, of which there are $4+4+1$ (4 vertices of T, the interior points on the 4 faces of T, and one interior point in T). Thus the dimension of this space is 27.

The divergence of an element of this space is piecewise constant with respect to $\mathcal{R}(T)$, which consists of 12 small tetrahedra. Thus $K(T)$ has dimension at least $27-12 = 15$. □

Lemma 2.3. *Consider a two-dimensional face F. We denote by $\mathcal{R}(F)$ the refinement of F stemming from $\mathcal{R}(T)$, that divides F into 3 triangles having W_F as vertex. Let $v \in X(T)$ and let e_F be the vector $e_F = W_F - W_T$, which is transverse to F. We let $(v \times e_F)_F$ be the tangential component of $v \times e_F$ on the face F. Then $(v \times e_F)_F$ is an affine vectorfield on F.*

Proof. (i) We use the notation $e = e_F$. We notice the formula (a special case of Cartan's formula for Lie derivatives):

$$\partial_e v = (\operatorname{div} v)e + \operatorname{curl}(v \times e), \tag{19}$$

which we consider inside the tetrahedron with base F and vertex W_T.

There, we see that $\partial_e v$ (the derivative of v along e) is piecewise constant with respect to $\mathcal{R}(T)$, yet continuous. Therefore it is constant. For the proof it is enough to consider the case $v \in K(T)$, that is $\operatorname{div} v = 0$. We then get that $\operatorname{curl}(v \times e)$ is constant. In particular its normal trace on F is constant. This proves that $\operatorname{rot}(v \times e)_F$ is constant on F.

(ii) We now proceed to show that if a tangent vectorfield u on F, which is continuous on F and piecewise affine with respect to $\mathcal{R}(F)$, has constant rot on F, then it is affine. For this purpose it is enough to show that if u is 0 at the 3 vertices of F then it is 0, so we consider this hypothesis. We first remark that u restricts to 0 on the boundary ∂F of F. This gives $\int \operatorname{rot} u = 0$ therefore $\operatorname{rot} u = 0$. Let now V_0, V_1, V_2 be the vertices of F.

We let $[W_F, V_i, V_{i+1}]$ be the triangle with vertices W_F, V_i, V_{i+1}. We have:

$$0 = \int_{[W_F,V_i,V_{i+1}]} \operatorname{rot} u = \int_{\partial[W_F,V_i,V_{i+1}]} u \cdot \tau, \tag{20}$$

$$= 1/2u(W_F) \cdot (V_i - W_F) - 1/2u(W_F) \cdot (V_{i+1} - W_F). \tag{21}$$

This shows that the three numbers $u(W_F) \cdot (W_F - V_i)$ are equal.

On the other hand, let α_i be the barycentric coordinate of W_F, in F, with respect to V_i. Then we have:

$$\sum_i \alpha_i u(W_F) \cdot (V_i - W_F) = u(W_F) \cdot (\sum_i \alpha_i V_i - W_F) = u(W_F) \cdot (W_F - W_F) = 0. \tag{22}$$

This shows that $u(W_F) \cdot (V_i - W_F) = 0$ for each i. Therefore $u(W_F) = 0$. This shows that $u = 0$. □

Lemma 2.4. *Let F be a face of T and let $\overline{F}$ be the tangent vector space of F. As in the previous lemma, we let $e_F = W_F - W_T$. For a vector $u \in \mathbb{R}^3$, we denote by $u_F = u - (u \cdot n_F)n_F \in \overline{F}$ its tangent component along F. We have an isomorphism:*

$$\begin{cases} \mathbb{R}^3 & \to & \overline{F} \times \mathbb{R} \\ u & \mapsto & ((u \times e_F)_F, u \cdot n_F) \end{cases} \tag{23}$$

Proof. From the point of view of differential forms, it is natural to prove that the pair $((u \times e)_F, u \cdot n_F)$ determines the antisymmetric form $\det(u, \cdot, \cdot)$ which in turn determines u.

One can introduce an orthonormal basis (τ_1, τ_2) of $\overline{F}$, so that $\tau_1 \times \tau_2 = n_F$. Then $\det(u, a, b)$ is known for $\{a, b\} \subseteq \{e_F, \tau_1, \tau_2\}$. □

Lemma 2.5. *Let $v \in X(T)$ and suppose that the 16 degrees of freedom are 0, namely vertex values and integrals of normal components on faces. Then $v = 0$.*

Proof. (i) The divergence of v is a constant, with integral 0, from Stokes' theorem, so $\operatorname{div} v = 0$.

Remark next that on each of the 4 faces F of T, $(v \times e_F)_F$ is affine and 0 at vertices, hence 0. Moreover $v \cdot n_F$ is continuous and piecewise affine with respect to $\mathcal{R}(F)$; being 0 at vertices of F and having 0 integral, it is 0.

By Lemma 2.4 it follows that v is 0 on ∂T. It remains to be proved that $v(W_T) = 0$.

(ii) Denote by V_0, V_1, V_2, V_3 the vertices of T.

For each edge $E = E_{ij} = [V_i, V_j]$ of T, we may consider the triangle $[W_T, E] = [W_T, V_i, V_j]$ and we let n_E be the unit normal vector. We then let c_E be the number:

$$c_E = \int_{[W_T,E]} u \cdot n_E. \tag{24}$$

We now show that $c_E = 0$.

Notice that u is affine on $[W_T, E]$, and that W_T is the only vertex where it is non-zero. We deduce:

$$c_E = \frac{1}{6} u(W_T) \cdot ((V_i - W_F) \times (V_j - W_F)). \tag{25}$$

For each triangular face F of T we write Stokes theorem on the tetrahedron with base F and top vertex W_T. This gives:

$$\sum_E o(F, E) c_E = 0, \tag{26}$$

where the edges E we sum over are those in the boundary of F. This can be interpreted as saying that the coboundary of the cochain $c_\bullet$ is 0.

For each vertex V_i of T we let α_i be the barycentric coordinate of W_T with respect to V_i. For each i we can write:

$$\sum_{j \neq i} \alpha_j o(E_{ij}, V_i) c_{E_{ij}} = \sum_{j \neq i} \alpha_j u(W_T) \cdot ((V_i - W_T) \times (V_j - W_T)), \tag{27}$$

$$= u(W_T) \cdot ((V_i - W_T) \times (\sum_{j \neq i} \alpha_j V_j - W_T)), \tag{28}$$

$$= u(W_T) \cdot ((V_i - W_T) \times (\sum_j \alpha_j V_j - W_T)), \tag{29}$$

$$= u(W_T) \cdot ((V_i - W_T) \times (W_T - W_T)) = 0. \tag{30}$$

$$\tag{31}$$

If we denote by $\alpha_{E_{ij}} = \alpha_i \alpha_j$ this gives, for each vertex V of T:

$$\sum_E o(E, V) \alpha_E c_E = 0, \tag{32}$$

where the sum extends over edges E containing V. This can be interpreted as saying that a boundary of a weighted version of the cochain $c_\bullet$ is 0. One now uses that the cochain complex of a simplex is exact.

Explicitely, from (26) it follows that there are numbers c_V attached to the vertices V of T such that:

$$c_E = \sum_V o(E, V) c_V. \tag{33}$$

Plugging this into (32) gives $c_E = 0$, for each E. It then follows that $u(W_T) = 0$.

□

We are now in a position to prove Theorem 2.1:

Proof. It follows from Lemma 2.2 that the dimension of $X(T)$ is at least 16. From Lemma 2.5 it then follows that the dimension of $X(T)$ is exactly 16, and that the degrees of freedom are linearly independent on $X(T)$. □

Along the way we have also assembled the results necessary to prove the following. We suppose that the mesh $\mathcal{T}_h$ has been refined by a Worsey-Farin split, which we denote by $\mathcal{R}_h$.

Let $\tilde{X}_h$ consist of elements of X that that are piecewise smooth with respect to $\mathcal{R}_h$, and likewise for $\tilde{Y}_h$. The gobal finite element space X_h consists of the elements in X such that the restriction to each tetrahedron T in $\mathcal{T}_h$ is in $X(T)$. Such fields are in $\tilde{X}_h$. Recall that Y_h consists of the piecewise constants.

Theorem 2.6. *The degrees of provided freedom define interpolators that map onto X_h and Y_h and satisfy the commuting diagram:*

$$\begin{array}{ccc} \tilde{X}_h & \xrightarrow{\operatorname{div}} & \tilde{Y}_h \\ \downarrow I_h^X & & \downarrow I_h^Y \\ X_h & \xrightarrow{\operatorname{div}} & Y_h \end{array} \tag{34}$$

Proof. We check that the degrees of freedom guarantee interelement continuity, so that I_h^X maps into X_h. Let tetrahedra T and T' in $\mathcal{T}_h$ share a triangular face F. By the hypothesis that the split is Worsey-Farin, the vectors e_F and e'_F defined by Lemma 2.3 for T and T' respectively, are proportional to each other. Recall that for an element v of $X(T)$ the tangent component $(v \times e_F)_F$ is affine and $v \cdot n_F$ is piecewise affine, and similarly for an element of $X(T')$. The degrees of freedom attached to F are unisolvent on such fields. By Lemma 2.4 this guarantees that the interpolate of an element v of $\tilde{X}_h$ has the same restrictions on F from T and T'. Therefore the interpolate is continuous.

The commutation follows from Stokes theorem. □

3 Finite element complexes

Complexes with minimal regularity A more complete point of view on the RTN spaces is as follows. We consider the Hilbert complex consisting of spaces W^k defined by:

$$W^0 = \mathrm{H}^1(S) = \{u \in \mathrm{L}^2(S) \ : \ \operatorname{grad} u \in \mathrm{L}^2(S)^3\}, \tag{35}$$

$$W^1 = \mathrm{H}^0(\mathrm{curl}, S) = \{u \in \mathrm{L}^2(S)^3 \ : \ \operatorname{curl} u \in \mathrm{L}^2(S)^3\}, \tag{36}$$

$$W^2 = \mathrm{H}^0(\mathrm{div}, S) = \{u \in \mathrm{L}^2(S)^3 \ : \ \operatorname{div} u \in \mathrm{L}^2(S)\}, \tag{37}$$

$$W^3 = \mathrm{L}^2(S). \tag{38}$$

The two middle spaces are spaces of vector fields adapted to the curl and div operators, while the first and last spaces are spaces of scalar fields. The Nédélec spaces provide subspaces $W_h^k \subseteq W^k$ equipped with interpolation operators I_h^k which are projections onto W_h^k that are defined on dense subspaces of W^k containing the piecewise smooth fields. Crucially, these operators commute with the relevant differential operators grad, curl, div. By combining these interpolation operators with a smoothing technique one can extend the domain of definition, to get projections $P_h^k : W^k \to W_h^k$ such that the following diagram commutes:

$$\begin{array}{ccccccc} W^0 & \xrightarrow{\text{grad}} & W^1 & \xrightarrow{\text{curl}} & W^2 & \xrightarrow{\text{div}} & W^3 \\ \downarrow P_h^0 & & \downarrow P_h^1 & & \downarrow P_h^2 & & \downarrow P_h^3 \\ W_h^0 & \xrightarrow{\text{grad}} & W_h^1 & \xrightarrow{\text{curl}} & W_h^2 & \xrightarrow{\text{div}} & W_h^3 \end{array} \tag{39}$$

These spaces have proved very useful for several PDEs describing fluid flow and electromagnetic waves, but are not convenient for the Stokes equation. The vectorfields in W_h^2 are adapted to the div operator, but are not continuous in the tangential direction on interfaces of the mesh. On the other hand the vectorfields in W_h^1 are continuous in the tangential direction, but not in the normal direction, on interfaces of the mesh. The language of differential forms enables one to extend the above complexes of finite element spaces to arbitrary dimension [12][6][1].

Complexes for the Stokes equation. The previous example motivates the search for a Hilbert complex setting for the Stokes equation, as in [14].

A natural complex to consider consists in replacing L^2 by H^1 throughout:

$$\mathrm{H}^2(S) \xrightarrow{\text{grad}} \mathrm{H}^1(\text{curl}, S) \xrightarrow{\text{curl}} \mathrm{H}^1(\text{div}, S) \xrightarrow{\text{div}} \mathrm{H}^1(S). \tag{40}$$

However this forces the pressure to become continuous, which is not always wise. One can therefore consider the alterative complex:

$$\mathrm{H}^2(S) \xrightarrow{\text{grad}} \mathrm{H}^1(\text{curl}, S) \xrightarrow{\text{curl}} \mathrm{H}^1(S) \xrightarrow{\text{div}} \mathrm{L}^2(S). \tag{41}$$

The goal is then to construct finite dimensional subcomplexes of these spaces, that can be equipped with uniformly bounded projections that commute with the differential operators at hand.

In the previous section we provided subspaces corresponding to the last part of the last complex, namely $\mathrm{H}^1(S) \to \mathrm{L}^2(S)$. We refer to [8] for a treatment of the rest of both of the above complexes, in arbitrary space dimension.

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