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# On optimality conditions in nonsmooth semi-infinite vector optimization problems[1]

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**Abstract.** In this paper, we establish optimality conditions (both necessary and sufficient) for a nonsmooth semi-infinite vector optimization problem by using the scalarization method.

## 1 Introduction

As we know, scalarization methods are regarded as very important tools to study (weakly/properly) efficient solutions to multiobjective optimization. The relevance of using scalarization methods to solve multiobjective optimization problems is that scalar problems can have more effective means of finding optimal solutions than vector problems. The reader can refer to the papers [12, 14], where surveys of methods for multiobjective optimization are reviewed. For deeper, the reader is referred to the books [1, 6, 7, 11] and the papers [2, 10, 13]. In this paper, we are interested in Chankong–Haimes method which is an effective method to solve multiobjective optimization problems for exact solutions via scalarization also. The reader is referred to the Chankong–Haimes's book [3] for more details. Mathematically speaking, consider the following nonsmooth semi-infinite multiobjective optimization problem:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) := \big(f_1(x), f_2(x), \ldots, f_m(x)\big) \\ & \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & & x \in C, \end{array}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}, i \in M := \{1, 2, \ldots, m\}$, $g_t, t \in T$ are locally Lipschitz functions, $T$ is an index set (possibly infinite), and $C$ is a nonempty closed subset of $\mathbb{R}^n$. The feasible set of (MP) is denoted by $F_M := \{x \in C \colon g_t(x) \leqq 0, t \in T\}$.

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In our research, due to Chankong–Haimes method, for $j \in M$ and $z \in C$, we formulate the following scalar problem associated to (MP),

$$\begin{array}{lll} (\mathrm{P_j(z)}) & \text{Minimize} & f_j(x) \\ & \text{subject to} & f_k(x) \leqq f_k(z), k \in M^j := M \setminus \{j\}, \\ & & g_t(x) \leqq 0, t \in T, \\ & & x \in C. \end{array}$$

First we give the necessary condition for an optimal solution of $(\mathrm{P_j(z)})$ by introducing a modified constraint qualification, then generalized necessary condition for an efficient solution of (MP) is established by using the modified constraint qualification. In addition, sufficient condition for the optimal solution of $(\mathrm{P_j(z)})$ and generalized sufficient condition for the efficient solution of (MP) are provided by using suitable generalized convexity conditions.

## 2 Preliminaries

The following notation will be used for vectors in $\mathbb{R}^n$:

$$\begin{aligned} x < y &\iff x_i < y_i,\ i = 1, 2, \cdots, n; \\ x \leqq y &\iff x_i \leqq y_i,\ i = 1, 2, \cdots, n; \\ x \leq y &\iff x_i \leqq y_i,\ i = 1, 2, \cdots, n \text{ but } x \neq y. \end{aligned}$$

Let us denote by $\mathbb{R}^{(T)}$ a following linear space (see [9]):

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For each $\lambda \in \mathbb{R}^{(T)}$, the supporting set corresponding to $\lambda$ is $T(\lambda) := \{t \in T \colon \lambda_t \neq 0\}$, which is a finite subset of $T$.

We denote $\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \colon \lambda_t \geqq 0, t \in T\}$, which is a nonnegative cone of $\mathbb{R}^{(T)}$.

For $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t \in T} \subset Z$, $Z$ being a real linear space, we understand that

$$\sum_{t \in T} \lambda_t z_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $g_t, t \in T$,

$$\sum_{t \in T} \lambda_t g_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t g_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

We also note that in $\mathbb{R}^{(T)}$, a norm formulated by (see [15])

$$\| \lambda \|_1 = \sum_{t \in T(\lambda)} |\lambda_t|.$$

Throughout this paper, $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, and $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$, are locally Lipschitz with respect to $x$ uniformly in $t \in T$, i.e.,

$$\forall x \in X, \exists U(x), \exists K > 0, \quad |g_t(u) - g_t(v)| \leqq K\|u - v\|, \quad \forall u, v \in U(x), \quad \forall t \in T.$$

We also suppose that the function $t \mapsto g_t(x)$ is upper semicontinuous on $T$ for every $x \in X$. Note that most of the following basic concepts are concerned with nonsmooth analysis theory, which can be found in [4, 5, 8].

Let $g : X \to \mathbb{R}$ be a locally Lipschitz function. The directional derivative of $g$ at $z \in X$ in direction $d \in X$, is

$$g'(z; d) = \lim_{t \to 0^+} \frac{g(z + td) - g(z)}{t}$$

if the limit exists.

The Clarke generalized directional derivative of $g$ at $z \in X$ in direction $d \in X$ is

$$g^c(z; d) := \limsup_{\substack{y \to z \\ t \to 0^+}} \frac{g(y + td) - g(y)}{t}.$$

The Clarke subdifferential of $g$ at $z \in X$, denoted by $\partial^c g(z)$, is defined by

$$\partial^c g(z) := \{v \in X^* : v(d) \leqq g^c(z; d), \forall d \in X\}.$$

A locally Lipschitz function $g$ is said to be regular (in the sense of Clarke) at $z \in X$ if $g'(z; d)$ exists and

$$g^c(z; d) = g'(z; d), \forall d \in X.$$

Let $D$ be a nonempty closed subset of $X$. The tangent cone to $D$ is defined by

$$T_D(x) = \{h \in X : d_D^c(x; h) = 0\},$$

where $d_D$ denotes the distance function to $D$. The normal cone to $D$ at a point $z \in D$ coincides with the normal cone in the sense of convex analysis and given by

$$N_D(z) := \{v \in X^* : v(x - z) \leqq 0, \forall x \in D\}.$$

**Definition 2.1** Let $C$ be a subset of $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.

(i) The function $h$ is said to be *pseudoconvex* at $x \in C$ if

$$h(y) < h(x) \Rightarrow u(y - x) < 0, \quad \forall u \in \partial^c h(x), y \in C,$$

equivalently,

$$u(y - x) \geqq 0 \Rightarrow h(y) \geqq h(x), \quad \forall u \in \partial^c h(x), y \in C.$$

(i)' The function $h$ is said to be pseudoconvex on C if it is pseudoconvex at every $x \in C$. Moreover, the function $f$ is said to be strictly pseudoconvex at $x \in C$ if

$$u(y-x) \geqq 0 \Rightarrow f(y) > f(x), \quad \forall u \in \partial^c f(x), \quad y \neq x \text{ and } y \in C.$$

(ii) The function $h$ is said to be *quasiconvex* at $x \in C$ if

$$h(y) \leqq h(x) \Rightarrow u(y-x) \leqq 0, \quad \forall u \in \partial^c h(x), y \in C,$$

equivalently,

$$u(y-x) > 0 \Rightarrow h(y) > h(x), \quad \forall u \in \partial^c h(x), y \in C.$$

(ii)' The function $h$ is said to be quasiconvex on C if it is quasiconvex at every $x \in C$.

Below, we recall the concept of efficient solution of (MP).

**Definition 2.2** A point $z \in F_M$ is said to be an *efficient solution* of (MP) if there exists no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(z), \text{ for all } i \in M$$

and

$$f_{i_0}(x) < f_{i_0}(z), \text{ for some } i_0 \in M,$$

it is equivalent to

$$f(x) \leq f(z).$$

Let us consider the following single objective optimization problem.

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(x) \\ & \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & & x \in C \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz function and functions $g_t, t \in T$ and $C$ are as above. Also, the feasible set of (P) is denoted by $F_M := \{x \in C \colon g_t(x) \leqq 0, t \in T\}$.

Let $x \in \mathbb{R}^n$. We need the following condition [16],

$(\mathcal{A}): \quad \exists d \in T_C(x) : g_t^c(x; d) < 0, \text{ for all } t \in I(x) := \{t \in T \colon g_t(x) = 0\}.$

Then we would like to derive the following KKT necessary optimality theorem for the case of the involved functions defined on $\mathbb{R}^n$ and index set $T$ is compact.

**Theorem 2.1** *Let $z$ be an optimal solution for* (P), *and assume that the condition* ($\mathcal{A}$) *holds for $z$. Then, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z), \quad g_t(z) = 0, \quad \forall t \in T(\lambda).$$

**Definition 2.3** Let $z \in C, \lambda \in \mathbb{R}_+^{(T)}$, $(z, \lambda)$ is said to satisfy *generalized KKT condition* if the following condition holds

$$0 \in \partial^c f(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z), \quad \lambda_t g_t(z) = 0, \quad \forall t \in T(\lambda).$$

**Remark 2.1** If $z$ is an optimal solution of (P) and the condition ($\mathcal{A}$) holds for $z$, then there exists $\lambda \in \mathbb{R}_+^{(T)}$ and $(z, \lambda) \in C \times \mathbb{R}_+^{(T)}$ satisfies obviously the generalized KKT condition from Theorem 2.1.

Recall that the criteria of Chankong–Haimes method [3] applied for a semi-infinite multiobjective optimization problem (MP) is as follows. The proof would be omitted.

**Lemma 2.1** *A feasible point $z$ of* (MP) *is an efficient solution if and only if it is an optimal solution of* ($\mathrm{P_j(z)}$) *for each $j \in M$.*

**Remark 2.2** *If $z$ is an efficient solution of* (MP), *then obviously, it is also an optimal solution of* ($\mathrm{P_j(z)}$) *for some $j \in M$, but the converse is not always true.*

## 3 Optimality Conditions

In this section we establish KKT and generalized KKT optimality conditions for ($\mathrm{P_j(z)}$) and (MP), successively. The following condition, which is a modified constraint qualification, is associated to the problem ($\mathrm{P_j(z)}$), and the feasible set of ($\mathrm{P_j(z)}$) is denoted by $F_j(z)$.

Let $x \in \mathbb{R}^n$, $I(x) = \{t \in T \colon g_t(x) = 0\}$, $H_j(x) = \{k \in M^j \colon f_k(x) = f_k(z)\}$, and $\bar{T}(x) = I(x) \cup H_j(x)$.

$$(\mathcal{A}_j): \quad \exists d \in T_C(x): \begin{cases} g_t^c(x; d) < 0, \text{ for all } t \in I(x), \\ f_k^c(x; d) < 0, \text{ for all } k \in H_j(x). \end{cases}$$

With the fulfilment of condition ($\mathcal{A}_j$), we now give the KKT necessary condition for ($\mathrm{P_j(z)}$).

**Theorem 3.1** *Let* $z$ *be an optimal solution for* $(\mathrm{P_j(z)})$ *and assume that the condition* $(\mathcal{A}_j)$ *holds for* $z$, *then there exist* $\bar{\tau}_k \geqq 0, k \in M^j$ *and* $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ *such that the following KKT condition holds,*

$$0 \in \partial^c f_j(z) + \sum_{k \in M^j} \bar{\tau}_k \partial^c f_k(z) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(z) + N_C(z), \tag{3.1}$$

$$g_t(z) = 0, \quad \forall t \in T(\bar{\lambda}). \tag{3.2}$$

KKT sufficient condition for $(\mathrm{P_j(z)})$ is proposed as follows by using suitable generalized convexity.

**Theorem 3.2** *Let* $z \in F_j(z)$. *Assume that the function* $f_j$ *is pseudoconvex, the functions* $f_k, k \in M^j$ *and* $g_t, t \in T$ *are quasiconvex. If there exist* $\bar{\tau}_k \geqq 0, k \in M^j$ *and* $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ *such that* (3.1) *and* (3.2) *hold. Then* $z$ *is an optimal solution for* $(\mathrm{P_j(z)})$.

We now give the following generalized KKT necessary and sufficient conditions for (MP).

**Theorem 3.3** *Let* $z \in F_M$ *be an efficient solution of (MP). If there exists* $j \in M$ *such that the condition* $(\mathcal{A}_j)$ *holds for* $z$, *then there exist* $\tau_j \geqq 0, j \in M, \sum_{j \in M} \tau_j = 1$ *and* $\lambda \in \mathbb{R}_+^{(T)}$ *such that the following generalized KKT condition holds,*

$$0 \in \sum_{j \in M} \tau_j \partial^c f_j(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z), \quad \lambda_t g_t(z) = 0, \quad \forall t \in T. \tag{3.3}$$

**Theorem 3.4** *Let* $z \in F_M$. *Assume that there exist* $\tau_j \geqq 0, j \in M, \sum_{j \in M} \tau_j = 1$ *and* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* (3.3) *holds. If* $\tau^T f$ *is strictly pseudoconvex and* $\sum_{t \in T} \lambda_t g_t$ *is quasiconvex. Then* $z$ *is an efficient solution of* (MP).

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