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NOTES ON G_κ-SUBSETS OF COMPACT-LIKE SPACES

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In 1980 Gryzlov ([3], see also Hodel [4]) proved that every compact T_1 space has cardinality $\leq 2^{\psi(x)}$, where $\psi(X)$ is the pseudo-character of X, and later Stephenson generalized Gryzlov's result as follows:

Theorem 1 (Stephenson [5]). *Let X be a 2^κ-total T_1 space with $\psi(X) \leq \kappa$. Then $|X| \leq 2^\kappa$ and X is compact.*

A topological space X is *κ-total* if for every subset H of X with $|H| \leq \kappa$, every filter base on H has an adherent point in X.

On the other hand, Gryzlov obtained a similar result for H-closed spaces. Recall that, a Hausdorff space X is *H-closed* if X is closed in every Hausdorff space containing X as a subspace. A subset $H \subseteq X$ is a *H-set* if for every family $\mathcal{V}$ of open sets which covers H, there are finitely many $V_0, \dots, V_n \in \mathcal{V}$ with $H \subseteq \overline{V_0} \cup \cdots \cup \overline{V_n}$. It is known that a Hausdorff space X is H-closed if and only if X is an H-set in X. For a space X, $\psi_c(X)$ denotes the closed pseudo-character of X, that is, $\psi_c(X)$ is the minimum infinite cardinal κ such that for every $x \in X$ there is a family $\mathcal{V}$ of open neighborhood of x with $|\mathcal{V}| \leq \kappa$ and $\{x\} = \bigcap\{\overline{V} \mid V \in \mathcal{V}\}$. Note that closed pseduo-character can be defined only for Hausdorff spaces.

Theorem 2 (Gryzlov [3]). *Let X be an H-closed set with $\psi_c(X) = \omega$, then $|X| \leq 2^\omega$.*

Dow and Porter [2] extended this result as that $|X| \leq 2^{\psi_c(X)}$ for every H-closed space X.

In this note we prove slightly general and strong results in term of G_κ-subsets. Recall that, for a topological space X and an infinite cardinal κ, a *G_κ-subset* is the intersection of $\leq \kappa$ many open subsets in X.

Proposition 3. *Let κ be an infinite cardinal. Let X be a 2^κ-total space (no separation axiom of assumed), and $\mathcal{G}$ a cover of X by G_κ-subsets. If for every $x \in X$, the set $\{G \in \mathcal{G} \mid x \in G\}$ has cardinality $\leq 2^\kappa$, then $\mathcal{G}$ has a subcover of size $\leq 2^\kappa$.*

Proof. Suppose to the contrary that $\mathcal{G}$ has no subcover of size $\leq 2^\kappa$. Let $\lambda = |\mathcal{G}|$, and $\{G_\alpha \mid \alpha < \lambda\}$ be an enumeration of $\mathcal{G}$. Let $[\kappa]^{<\omega}$ denote the set of all finite subsets of κ. For $\alpha < \lambda$, we can take open sets W_a^α ($a \in [\kappa]^{<\omega}$) such that $G_\alpha = \bigcap_{a \in [\kappa]^{<\omega}} W_a^\alpha$ and whenever $b \supseteq a$ we have $W_b^\alpha \subseteq W_a^\alpha$.

Take a sufficiently large regular cardinal χ, and take $M \prec H(\chi)$ containing all relevant objects such that $|M| = 2^\kappa \subseteq M$ and $[M]^\kappa \subseteq M$. Since $|M \cap \lambda| = 2^\kappa$, we have $X \neq \bigcup_{\alpha \in M \cap \lambda} G_\alpha$. Fix $x^* \in X \setminus \bigcup_{\alpha \in M \cap \lambda} G_\alpha$. For $\alpha \in M \cap \lambda$, there is $a_\alpha \in [\kappa]^{<\omega}$ with $x^* \notin W^\alpha_{a_\alpha}$.

Now we claim that there are finitely many $\alpha_0, \ldots, \alpha_k \in M \cap \lambda$ such that $M \cap X \subseteq \bigcup_{i \leq k} W^{\alpha_i}_{a_{\alpha_i}}$. We can derive a contradiction by this claim; If $M \cap X \subseteq \bigcup_{i \leq k} W^{\alpha_i}_{a_{\alpha_i}}$, by the elementarity of M we have that $\{W^{\alpha_i}_{a_{\alpha_i}} \mid i \leq k\}$ is a cover of X. Hence there is $i \leq k$ with $x^* \in W^{\alpha_i}_{a_{\alpha_i}}$, this is a contradiction.

Suppose that $M \cap X \not\subseteq \bigcup_{i \leq k} W^{\alpha_i}_{a_{\alpha_i}}$ for every finitely many $\alpha_0, \ldots, \alpha_k \in M \cap \lambda$. Let $\mathcal{F} = \{(M \cap X) \setminus W^\alpha_{a_\alpha} \mid \alpha \in M \cap \lambda\}$. By the assumption, $\mathcal{F}$ has the finite intersection property. In addition if $x \in \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$ then $x \notin W^\alpha_{a_\alpha}$ for every $\alpha \in M \cap \lambda$. Take a family $\mathcal{F}' \subseteq \mathcal{P}(M \cap X)$ such that:

(1) $\mathcal{F} \subseteq \mathcal{F}'$ and every element of $\mathcal{F}'$ is closed in $M \cap X$.
(2) $\mathcal{F}'$ is a filter on $M \cap X$, hence has the finite intersection property.
(3) $\mathcal{F}'$ is a maximal family satisfying (1) and (2).

Since X is 2^κ-total and $|M \cap X| \leq 2^\kappa$, we can fix $y \in \bigcap\{\overline{F} \mid F \in \mathcal{F}'\}$, and take $\beta < \lambda$ with $y \in G_\beta$. Then we have $\beta \notin M \cap \lambda$.

For every $a \in [\kappa]^{<\omega}$, we know that the family $\{(M \cap X) \setminus W^\beta_a\} \cup \mathcal{F}'$ cannot have the finite intersection property; Otherwise, by the maximality of $\mathcal{F}'$, we have $(M \cap X) \setminus W^\beta_a \in \mathcal{F}'$. This contradicts to the choice of y. Hence there is $C_a \in \mathcal{F}'$ with $C_a \subseteq W^\beta_a$. We may assume $C_b \subseteq C_a$ for every $b \supseteq a$. Fix $z_a \in C_a$ for each $a \in [\kappa]^{<\omega}$.

Let $H = \{z_a \mid a \in [\kappa]^{<\omega}\}$. Then $H \subseteq M \cap X$ with $|H| \leq \kappa$, so $H \in M$. Put $B_a = \{z_b \mid b \supseteq a\}$ for $a \in [\kappa]^{<\omega}$. We know that $\{B_a \mid a \in [\kappa]^{<\omega}\}$ is a filter base on H. By the 2^κ-totality of X, we can pick $z \in \bigcap_{a \in [\kappa]^{<\omega}} \overline{B_a}$. Since $H \in M$, we may assume $z \in M \cap X$. Then we have $z \in \bigcap_{a \in [\kappa]^{<\omega}} C_a$; If $z \notin C_a$ for some a, since C_a is closed in $M \cap X$, pick an open neighborhood O of z with $O \cap C_a = \emptyset$. Because $z \in \overline{B_a}$, there is $b \supseteq a$ with $z_b \in O$. However $z_b \in C_b \subseteq C_a$, this is a contradiction.

We have known $z \in \bigcap_{a \in [\kappa]^{<\omega}} C_a \subseteq \bigcap_{a \in [\kappa]^{<\omega}} W^\beta_a = G_\beta$. The set $\{\alpha < \lambda \mid z \in G_\alpha\}$ is definable in M and has cardinality $\leq 2^\kappa$, hence $\beta \in \{\alpha < \lambda \mid z \in G_\alpha\} \subseteq M \cap \lambda$ and $\beta \in M \cap \lambda$. This is a contradiction. $\square$

For a topological space X and an infinite cardinal κ, let X_κ be the space X with topology generated by all G_κ-subsets. Let $L(X)$ denote the Lindelöf degree of X.

Corollary 4. *Let κ be an infinite cardinal, and X a 2^κ-total space. Then the following are equivalent:*

(1) $L(X_\kappa) \leq 2^\kappa$.
(2) For every cover $\mathcal{G}$ of X by G_κ-subsets, there is a subcover $\mathcal{G}'$ of $\mathcal{G}$ such that $|\{G \in \mathcal{G}' \mid x \in G\}| \leq 2^\kappa$ for every $x \in X$.
(3) For every cover $\mathcal{G}$ of X by G_κ-subsets, there is a refinement cover $\mathcal{G}'$ of $\mathcal{G}$ by G_κ-subsets such that $|\{G \in \mathcal{G}' \mid x \in G\}| \leq 2^\kappa$ for every $x \in X$.

Note that there is a compact T_2 space X such that $L(X_\omega)$ is much greater than 2^ω, e.g., see Usuba [6].

Corollary 5. *If X is a 2^κ-total space and $\mathcal{G}$ is a partition of X by G_κ-subsets, then $|\mathcal{G}| \le 2^\kappa$.*

Note 6. Arhangel'skiĭ [1] proved that if X is a compact Hausdorff space, then X cannot be partitioned into more than 2^ω-many closed G_δ-subsets. The above corollary is a generalization of this result.

Now Stephenson's theorem is immediate from this corollary.

Corollary 7. *If X is a $2^{\psi(X)}$-total T_1 space, then $|X| \le 2^{\psi(X)}$ and X is compact.*

For H-closed spaces, we use the following easy observation:

Lemma 8. *For a Hausdorff space X, the following are equivalent:*

(1) X is H-closed.

(2) For every upward directed set $D = \langle D, \le \rangle$ and net $\{x_a \mid a \in D\} \subseteq X$, there is $x \in X$ such that for every open neighborhood V of x and every $a \in D$, there is $b \ge a$ with $x_b \in \overline{V}$.

For a space X and $A \subseteq X$, the *θ-closure of* A, $\overline{A}^\theta$, is the set $\{x \in X \mid A \cap \overline{V} \neq \emptyset$ for every open neighborhood V of $x\}$. A subset $A \subseteq X$ is *θ-closed* if $\overline{A}^\theta = A$. Note that the following:

(1) For every $A \subseteq X$, $\overline{A}^\theta$ is θ-closed.

(2) Every θ-closed set is closed in X, and if X is regular then the converse holds.

(3) If $O \subseteq X$ is open, then $\overline{O}$ is θ-closed.

(4) Even if X is H-closed, every closed subset of X needs not be an H-set, but every θ-closed subset of X is an H-set.

Proposition 9. *Let κ be an infinite cardinal. Let X be an H-closed space, and $\mathcal{G}$ a cover of X by G_κ-sets such that for every $G \in \mathcal{G}$, there is a family $\{W_\xi \mid \xi < \kappa\}$ of open sets with $G = \bigcap_{\xi<\kappa} W_\xi = \bigcap_{\xi<\kappa} \overline{W_\xi}$. If for every $x \in X$, the set $\{G \in \mathcal{G} \mid x \in G\}$ has cardinality $\le 2^\kappa$, then $\mathcal{G}$ has a subcover of size $\le 2^\kappa$.*

Proof. Suppose to the contrary that $\mathcal{G}$ has no such a subcover, and let $\{G_\alpha \mid \alpha < \lambda\}$ be an enumeration of $\mathcal{G}$. For $\alpha < \lambda$, take open sets $\{W^\alpha_\xi \mid \xi < \kappa\}$ with $G_\alpha = \bigcap_{\xi<\kappa} W^\alpha_\xi = \bigcap_{\xi<\kappa} \overline{W^\alpha_\xi}$.

Take a sufficiently large regular cardinal χ, and take $M \prec H(\chi)$ containing all relevant objects such that $|M| = 2^\kappa \subseteq M$ and $[M]^\kappa \subseteq M$. We have $X \neq \bigcup_{\alpha \in M\cap\lambda} G_\alpha$. Fix $x^* \in X \setminus \bigcup_{\alpha\in M\cap\lambda} G_\alpha$. For $\alpha \in M \cap \lambda$, fix $\xi_\alpha < \kappa$ with $x^* \notin \overline{W^\alpha_{\xi_\alpha}}$.

Now we claim that there are finitely many $\alpha_0, \dots, \alpha_k \in M \cap \lambda$ such that $M \cap X \subseteq \bigcup_{i\le n} \overline{W^{\alpha_i}_{\xi_{\alpha_i}}}$. As before, however, this is impossible.

Suppose that $M \cap X \not\subseteq \bigcup_{i\leq k} \overline{W^{\alpha_i}_{\xi_{\alpha_i}}}$ for every finitely many $\alpha_0, \dots, \alpha_k \in M \cap \lambda$. Let $\mathcal{F} = \{(M \cap X) \setminus \overline{W^{\alpha}_{\xi_\alpha}} \mid \alpha \in M \cap \lambda\}$. By the assumption, $\mathcal{F}$ has the finite intersection property. Take a family $\mathcal{F}' \subseteq \mathcal{P}(M \cap X)$ such that:

(1) $\mathcal{F} \subseteq \mathcal{F}'$.
(2) $\mathcal{F}'$ is a filter over $M \cap X$, hence has the finite intersection property.
(3) $\mathcal{F}'$ is a maximal family satisfying (1) and (2).

For $C \in \mathcal{F}'$, take $y_C \in C$. Let $D = \langle \mathcal{F}', \supseteq \rangle$, this is an upward directed set. Hence by Lemma 8, we can find $y \in X$ such that for every open neighborhood V of y and $C \in \mathcal{F}$, there is $C' \in \mathcal{F}$ with $C' \subseteq C$ and $y_{C'} \in \overline{V}$.

Choose $\beta < \lambda$ with $y \in G_\beta$. As before, we have $\beta \notin M \cap \lambda$; If $\beta \in M \cap \lambda$, then $(M \cap X) \setminus \overline{W^{\beta}_{\xi_\beta}} \in \mathcal{F}'$, but $y \in W^{\beta}_{\xi_\beta}$ and $\overline{W^{\beta}_{\xi_\beta}} \cap ((M \cap X) \setminus \overline{W^{\beta}_{\xi_\beta}}) = \emptyset$. This is impossible.

For $\xi < \kappa$, we have that $\{(M \cap X) \setminus \overline{W^{\beta}_{\xi}}\} \cup \mathcal{F}'$ cannot have the finite intersection property; If so, then $(M \cap X) \setminus \overline{W^{\beta}_{\xi}} \in \mathcal{F}'$ by the maximiality of $\mathcal{F}'$. Put $C = (M \cap X) \setminus \overline{W^{\beta}_{\xi}}$. By the choice of y, we can find $C' \in \mathcal{F}$ with $z'_C \in C' \subseteq C$ and $z_{C'} \in \overline{W^{\beta}_{\xi}}$, this is impossible. Hence there is $C_\xi \in \mathcal{F}'$ with $C_\xi \subseteq \overline{W^{\beta}_{\xi}}$. For $a \in [\kappa]^{<\omega}$, let $C_a = \bigcap_{\xi \in a} C_\xi \in \mathcal{F}'$. We have that $C_b \subseteq C_a$ for every $b \supseteq a$. Fix $z_a \in O_a$ for each $a \in [\kappa]^{<\omega}$.

Let $H = \{z_a \mid a \in [\kappa]^{<\omega}\}$. We have $H \in M$. Then H is a net associated with the directed set $[\kappa]^{<\omega}$, hence we can find z such that for every open neighborhood V of z and $a \in [\kappa]^{<\omega}$, there is $b \supseteq a$ with $z_b \in \overline{V}$. Since $H \in M$, we may assume that $z \in M \cap X$. Then we have $z \in \bigcap_{\xi<\kappa} \overline{W^{\beta}_{\xi}} = G_\beta$; Suppose $z \notin \overline{W^{\beta}_{\xi}}$ for some $\xi < \kappa$. Since W^{β}_{ξ} is open, we have that $\overline{W^{\beta}_{\xi}}$ is θ-closed. Hence we can pick an open neighborhood V of z with $\overline{V} \cap \overline{W^{\beta}_{\xi}} = \emptyset$. On the other hand we can choose $b \supseteq \{\xi\}$ with $z_b \in \overline{V}$. $z_b \in O_b \subseteq \overline{W^{\beta}_{\xi}}$, this is impossible.

The set $\{\alpha < \lambda \mid z \in G_\alpha\}$ is definable in M and has cardinality $\leq 2^\kappa$, hence $\beta \in \{\alpha < \lambda \mid z \in G_\alpha\} \subseteq M \cap \lambda$ and $\beta \in M \cap \lambda$. This is a contradiction. $\square$

For a θ-closed set $G \subseteq X$, let $\psi_c(G, X)$ denote the minimum infinite cardinal κ such that there is an open sets $\{V_\alpha \mid \alpha < \kappa\}$ with $G = \bigcap_{\alpha<\kappa} V_\alpha = \bigcap_{\alpha<\kappa} \overline{V_\alpha}$. It is clear that $\psi_c(G, X) \leq \chi(G, X)$.

Corollary 10. *Let X be an H-closed space, and κ an infinite cardinal.*

(1) For every partition $\mathcal{G}$ of X by θ-closed sets, if $\psi_c(G, X) \leq \kappa$ for every $G \in \mathcal{G}$ then $|\mathcal{G}| \leq 2^\kappa$.
(2) (Gryzlov [3]*, Dow-Porter* [2]*) Let X be an H-closed space. Then $|X| \leq 2^{\psi_c(X)}$.*

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