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Viscous shock wave and singular limit for some hyperbolic system with relaxation

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1 Introduction

This paper is a survey of the papers [6, 9, 10, 12] on large time behavior of solutions to a scalar conservation laws with an artificial heat flux

$$u_t + f(u)_x + q_x = 0 \tag{1.1}$$

over a one-dimensional full space $\mathbb{R} := (-\infty, \infty)$. Here $u = u(t, x) \in \mathbb{R}$ is an unknown function; $f(u) \in \mathbb{R}$ is a flux function which is a smooth given function of u; $q = q(t, x) \in \mathbb{R}$ is an artificial heat flux. We assume that $f(u)$ is strictly convex, that is, there exists a positive constant c such that

$$f''(u) \geq c > 0 \tag{1.2}$$

holds for an arbitrary u. In the case where the heat flux q is given by Fourier's law

$$\mu u_x + q = 0,$$

where $\mu > 0$ is a viscosity coefficient, we get the system of Fourier-type:

$$u_t + f(u)_x + q_x = 0, \tag{1.3a}$$
$$\mu u_x + q = 0, \tag{1.3b}$$
$$u(0, x) = u_0(x) \to u_\pm \quad (x \to \pm\infty). \tag{1.3c}$$

Notice that the system of Fourier-type (1.3a) and (1.3b) is deduced to a scalar viscous conservation laws for u as

$$u_t + f(u)_x = \mu u_{xx}. \tag{1.4}$$

On the other hand, by prescribing Cattaneo's law

$$\varepsilon q_t + \mu u_x + q = 0$$

in stead of Fourier's law, where $\varepsilon > 0$ is a relaxation time, we have the system of Cattaneo-type:

$$u_t^\varepsilon + f(u^\varepsilon)_x + q_x^\varepsilon = 0, \tag{1.5a}$$
$$\varepsilon q_t^\varepsilon + \mu u_x^\varepsilon + q^\varepsilon = 0, \tag{1.5b}$$
$$(u^\varepsilon, q^\varepsilon)(0, x) = (u_0, q_0)(x) \to (u_\pm, 0) \quad (x \to \pm\infty). \tag{1.5c}$$

Here $u_\pm$ are constants satisfying

$$u_+ < u_-.$$

From this condition as well as the convexity condition (1.2), we see

$$f'(u_+) < f'(u_-).$$

For the scalar viscous conservation laws (1.4), asymptotic stability of a viscous shock wave has been studied. The pioneering work was done by Il'in and Oleinik [5]. It was shown in [5] that the viscous shock wave is asymptotically stable with exponential decay if the initial disturbance decays exponentially as $|x| \to \infty$. The proof is based on the maximum principle. For the isentropic model of compressible viscous fluid, Matsumura and Nishihara [9] proved asymptotic stability of the viscous shock wave by using the L^2 energy method for the integrated system. Goodman [3] also used the L^2 energy method for the uniformly parabolic system and showed asymptotic stability of the viscous shock wave. The L^2 energy method for the integrated system was generalized to the full system of an ideal polytropic gases and the Broadwell model of the discrete Boltzmann equation by Kawashima and Matsumura [6]. The case where the flux function $f(u)$ is non-convex was handled in [7, 8, 10, 11]. Especially in [10], the technical weight function with using the viscous shock wave was developed in order to obtain the convergence rate.

In place of Fourier's law, Cattaneo's law has been widely used for describing the finite speed of heat conduction. As for the model systems with Cattaneo's law, see [2, 15] for the thermoelasticity and [4] for the compressible viscous fluid. For the Cattaneo-type system (1.5), existence and asymptotic stability of the viscous shock wave are proved in [12]. By letting $\varepsilon \to 0$ in Cattaneo-type (1.5), we formally obtain Fourier-type (1.3). This is a relaxation limit from a 2×2 hyperbolic system to a scalar parabolic equation. Since the initial data in (1.5c) does not necessarily satisfy the relation $q_0 = -\mu u_{0x}$, the difference $q_0 + \mu u_{0x}$ remains as an initial layer. Thus this problem is a singular limit problem. The relaxation limit problem is also studied in [12].

Notations. For $p \in [1, \infty]$, $L^p = L^p(\mathbb{R})$ denotes a standard Lebesgue space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_{L^p}$. For a non-negative integer s, $H^s = H^s(\mathbb{R})$ denotes an s-th order Sobolev space over $\mathbb{R}$ in the L^2 sense with a norm $\|\cdot\|_{H^s}$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^2 space by $L^2_\alpha := L^2(e^{\alpha|x|})$ of which norm is given by

$$\|u\|_{L^2_\alpha} := \Big(\int_{\mathbb{R}} e^{\alpha|x|} |u(x)|^2 \, dx \Big)^{1/2}.$$

We define the exponentially weighted H^s space by $H^s_\alpha := H^s(e^{\alpha|x|})$ of which norm is given by

$$\|u\|_{H^s_\alpha} := \Big(\sum_{k=0}^{s} \|\partial_x^k u\|^2_{L^2_\alpha} \Big)^{1/2}.$$

Through the paper, c and C denote several generic positive constants.

2 Fourier-type : scalar viscous conservation laws

In this section, we consider existence and asymptotic stability of the viscous shock wave for Fourier-type (1.3) by introducing the results in [6, 9, 10].

2.1 Existence of viscous shock wave

We firstly show the existence of the viscous shock wave. Let $(\tilde{u}, \tilde{q})(\xi)$ be a smooth traveling wave solution to (1.3) satisfying $\tilde{u}(\xi) \to u_\pm$ $(\xi \to \pm\infty)$, where $\xi := x - st$ and s is a shock speed. Thus the equations for $(\tilde{u}, \tilde{q})$ are given by

$$-s\tilde{u}_\xi + f(\tilde{u})_\xi + \tilde{q}_\xi = 0, \tag{2.1a}$$

$$\mu\tilde{u}_x + \tilde{q} = 0. \tag{2.1b}$$

Substituting (2.1b) in (2.1a), we get a single equation for $\tilde{u}$ as

$$-s\tilde{u}_\xi + f(\tilde{u})_\xi = \mu\tilde{u}_{\xi\xi}, \tag{2.2a}$$

$$\tilde{u}(\xi) \to u_\pm \quad (\xi \to \pm\infty). \tag{2.2b}$$

Integrating (2.2a) over $\mathbb{R}$, we have

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0,$$

which gives the Rankine–Hugoniot condition

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.3}$$

Integrating (2.2a) over $(\pm\infty, \xi)$, we get the ordinary differential equation of first order for $\tilde{u}$ as

$$\mu\tilde{u}_\xi = h(\tilde{u}) := -s\tilde{u} + f(\tilde{u}) - (-su_\pm + f(u_\pm)), \tag{2.4a}$$

$$\tilde{u}(0) = u_*, \qquad \tilde{u}(\xi) \to u_\pm \quad (\xi \to \pm\infty), \tag{2.4b}$$

where $u_* \in (u_+, u_-)$ is a constant satisfying

$$h'(u_*) = 0. \tag{2.5}$$

Due to the uniform convexity (1.2) of $f(u)$, we have the Lax shock condition

$$f'(u_+) < s < f'(u_-) \tag{2.6}$$

and hence $h'(u_+) < 0$ and $h'(u_-) > 0$. Therefore we obtain the existence of the non-degenerate viscous shock wave which converges to $u_\pm$ exponentially fast as $\xi \to \pm\infty$. Notice that (1.2) and (2.6) give the unique existence of u_* satisfying (2.5).

Theorem 2.1 ([6, 9])**.** *The problem* (2.4) *has a unique smooth solution* $\tilde{u}(\xi)$ *satisfying*

$$|\partial_\xi^k(\tilde{u}(\xi) - u_-)| \le C\delta e^{c\delta\xi} \ (\xi \le 0), \qquad |\partial_\xi^k(\tilde{u}(\xi) - u_+)| \le C\delta e^{-c\delta\xi} \ (\xi \ge 0) \tag{2.7}$$

for $k = 0, 1, \ldots,$ *where* $\delta := |u_+ - u_-|$.

2.2 Asymptotic stability

We next consider asymptotic stability of $\tilde{u}$ obtained in Theorem 2.1 by introducing the L^2 energy method for the integrated equation developed in [6, 9]. Define a perturbation φ of the solution u to Fourier-type (1.3) from $\tilde{u}$ as

$$\varphi(t,\xi) = u(t,\xi+st) - \tilde{u}(\xi+x_0),$$

where $x_0 \in \mathbb{R}$ is a shift to be determined later. Thus the equation for φ is given by

$$\varphi_t - s\varphi_\xi + (f(\tilde{u}+\varphi) - f(\tilde{u}))_\xi - \mu\varphi_{\xi\xi} = 0. \tag{2.8}$$

Integrating (2.8) over $(0,t)\times\mathbb{R}$, we formally get

$$\int_{\mathbb{R}} \varphi(t,\xi)\,d\xi = \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi+x_0))\,d\xi.$$

We determine the shift x_0 to satisfy

$$I(x_0) := \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi+x_0))\,d\xi = 0 \tag{2.9}$$

provided that $u_0 - \tilde{u} \in L^1$. Since we have $I'(x_0) = -(u_+ - u_-)$, it holds that $I(x_0) = I(0) - (u_+ - u_-)x_0$. Therefore, by determining x_0 as

$$x_0 = \frac{1}{u_+ - u_-} I(0) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi))\,d\xi, \tag{2.10}$$

we get (2.9) and hence $\int_{\mathbb{R}} \varphi(t,\xi)\,d\xi = 0$. Then we define an anti-derivative of φ by

$$\Phi(t,\xi) = \int_{-\infty}^{\xi} (u(t,\xi+st) - \tilde{u}(\xi+x_0))\,d\xi.$$

Notice that $\Phi_\xi = \varphi$. The initial value problem for Φ is derived by integrating (2.8) as

$$\Phi_t - s\Phi_\xi + f(\tilde{u}+\Phi_\xi) - f(\tilde{u}) - \mu\Phi_{\xi\xi} = 0, \tag{2.11a}$$

$$\Phi(0,\xi) = \Phi_0(\xi) := \int_{-\infty}^{\xi} (u_0(\xi) - \tilde{u}(\xi+x_0))\,d\xi. \tag{2.11b}$$

The asymptotic stability of the viscous shock wave $\tilde{u}$ is shown in the next theorem by deriving the a priori estimate in the function space

$$X(0,T) := \bigcap_{k=0}^{1} C^k([0,T]; H^{3-2k}).$$

Theorem 2.2 ([6, 9]). *Let $u_0 - \tilde{u} \in L^1$ and $\Phi_0 \in H^3$. Then there exists a positive constant η such that if $\|\Phi_0\|_{H^3} \leq \eta$, the problem (2.11) has a unique solution $\Phi \in X(0,\infty)$. Moreover, the solution $u(t,x)$ to (1.3) converges to the viscous shock wave $\tilde{u}(x-st+x_0)$ as $t\to\infty$:*

$$\sup_{x\in\mathbb{R}} |u(t,x) - \tilde{u}(x-st+x_0)| \to 0 \quad (t\to\infty). \tag{2.12}$$

Theorem 2.2 is proved by combining the uniform a priori estimate of Φ with the existence of the solution locally in time. To show the a priori estimate, we define the energy norm defined by

$$E(t) := \sup_{\tau\in[0,t]} \|\Phi(\tau)\|_{H^3}.$$

Proposition 2.3. *Let $\Phi \in X(0,T)$ be a solution to* (2.11) *for a certain $T > 0$. Then there exists a positive constant η such that if $E(T) \le \eta$, the solution satisfies*

$$\|\Phi(t)\|_{H^3}^2 + \int_0^t \|\Phi_\xi(\tau)\|_{H^3}^2\, d\tau \le C\|\Phi_0\|_{H^3}^2 \tag{2.13}$$

for $t \in [0,T]$.

From the uniform estimate (2.13) as well as the standard continuity argument, we obtain the existence of the solution Φ globally in time. Moreover, the dissipative estimate in (2.13) gives the convergence $\|\Phi_\xi(t)\|_{L^\infty} \to 0$ $(t \to \infty)$ which yields the asymptotic stability (2.12). Proposition 2.3 is proved by the L^2 energy method. For details, see [6, 9, 12].

2.3 Convergence rate

We next obtain the convergence rate for asymptotic stability in Theorem 2.2 by introducing the results in [6, 10]. To obtain the convergence rate, we derive the weighted energy estimate with employing an weight function in terms of $\tilde{u}$ developed in [10] defined by

$$\omega(\tilde{u}) := \frac{(-g(\tilde{u}))^{1-\beta\delta^2}}{-h(\tilde{u})} \quad (0 \le \beta \le 1), \tag{2.14}$$

where $g(\tilde{u}) := (\tilde{u} - u_+)(\tilde{u} - u_-)$ and $h(\tilde{u})$ is defined in (2.4a). In order to obtain the weighted energy estimate, we utilize property of the weight function $\omega(\tilde{u})$ summarized in the next lemma. The proof of this lemma is given in the paper [12].

Lemma 2.4. *Let $\tilde{u}$ be a viscous shock wave obtained in Theorem* 2.1. *Then we have*

(i) $c \le \dfrac{h(\tilde{u})}{g(\tilde{u})} \le C$,

(ii) $0 < \omega(\tilde{u}) \le Ce^{C\beta|\xi|}$ $(\xi \in \mathbb{R})$,

(iii) $-(\omega(\tilde{u})h(\tilde{u}))''\tilde{u}_\xi \ge c(\beta\delta^4 - \tilde{u}_\xi)\omega(\tilde{u})$, *and*

(iv) $|\omega(\tilde{u})_\xi| \le C(\beta\delta^2 - \tilde{u}_\xi)\omega(\tilde{u})$,

where c and C are positive constants independent of δ.

By using the weighted energy method, we obtain the weighted energy estimate which yields the convergence rate.

Theorem 2.5 ([6, 10]). *Let $u_0 - \tilde{u} \in L^1$ and $\Phi_0 \in H^3 \cap L^2_\alpha$ for a certain $\alpha > 0$. Then there exist positive constants η and γ such that if $\|\Phi_0\|_{H^3} \leq \eta$, the solution Φ to* (2.11) *verifies*

$$\|\Phi(t)\|_{L^2} \leq Ce^{-\gamma\delta^4 t} \quad (t \geq 0). \tag{2.15}$$

From (2.13) and (2.15) with the aid of the interpolation inequality

$$\|\partial_\xi^k \Phi\|_{L^2} \leq C\|\partial_\xi^3 \Phi\|_{L^2}^\theta \|\Phi\|_{L^2}^{1-\theta}, \quad \theta = \frac{k}{3}, \quad k = 1, 2,$$

we have the convergence rate for the higher derivatives. Namely, there exists a positive constant γ such that we have

$$\|\Phi(t)\|_{H^2} \leq Ce^{-\gamma\delta^4 t} \quad (t \geq 0). \tag{2.16}$$

3 Cattaneo-type : system of hyperbolic equations

In this section, we consider the system of Cattaneo-type (1.5) and show existence and asymptotic stability of the viscous shock wave.

3.1 Existence of viscous shock wave

Let $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi)$ be a viscous shock wave, where $\xi = x - st$. Thus $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi)$ is a smooth traveling wave solution to (1.5) and satisfies

$$-s\tilde{u}^\varepsilon_\xi + f(\tilde{u}^\varepsilon)_\xi + \tilde{q}^\varepsilon_\xi = 0, \tag{3.1a}$$

$$-\varepsilon s\tilde{q}^\varepsilon_\xi + \mu\tilde{u}^\varepsilon_\xi + \tilde{q}^\varepsilon = 0, \tag{3.1b}$$

$$(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi) \to (u_\pm, 0) \quad (\xi \to \pm\infty). \tag{3.1c}$$

Integrating (3.1a) over $\mathbb{R}$, we have

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0.$$

Therefore the shock speed s is determined by the same condition (2.3) as the case of Fourier-type. Integrating (3.1a) over $(\pm\infty, \xi)$ gives

$$\tilde{q}^\varepsilon = s\tilde{u}^\varepsilon - f(\tilde{u}^\varepsilon) - (su_\pm - f(u_\pm)) = -h(\tilde{u}^\varepsilon). \tag{3.2}$$

Substituting (3.2) in (3.1b), we have the differential equation of first order for $\tilde{u}^\varepsilon$ as

$$\mu\tilde{u}^\varepsilon_\xi = h_\varepsilon(\tilde{u}^\varepsilon) := \frac{\mu h(\tilde{u}^\varepsilon)}{\mu + \varepsilon s h'(\tilde{u}^\varepsilon)}, \tag{3.3a}$$

$$\tilde{u}^\varepsilon(0) = u_*, \qquad \tilde{u}^\varepsilon(\xi) \to u_\pm \quad (\xi \to \pm\infty), \tag{3.3b}$$

where $u_* \in (u_+, u_-)$ is a constant satisfying (2.5). Note that $u_\pm$ are equilibrium points of (3.3) since $h^\varepsilon(u_\pm) = 0$. Under an assumption

$$\varepsilon < \inf_{u \in [u_+, u_-]} \frac{\mu}{|sh'(u)|}, \tag{3.4}$$

we see that $\mu + \varepsilon sh'(u) > 0$ for $u \in [u_+, u_-]$. Moreover, since we have

$$h'_\varepsilon(u_\pm) = \frac{\mu h'(u)(\mu + \varepsilon sh'(u)) - \mu\varepsilon sh(u)h''(u)}{(\mu + \varepsilon sh'(u))^2}\Big|_{u=u_\pm} = \frac{\mu h'(u_\pm)}{\mu + \varepsilon sh'(u_\pm)},$$

the Lax condition (2.6) yields $h'_\varepsilon(u_+) < 0$ and $h'_\varepsilon(u_-) > 0$. Moreover we have the asymptotic expansion of $h'_\varepsilon(u_\pm)$ as

$$h'_\varepsilon(u_\pm) = \mp\frac{1}{2} f''(u_\pm)\delta + O(\delta^2) \quad (\delta \to 0).$$

Therefore we get the existence of the non-degenerate viscous shock wave for (3.1) summarized in Theorem 3.1.

Theorem 3.1 ([12]). *For a small ε satisfying* (3.4)*, the problem* (3.3) *has a unique smooth solution $\tilde{u}^\varepsilon(\xi)$. Moreover, if δ is sufficiently small, the solution $\tilde{u}^\varepsilon(\xi)$ satisfies*

$$|\partial_\xi^k(\tilde{u}^\varepsilon(\xi) - u_-)| \le C\delta e^{c_0\delta\xi} \ (\xi < 0), \qquad |\partial_\xi^k(\tilde{u}^\varepsilon(\xi) - u_+)| \le C\delta e^{-c_0\delta\xi} \ (\xi > 0) \tag{3.5}$$

for $k = 0, 1, \ldots,$ where c_0 is a positive constant independent of δ and ε.

3.2 Asymptotic stability

We next show the asymptotic stability of the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$ to (3.1) of Cattaneo-type. Define a perturbation $(\varphi^\varepsilon, \psi^\varepsilon)$ of the solution $(u^\varepsilon, q^\varepsilon)$ to (1.5) from the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$ as

$$(\varphi^\varepsilon, \psi^\varepsilon)(t, \xi) = (u^\varepsilon, q^\varepsilon)(t, \xi + st) - (\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi + x_0^\varepsilon),$$

where x_0^ε is a shift to be determined later. From (1.5) and (3.1), the equations for $(\varphi^\varepsilon, \psi^\varepsilon)$ are given by

$$\varphi_t^\varepsilon - s\varphi_\xi^\varepsilon + (f(\tilde{u}^\varepsilon + \varphi^\varepsilon) - f(\tilde{u}^\varepsilon))_\xi + \psi_\xi^\varepsilon = 0, \tag{3.6a}$$

$$\varepsilon\psi_t^\varepsilon - \varepsilon s\psi_\xi^\varepsilon + \psi^\varepsilon + \mu\varphi_\xi^\varepsilon = 0. \tag{3.6b}$$

We determine the shift x_0^ε in the similar way to (2.10). Namely, under an assumption that $u_0 - \tilde{u}^\varepsilon \in L^1(\mathbb{R})$, we integrate (3.6a) over $(0, t) \times \mathbb{R}$ to get

$$\int_{\mathbb{R}} \varphi^\varepsilon(t, \xi)\, d\xi = \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon))\, d\xi.$$

Then we determine x_0^ε to satisfy

$$I^\varepsilon(x_0^\varepsilon) := \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon))\, d\xi = 0. \tag{3.7}$$

Since we see $I^\varepsilon(x_0^\varepsilon) = I^\varepsilon(0) - (u_+ - u_-)x_0^\varepsilon$, letting x_0^ε as

$$x_0^\varepsilon = \frac{1}{u_+ - u_-} I^\varepsilon(0) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi))\, d\xi \tag{3.8}$$

yields (3.7) and $\int_{\mathbb{R}} \varphi^\varepsilon(t,\xi)\, d\xi = 0$. Then we employ an anti-derivative of φ^ε by

$$\Phi^\varepsilon(t,\xi) = \int_{-\infty}^{\xi} (u^\varepsilon(t, \xi + st) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon))\, d\xi,$$

and deduce the problem (3.6) to that for $(\Phi^\varepsilon, \psi^\varepsilon)$ as

$$\Phi_t^\varepsilon - s\Phi_\xi^\varepsilon + f(\tilde{u}^\varepsilon + \Phi_\xi^\varepsilon) - f(\tilde{u}^\varepsilon) + \psi^\varepsilon = 0, \tag{3.9a}$$

$$\varepsilon\psi_t^\varepsilon - \varepsilon s \psi_\xi^\varepsilon + \psi^\varepsilon + \mu\Phi_{\xi\xi}^\varepsilon = 0, \tag{3.9b}$$

$$(\Phi^\varepsilon, \psi^\varepsilon)(0,\xi) = (\Phi_0^\varepsilon, \psi_0^\varepsilon)(\xi), \tag{3.9c}$$

where $(\Phi_0^\varepsilon, \psi_0^\varepsilon)$ is an initial perturbation defined by

$$\Phi_0^\varepsilon(\xi) := \int_{-\infty}^{\xi} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon))\, d\xi, \qquad \psi_0^\varepsilon(\xi) := q_0(\xi) - \tilde{q}^\varepsilon(\xi + x_0^\varepsilon).$$

To show asymptotic stability of the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$, we define a function space

$$Y(0,T) := \bigcap_{k=0}^{2} C^k([0,T]; H^{3-k} \times H^{2-k})$$

and a norm of the initial perturbation

$$E_0^\varepsilon := \sqrt{\|\Phi_0^\varepsilon\|_{H^3}^2 + \|\psi_0^\varepsilon\|_{H^2}^2}.$$

Theorem 3.2 ([12]). *Let $u_0 - \tilde{u}^\varepsilon \in L^1$ and $(\Phi_0^\varepsilon, \psi_0^\varepsilon) \in H^3 \times H^2$. Then there exists a positive constant η such that if $E_0^\varepsilon + \delta \le \eta$, the problem* (3.9) *has a unique solution $(\Phi^\varepsilon, \psi^\varepsilon) \in Y(0,\infty)$. Moreover, the solution $(u^\varepsilon, q^\varepsilon)$ to* (1.3) *converges to the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(x - st + x_0^\varepsilon)$ as $t \to \infty$:*

$$\sup_{x\in\mathbb{R}} |(u^\varepsilon, q^\varepsilon)(t,x) - (\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(x - st + x_0^\varepsilon)| \to 0 \quad (t \to \infty).$$

To show theorem 3.2, we combine the uniform a priori estimate of $(\Phi^\varepsilon, \psi^\varepsilon)$ with the existence of the solution locally in time which is shown by a standard iteration method. Thus it suffices to obtain the uniform a priori estimate for $(\Phi^\varepsilon, \psi^\varepsilon)$. To this end, we employ an energy norm defined by

$$E^\varepsilon(t) := \sup_{\tau\in[0,t]} \|\Phi^\varepsilon(\tau)\|_{H^3}.$$

Proposition 3.3. *Let* $(\Phi^\varepsilon, \psi^\varepsilon) \in Y(0,T)$ *be a solution to* (3.9) *for a certain* $T > 0$*. Then there exists a positive constant* η *such that if* $E^\varepsilon(T) + \delta \le \eta$*, the solution satisfies*

$$\|\Phi^\varepsilon(t)\|_{H^3}^2 + \varepsilon\|\psi^\varepsilon(t)\|_{H^2}^2 + \int_0^t \|(\Phi_\xi^\varepsilon, \psi^\varepsilon)(\tau)\|_{H^2}^2\, d\tau \le C\|\Phi_0^\varepsilon\|_{H^3}^2 + C\|\psi_0^\varepsilon\|_{H^2}^2 \quad (3.10)$$

for $t \in [0, T]$*.*

The a priori estimate (3.10) and the local existence as well as the continuity argument give the global existence. Moreover, the estimate (3.10) gives a convergence $\|(\Phi_\xi^\varepsilon, \psi^\varepsilon)(t)\|_{L^\infty} \to 0$ $(t \to \infty)$ which yields the asymptotic stability in Theorem 3.2. Proposition 3.3 is proved by the L^2 energy method. For details, see [12].

3.3 Convergence rate

We next show the convergence rate associated with the asymptotic stability in Theorem 3.2. To do this, we employ the weight function $\omega(\tilde{u}^\varepsilon)$ defined in (2.14). Namely, $\omega(\tilde{u}^\varepsilon)$ is given by

$$\omega(\tilde{u}^\varepsilon) = \frac{(-g(\tilde{u}^\varepsilon))^{1-\beta\delta^2}}{-h(\tilde{u}^\varepsilon)}. \quad (3.11)$$

In the same way as Lemma 2.4, we see that $\omega(\tilde{u}^\varepsilon)$ satisfies

$$0 < \omega(\tilde{u}^\varepsilon) \le Ce^{C\beta|\xi|} \quad (\xi \in \mathbb{R}), \quad (3.12)$$

$$-(\omega(\tilde{u}^\varepsilon)h(\tilde{u}^\varepsilon))''\tilde{u}_\xi^\varepsilon \ge c(\beta\delta^4 - \tilde{u}_\xi^\varepsilon)\omega(\tilde{u}^\varepsilon), \quad (3.13)$$

$$|\omega(\tilde{u}^\varepsilon)_\xi| \le C(\beta\delta^2 - \tilde{u}_\xi^\varepsilon)\omega(\tilde{u}^\varepsilon), \quad (3.14)$$

where the positive constants c and C in the above estimates are independent of δ and ε.

Theorem 3.4 ([12]). *Let* $u_0 - \tilde{u}^\varepsilon \in L^1$ *and* $(\Phi_0^\varepsilon, \psi_0^\varepsilon) \in (H^3 \times H^2) \cap (H_\alpha^1 \times L_\alpha^2)$ *for a certain* $\alpha > 0$*. Then there exist positive constants* η *and* γ *such that if* $E_0^\varepsilon + \delta \le \eta$*, the solution* $(\Phi^\varepsilon, \psi^\varepsilon)$ *to* (3.9) *verifies*

$$\|\Phi^\varepsilon(t)\|_{H^1}^2 + \varepsilon\|\psi^\varepsilon(t)\|_{L^2}^2 \le Ce^{-\gamma\delta^4 t} \quad (t \ge 0). \quad (3.15)$$

Theorem 3.4 is proved by the weighted energy method. The detailed proof is given in [12].

In the same way as the case of Fourier-type, the convergence (3.15) and the interpolation inequality give

$$\|\Phi^\varepsilon(t)\|_{H^2}^2 + \varepsilon\|\psi^\varepsilon(t)\|_{H^1}^2 \le Ce^{-\gamma\delta^4 t} \quad (t \ge 0). \quad (3.16)$$

4 Relaxation limit

In this section, we consider the relaxation limit $\varepsilon \to 0$. We firstly show that the viscous shock wave $\tilde{u}^\varepsilon$ of Cattaneo-type tends to $\tilde{u}$ of Fourier-type as $\varepsilon \to 0$ by obtaining the estimate of $\tilde{u}^\varepsilon - \tilde{u}$ in terms of ε. We next consider the singular limit $\varepsilon \to 0$ for the solutions to the initial value problem. Namely, we show in Theorem 4.2 that the solution $(u^\varepsilon, q^\varepsilon)$ to (1.5) tends to (u, q) to (1.3) as $\varepsilon \to 0$ uniformly in t.

In the paper [1], Caflisch obtained the estimate of shock profiles between the Boltzmann equation and the Navier–Stokes equations. Related to this result, we obtain the estimate of the difference of viscous shock waves $\tilde{u}^\varepsilon - \tilde{u}$ in L^p norm in terms of ε, which is summarized in the following theorem.

Theorem 4.1 ([12]). *Under the same assumptions as in Theorems* 2.1 *and* 3.1*, the viscous shock waves $\tilde{u}^\varepsilon$ and $\tilde{u}$ satisfy*

$$\|\tilde{u}^\varepsilon - \tilde{u}\|_{L^p} \le C\varepsilon\delta^{2-1/p} \quad \text{for} \quad p \in [1, \infty]. \tag{4.1}$$

Theorem 4.1 is proved by the energy computation for $\tilde{u}^\varepsilon - \tilde{u}$ with the aid of Gronwall's inequality and the exponential convergence of the viscous shock waves.

In the papers [14, 16], singular limit problem with initial layer is considered between the Boltzmann equation and the compressible Euler equations obtained as the first approximation of the Chapman–Enskog expansion. For model systems of semi-conductors, the singular limit problem from hydrodynamic model to drift-diffusion model associated with stationary waves is considered in the paper [13].

We next show that the solution $(u^\varepsilon, q^\varepsilon)$ to (1.5) tends to the solution (u, q) to (1.3) as $\varepsilon \to 0$. Since the relation $q = -\mu u_x$ does not holds for the system of Cattaneo-type, the initial data q_0 is not necessarily equal to $-\mu u_{0x}$. Thus the difference $q_0 + \mu u_{0x}$ appears as the initial layer and hence the relaxation limit is a singular limit. We also show that the initial layer decays as $t \to \infty$ or $\varepsilon \to 0$.

Theorem 4.2 ([12]). *Suppose that the same assumptions as in Theorems* 2.5 *and* 3.4 *hold. Then the solutions (u, q) to* (1.3) *and $(u^\varepsilon, q^\varepsilon)$ to* (1.5) *satisfy*

$$\|(u^\varepsilon - u)(t)\|_{H^1}^2 \le C\varepsilon^{\lambda_0}, \tag{4.2}$$

$$\|(q^\varepsilon - q)(t)\|_{L^2}^2 \le \|q_0 + \mu u_{0x}\|_{L^2}^2 e^{-t/\varepsilon} + C\varepsilon^{\lambda_0} \tag{4.3}$$

for $t \ge 0$, where λ_0 and C are independent of ε and t.

In the proof of Theorem 4.2, we use the estimate of $|x_0^\varepsilon - x_0|$ in terms of ε summarized in the next lemma.

Lemma 4.3. *We have*

(i) $\int_{\mathbb{R}} |\tilde{u}(\xi + y) - \tilde{u}(\xi)|^2 \, d\xi \le Cy$ *for* $y \in (0, 1)$*, and*

(ii) $|x_0^\varepsilon - x_0| \le C\varepsilon$.

To prove Theorem 4.2, we firstly use Gronwall's inequality. Next, to show independence of (4.2) and (4.3) in time t, We utilize the exponential convergence of the perturbation. For details, see [12].

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