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# ON SEQUENTIAL OPTIMALITY THEOREMS FOR CONVEX OPTIMIZATION PROBLEMS

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ABSTRACT. In this paper, we review two kinds of sequential optimality theorems for a convex optimization problem [11]. The involved functions of the problem are proper, lower semicontinuous and convex. Moreover, we give sufficient conditions for the closedness of characterization cones for the problem.

## 1. INTRODUCTION

Consider the following convex programming problem

$$\begin{array}{lll} \text{(CP)} & \min & f(x) \\ & \text{s.t.} & g_i(x) \leq 0,\ i = 1, \ldots, m, \end{array}$$

where $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, are proper lower semicontinuous convex functions, and $\overline{\mathbb{R}} = [-\infty, +\infty]$.

Recently, new sequential Lagrange multiplier conditions characterizing optimality without any constraint qualification for convex programs were presented in terms of the subgradients and the $\epsilon$-subgradients [6, 7, 10]. It was also shown how the sequential conditions arerelated to the standard Lagrange multiplier condition [7, 10].

The characterization of the solution set of all optimal solutions of optimization problems is very important for understanding the behavior of solution methods for optimization programming problems that have multiple solutions [3, 4, 8, 9, 12, 14]. Recently, various characterizations of the solution set of the convex optimization problem have been developed [3, 5, 12].

In this paper, we review two kinds of sequential optimality theorems for a convex optimization problem (which was in the paper [11]). The involved functions of the problem are proper, lower semi-continuous and convex functions.

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Finally, we give sufficient conditions for the closedness of characterization cones for the problem.

The outline of the paper is as follows. In Section 2, some basic definitions and preliminary results are given. In Section 3 and 4, we establish two kinds of sequential optimality theorems for a convex optimization problem. In Section 5, we give sufficient conditions for the closedness of a characteristic cone.

## 2. Preliminaries

Let us first recall some notations and preliminary results which will be used throughout this thesis.

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set $A$ in $\mathbb{R}^n$ is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. For a given set $A \subset \mathbb{R}^n$, we denote the closure and the convex hull generated by $A$, by $\mathrm{cl}A$ and $\mathrm{co}A$, respectively.

Let $f$ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Here, $f$ is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of $f$ by $\mathrm{dom}f$, that is, $\mathrm{dom}f\colon = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The epigraph of $f$, $\mathrm{epi}f$, is defined as $\mathrm{epi}f\colon = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$, and $f$ is said to be convex if $\mathrm{epi}f$ is convex. We say that $f$ is a lower semicontinuous function if $\liminf_{y\to x} f(y) \geq f(x)$ for all $x \in \mathbb{R}^n$. As usual, for any proper convex function $g$ on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is defined by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in \mathbb{R}^n\}$ for any $x^* \in \mathbb{R}^n$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a convex function. The subdifferential of $f$ at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & x \in \mathrm{dom}f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geq 0$, the $\epsilon$-subdifferential of $f$ at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}, & x \in \mathrm{dom}f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We recall a version of the Brondsted-Rockafellar theorem which was established in [13].

**Proposition 2.1.** [1, 13] **(Brondsted-Rockafellar Theorem)** *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semi-continuous convex function. Then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$ there exist $x_\epsilon \in \mathbb{R}^n$, $x^*_\epsilon \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - \bar{x}\| \leq \sqrt{\epsilon}, \quad \|x^*_\epsilon - x^*\| \leq \sqrt{\epsilon} \quad \textit{and} \quad |f(x_\epsilon) - \langle x^*_\epsilon, x_\epsilon - \bar{x}\rangle - f(\bar{x})| \leq 2\epsilon.$$

## 3. Sequential Optimality Theorems I

Now we give sequential optimality theorems for (CP), which are expressed sequences of $\epsilon$-subgradients of involved functions. The involved functions of the problem are proper, lower semi-continuous and convex functions.

**Theorem 3.1.** *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,\dots,m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R} : g(x) \leq 0\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*

(ii) *there exist $\delta_k \geq 0$, $\gamma_k \geq 0$, $\lambda_i^k \geq 0$, $i = 1,\dots,m$, $\xi_k \in \partial_{\delta_k} f(\bar{x})$ and $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\lim_{k\to\infty}(\xi_k + \zeta_k) = 0, \quad \lim_{k\to\infty}(\delta_k + \gamma_k) = 0 \quad \textit{and} \quad \lim_{k\to\infty}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

**Theorem 3.2.** *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,\dots,m$, be proper lower semi-continuous convex functions. Let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$ and* $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*

(ii) *there exist $\gamma_k \geq 0$, $\lambda_i^k \geq 0$, $i = 1,\dots,m$, $\xi \in \partial f(\bar{x})$, and $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\xi + \lim_{k\to\infty} \zeta_k = 0, \quad \lim_{k\to\infty} \gamma_k = 0 \text{ and } \lim_{k\to\infty}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

**Theorem 3.3.** *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,\dots,m$, be proper lower semi-continuous convex functions. Let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$ and* $\mathrm{epi} f^* + \bigcup_{\lambda_i \geq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*

(ii) *there exist $\bar{\lambda}_i \geq 0$, $i = 1,\dots,m$, such that*

$$0 \in \partial f(\bar{x}) + \partial(\sum_{i=1}^m \bar{\lambda}_i g_i)(\bar{x}) \quad \textit{and} \quad \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) = 0.$$

**Remark 3.4.** Theorem 3.3 can be regarded as one which is sharper than Theorem 4.2 in [2] in the case that the involved geometric set is empty.

## 4. Sequential Optimality Theorems II

By using Proposition 2.1 (a version of Brondsted-Rockafellar Theorem) and Theorem 3.1, we can obtain the following sequential optimality theorem for (CP) which involve only the subgradients at nearby points to a minimizer of (CP). So the sequential optimality condition in Theorem 3.1 is different from the one in the following theorem.

**Theorem 4.1.** *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, *be proper lower semi-continuous convex functions. Let* $\bar{x} \in A$. *Assume that* $A \cap \mathrm{dom} f \neq \emptyset$. *Then the following statements are equivalent:*

(i) $\bar{x}$ *is an optimal solution of* (CP)*;*
(ii) *there exist* $x_k \in \mathbb{R}^n$, $\lambda_i^k \geq 0$, $i = 1, \ldots, m$, $\bar{\xi}_k \in \partial f(x_k)$, *and* $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ *such that*

$$\lim_{k\to\infty} x_k = \bar{x}, \quad \lim_{k\to\infty} (\bar{\xi}_k + \bar{\zeta}_k) = 0,$$
$$\text{and} \quad \lim_{k\to\infty} \left[ f(x_k) + (\sum_{i=1}^m \lambda_i^k g_i)(x_k) - f(\bar{x}) \right] = 0.$$

By using Proposition 2.1 and Theorem 3.2, we can obtain the following sequential optimality theorem for (CP). The sequential optimality condition in Theorem 3.2 is different from the one in the following theorem.

**Theorem 4.2.** *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, *be proper lower semi-continuous convex functions. Let* $\bar{x} \in A$. *Assume that* $A \cap \mathrm{dom} f \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geq 0} \mathrm{epi}(\sum_{i=1}^n \lambda_j g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an optimal solution of* (CP)*;*
(ii) *there exist* $x_k \in \mathbb{R}^n$, $\lambda_i^k \geq 0$, $i = 1, \ldots, m$, $\bar{\xi} \in \partial f(\bar{x})$, *and* $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ *such that*

$$\lim_{k\to\infty} x_k = \bar{x}, \ \bar{\xi} + \lim_{k\to\infty} \bar{\zeta}_k = 0 \ \text{and} \ \lim_{k\to\infty} (\sum_{i=1}^m \lambda_i^k g_i)(x_k) = 0.$$

## 5. Closedness of Characterization Cones

The set $\bigcup_{\lambda_i \geq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ is called the characterization cone of (CP). Closedness of the set is important in Theorem 3.3 since the set is related to the constraint qualification for (CP) (see, e.g., [7]). Now we give sufficient conditions for the set to be closed.

**Proposition 5.1.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,m$ be convex functions. Let $\bar{x} \in A$. Assume that $h\colon \mathbb{R}^n \to \mathbb{R}$ is a positive homogeneous convex function such that $g^* \geq h$ and $0 \notin \partial h(0)$. Then*

$$\Lambda := \bigcup_{\lambda \geq 0} \operatorname{epi}(\lambda g)^* = \bigcup_{\lambda > 0} \operatorname{epi}(\lambda g)^* \cup \{0\} \times \mathbb{R}_+$$

*is closed.*

**Proposition 5.2.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive homogeneous convex function which is separable, that is , $g(x) = \sum_{i=1}^m g_i(x_i)$, where $g_i\colon \mathbb{R} \to \mathbb{R}$ is a function, $i = 1,2,\dots,m$. Assume that $g_i(0) = 0$, $i = 1,2,\dots,m$. Then $\bigcup_{\lambda\geq 0}(\lambda g)^*$ is closed.*

**Proposition 5.3.** *Let $g_i\colon \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2$, be a function such that $g_i = \max\{a_i^j x_i + b_i^j \mid j = 1,2\}$, $i = 1,2$. Let $g = g_1 + g_2$. Then $\bigcup_{\lambda\geq 0} \operatorname{epi}(\lambda g)^*$ is closed.*

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