Remark on skew $m$-complex symmetric operators (Research on structure of operators using operator means and related topics)

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Citation
数理解析研究所講究録 = RIMS Kokyuroku (2019), 2113: 1-12

Issue Date
2019-05

URL
http://hdl.handle.net/2433/252015

Type
Departmental Bulletin Paper

Textversion
publisher
Kyoto University
Remark on skew $m$-complex symmetric operators

by

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Abstract

In this paper we study skew $m$-complex symmetric operators. In particular, we prove that if $T \in \mathcal{L}(\mathcal{H})$ is a skew $m$-complex symmetric operator with a conjugation $C$, then $e^{itT}$, $e^{-itT}$, and $e^{-itT^*}$ are $(m, C)$-isometric for every $t \in \mathbb{R}$. Moreover, we investigate some conditions for skew $m$-complex symmetric operators to be skew $(m - 1)$-complex symmetric.

1 Introduction

The results in this paper will be appeared in other journals. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$.

Definition 1.1 An operator $C$ is said to be a conjugation on $\mathcal{H}$ if the following conditions hold:

(i) $C$ is antilinear; $C(ax + by) = \overline{a}Cx + \overline{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$,
(ii) $C$ is isometric; $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$, and
(iii) $C$ is involutive; $C^2 = I$.

Moreover, if $C$ is a conjugation on $\mathcal{H}$, then $\|C\| = 1$, $(CTC)^* = CT^*C$ and $(CTC)^k = CT^kC$ for every positive integer $k$. For any conjugation $C$, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ (see [11] for more details). We first consider the following examples for conjugations.

Example 1.2 Let’s define an operator $C$ as follows:

(i) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
(ii) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.
(iii) $[Cf](x) = \overline{f(x)}$ on $\mathcal{L}^2(\mathcal{X}, \mu)$.
(iv) $[Cf](x) = f(1 - x)$ on $L^2([0, 1])$.
(v) $[Cf](x) = \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.
(vi) $Cf(z) = zf(z)u(z) \in \mathcal{K}_u^2$ for all $f \in \mathcal{K}_u^2$ where $u$ is inner function and $\mathcal{K}_u^2 = H^2 \ominus uH^2$ is Model space.

Then each $C$ in (i)-(vi) is a conjugation.

This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.
In 1970, J. W. Helton [15] initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form:

$$
\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^{m-j} = 0. \quad (1)
$$

Using the identity (1) and a conjugation operator, we define skew $m$-complex symmetric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a skew $m$-complex symmetric operator if there exists some conjugation $C$ such that

$$
\sum_{j=0}^{m} \left( \begin{array}{l} m \\
\dot{j} 
\end{array} \right) T^{*j} CT^{m-j} C = 0
$$

for some positive integer $m$. In this case, we say that $T$ is skew $m$-complex symmetric with conjugation $C$. In particular, if $m = 1$, then $T$ is said to be skew complex symmetric, i.e., $T = -CTC$. Set $\Gamma_m(T; C) := \sum_{j=0}^{m} \left( \begin{array}{l} m \\
\dot{j} 
\end{array} \right) T^{*j} CT^{m-j} C$. Then $T$ is a skew $m$-complex symmetric operator with conjugation $C$ if and only if $\Gamma_m(T; C) = 0$. Note that

$$
T^* \Gamma_m(T; C) + \Gamma_m(T; C)(CTC) = \Gamma_{m+1}(T; C). \quad (2)
$$

From (2), if $T$ is skew $m$-complex symmetric with conjugation $C$, then $T$ is skew $n$-complex symmetric with conjugation $C$ for $n \geq m$. In general, skew $m$-complex symmetric operators are not skew $(m - 1)$-complex symmetric.

**Example 1.3** Let $Cx = (\frac{x_2}{x_1})$ for $x = (x_1, x_2)$ and $T = \left( \begin{array}{cc} 0 & 1 \\
0 & 0 \end{array} \right)$ on $\mathbb{C}^2$. Then $T^* = CTC = \left( \begin{array}{cc} 0 & 0 \\
1 & 0 \end{array} \right)$ and so $CT^2C + 2T^* CTC + T^{*2} = 0$. But, $CTC + T^* = 2 \left( \begin{array}{cc} 0 & 0 \\
0 & 1 \end{array} \right) \neq 0$. Hence $T$ is a skew 2-complex symmetric operator which is not skew complex symmetric (see [3]).

In 1995, Agler and Stankus ([1]) studied the following operator. For a fixed $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $m$-isometric operator if it satisfies an identity;

$$
\sum_{j=0}^{m} (-1)^{j} \binom{m}{j} T^{*m-j} T^{m-j} = 0. \quad (3)
$$

Using the identity (3) and a conjugation $C$, the authors of [9] define the following operator; An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $(m, C)$-isometric operator if there exists some conjugation $C$ such that

$$
\sum_{j=0}^{m} (-1)^{j} \binom{m}{j} T^{*m-j} CT^{m-j} C = 0 \quad (4)
$$

for some $m \in \mathbb{N}$. In particular, if $T = CTC$, then $T$ is an $m$-isometric operator. Put $\Lambda_m(T) := \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} T^{*m-j} CT^{m-j} C$. Thus $T$ is an $(m, C)$-isometric operator if and only if $\Lambda_m(T) = 0$. Note that

$$
T^* \Lambda_m(T)(CTC) - \Lambda_m(T) = \Lambda_{m+1}(T). \quad (5)
$$
From (5), if $\Lambda_m(T) = 0$, then $\Lambda_n(T) = 0$ for all $n \geq m$. Moreover, $T$ is an $(m, C)$-isometry if and only if $CTC$ is an $(m, C)$-isometry (see [9]).

Next, we provide several examples of $(m, C)$-isometric operators with a conjugation $C$.

**Example 1.4** ([9]) Let $C$ be the canonical conjugation on $\mathcal{H}$ given by

$$C\left(\sum_{n=0}^{\infty} x_n e_n\right) = \sum_{n=0}^{\infty} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ with $Ce_n = e_n$ for all $n$. Assume that $W$ is the weighted shift given by $W e_n = \alpha_n e_{n+1}$ where $\alpha_n = \sqrt{\frac{n+\alpha}{n+1}}$ for $\alpha > 0$. If $\alpha = 1$, then $W = S$ is the unilateral shift. Hence $S$ is $(1, C)$-isometry. If $\alpha = 2$, then, since $W = CWC$, it holds that

$$I - 2W^* CW C + W^* 2 CW^2 C = 0.$$ 

Therefore, $W$ is an $(2, C)$-isometric operator which is called the Dirichlet shift. On the other hand, if $\alpha = m$, then, since $W = CWC$, it holds that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} W^{*m-j} CW^{m-j} C = 0.$$ 

So, $W$ is an $(m, C)$-isometric operator.

**Example 1.5** ([9]) Let $C$ be a conjugation defined by $C f(z) = \overline{f(\overline{z})}$ and let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $H^2$. Set $C = C \oplus C$. Then $C$ is clearly a conjugation on $H^2 \oplus H^2$. Assume that

$$T = \begin{pmatrix} S & e_0 \otimes e_0 \\ 0 & I \end{pmatrix} \in \mathcal{L}(H^2 \oplus H^2)$$

where $S$ is the unilateral shift on $H^2$. Then

$$\Lambda_2(T) = T^* (T^* CTC - I) CTC - (T^* CTC - I) = \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix} = 0.$$ 

Hence $T$ is an $(2, C)$-isometric operator. If $R = S + e_0 \otimes e_0$, then

$$CRC = CSC + C(e_0 \otimes e_0)C = S + e_0 \otimes e_0.$$ 

Since $S^* e_0 = 0$, it follows that $R^* CRC = (S^* + e_0 \otimes e_0)(S + e_0 \otimes e_0) = I + e_0 \otimes e_0$ and so

$$\Lambda_2(R) = R^* (R^* CRC - I) CRC - (R^* CRC - I) = (S^* + e_0 \otimes e_0)(e_0 \otimes e_0)(S + e_0 \otimes e_0) - e_0 \otimes e_0 = 0.$$ 

Therefore, $R$ is an $(2, C)$-isometric operator.
2 \ ((m, C)\text{-isometric operators})

In this section, we state properties of \((m, C)\text{-isometric operators which are the known results in [9].}

**Theorem 2.1** Let \(T \in \mathcal{L}(\mathcal{H})\) and let \(C\) be a conjugation on \(\mathcal{H}\). Then the following statements hold.

(i) If \(T\) is an invertible, then \(T\) is an \((m, C)\text{-isometric operator if and only if } T^{-1}\) is an \((m, C)\text{-isometry.}

(ii) If \(T\) is an \((m, C)\text{-isometric operator with the conjugation } C\) and \(T\) is complex symmetric, i.e., \(T = CT^*C\), then \(T\) is an algebraic operator of order at most \(2m\).

(iii) If \(\{T_k\}\) is a sequence of \((m, C)\text{-isometric operators with conjugation } C\) such that \(\lim_{k \to \infty} \|T_k - T\| = 0\), then \(T\) is also an \((m, C)\text{-isometric operator.}

(iv) If \(T\) is an \((m, C)\text{-isometric operator, then } T^n\) is also an \((m, C)\text{-isometric operator for any } n \in \mathbb{N}.

If \(T \in \mathcal{L}(\mathcal{H})\), we write \(\sigma(T), \sigma_p(T)\) and \(\sigma_a(T)\) for the spectrum, the point spectrum and the approximate point spectrum of \(T\), respectively.

**Lemma 2.2** Let \(T \in \mathcal{L}(\mathcal{H})\) be an \((m, C)\text{-isometric operator where } C\) is a conjugation on \(\mathcal{H}\). Then \(0 \notin \sigma_a(T)\).

**Theorem 2.3** Let \(T \in \mathcal{L}(\mathcal{H})\) be an \((m, C)\text{-isometric operator where } C\) is a conjugation on \(\mathcal{H}\). If \(\lambda \in \sigma_a(T)\), then \(\frac{1}{\overline{\lambda}} \in \sigma_a(T^*)\). In particular, if \(\lambda\) is an eigenvalue of \(T\), then \(\frac{1}{\overline{\lambda}}\) is an eigenvalue of \(T^*\).

**Theorem 2.4** Let \(T \in \mathcal{L}(\mathcal{H})\) be an \((m, C)\text{-isometric operator where } C\) is a conjugation on \(\mathcal{H}\). Let \(\lambda, \mu \in \mathbb{C}\) with \(\lambda \mu \neq 1\). If \(\{x_n\}\) and \(\{y_n\}\) are sequences of unit vectors such that \(\lim_{n \to \infty} (T - \lambda)x_n = 0\) and \(\lim_{n \to \infty} (T - \mu)y_n = 0\), then \(\lim_{n \to \infty} \langle Cx_n, y_n \rangle = 0\). In particular, if \((T - \lambda)x = 0\) and \((T - \mu)y = 0\), then \(\langle Cx, y \rangle = 0\).

**Corollary 2.5** Let \(C\) be a conjugation on \(\mathcal{H}\). If \(T \in \mathcal{L}(\mathcal{H})\) is an \((m, C)\text{-isometric operator with a conjugation } C\), then \(\ker(T - \lambda) \subseteq C \ker((T^* - \frac{1}{\overline{\lambda}})^n)\).
3 Skew $m$-complex symmetric operators

In this section, we study properties of skew $m$-complex symmetric operators. In [7], if $T$ is an $m$-complex symmetric operator, then $T^n$ is also $m$-complex symmetric for some $n$. Unlike an $m$-complex symmetric operator (see [7] and [9]), the power of a skew $m$-complex symmetric operator is not skew $m$-complex symmetric.

Example 3.1 If $T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & -1 \end{pmatrix}$ for $a \in \mathbb{C}$, then $T$ is skew complex symmetry with the conjugation $C(z_1, z_2, z_3) = (-\overline{z_3}, \overline{z_2}, -\overline{z_1})$ from [18]. A simple calculation shows that $T^2 = \begin{pmatrix} 1 & a & a^2 \\ 0 & 0 & -a \\ 0 & 0 & 1 \end{pmatrix}$ and $-CT^2C = \begin{pmatrix} -1 & 0 & 0 \\ -a & 0 & 0 \\ -a^2 & a & -1 \end{pmatrix}$. Hence $T^2$ is not skew complex symmetric with the conjugation $C$.

Example 3.2 Let $C$ be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on $\mathbb{C}^3$. If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $T^* \neq CTC = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $T^{*2} = CT^2C = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hence $T^2$ is a 1-complex symmetric operator but $T$ is not a 1-complex symmetric operator with conjugation $C$.

Now we will introduce exponential operators $T := e^{-iA}$ which act on a wave function to move it in time and space (see [1]). Note that $T$ is a function of an operator $f(A)$ which is defined its expansion in a Taylor series

\[ T = \exp(-iA) = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = 1 - iA + \frac{(-iA)^2}{2!} + \cdots. \]

The most common one is the time-propagator or time-evolution operator $U$ which is the Hamiltonian function and propagates the wave function forward in time;

\[ U = \exp\left(\frac{-iHt}{\hbar}\right) = 1 + \frac{-iHt}{\hbar} + \frac{1}{2!}\left(\frac{-iHt}{\hbar}\right)^2 + \cdots. \]

For an operator $T \in \mathcal{L}(\mathcal{H})$, if $t \in \mathbb{R}$, then

\[ e^{itT} = I + itT + \frac{(it)^2}{2!}T^2 + \frac{(it)^3}{3!}T^3 + \cdots. \] (6)

**Theorem 3.3** If $T \in \mathcal{L}(\mathcal{H})$ is a skew $m$-complex symmetric operator with a conjugation $C$, then $e^{itT}$, $e^{-itT}$, and $e^{-itT^*}$ are $(m, C)$-isometric for every $t \in \mathbb{R}$. 
In general, the converse of the previous theorem may not be hold. But, if $e^{itT}$ is $(1, C)$-isometric operator and $T$ is a skew 2-complex symmetric operator with the conjugation $C$, then $T$ is a skew complex symmetric operator.

**Corollary 3.4** Let $T \in \mathcal{L}(\mathcal{H})$. Then the following statements hold:

(i) Assume that $T$ is skew $m$-complex symmetric with a conjugation $C$. If $\lambda \in \sigma_a(e^{itT})$, then $\frac{1}{\overline{\lambda}} \in \sigma_a(e^{-itT^*})$. In particular, if $\lambda \in \sigma_p(e^{itT})$, then $\frac{1}{\overline{\lambda}} \in \sigma_p(e^{-itT^*})$.

(ii) If $T$ is skew $m$-complex symmetric with a conjugation $C$, then $e^{itnT}$ is an $(m, C)$-isometric operator for any $n \in \mathbb{N}$.

(iii) Let $\{T_k\}$ be a sequence of skew $m$-complex symmetric operators with a conjugation $C$ such that $\lim_{k \to \infty} \|e^{itT_k} - e^{itT}\| = 0$. Then $e^{itT}$ is an $(m, C)$-isometric operator.

Recall that
$$
\cos(tT) = \frac{e^{itT} + e^{-itT}}{2} \quad \text{and} \quad \sin(tT) = \frac{e^{itT} - e^{-itT}}{2i}
$$
for every $t \in \mathbb{R}$.

**Corollary 3.5** Let $T \in \mathcal{L}(\mathcal{H})$ be skew complex symmetric with a conjugation $C$ and let $t \in \mathbb{R}$. Then the following statements hold.

(i) $\cos(tT)$ is a $(1, C)$-isometric operator if and only if $\cos(2tT^*) = I$.

(ii) $\sin(tT)$ is a $(1, C)$-isometric operator if and only if $\cos(2tT^*) = -I$.

A closed subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for $T$ if $TM \subset \mathcal{M}$.

**Corollary 3.6** If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric and complex symmetric with a conjugation $C$, i.e., $T^* = CTC$, then the following statements hold:

(i) $e^{itT}$ is an algebraic operator of order at most $2m$.

(ii) $Cker(\Gamma_{m-1}(e^{itT};C))$ is invariant for $e^{itT}$.

**Corollary 3.7** If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric and complex symmetric with a conjugation $C$, then the following statements hold.

(i) $e^{itT}$ is unitarily equivalent to a finite operator matrix of the form:

$$
\begin{pmatrix}
\alpha_1 & A_{12} & \cdots & \cdots & A_{1,2m} \\
0 & \alpha_2 & A_{23} & \cdots & A_{2,2m} \\
0 & 0 & \alpha_3 & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & A_{2m-1,2m} \\
0 & 0 & \cdots & \cdots & \alpha_{2m}
\end{pmatrix}
$$

where $\alpha_j$ are the roots of the polynomial $p(z)$ of degree at most $2m$.

(ii) The dimension of $\bigcap_{k=0}^{\infty}\{(e^{itT})^kx\}$ is less than or equals to $2m$. 
It is known from [15] that if $T$ is $m$-symmetric and $m$ is even, then $T$ is $(m-1)$-symmetric. In 2012, M. Cho, S. Ôta, K. Tanahashi, and A. Uchiyama proved that if $T$ is an invertible $m$-isometric operator and $m$ is even, then $T$ is an $(m-1)$-isometric operator (see [6] for more details). In view of these results, we will consider the following question: if $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric with a conjugation $C$ and $m$ is even, is it skew $(m-1)$-complex symmetric? In the next theorem, we give a partial solution for the previous question.

**Theorem 3.8** Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Suppose that $A_{m-1}(e^{itT};C)$ and $((e^{itT})^{*})^{m-1}A_{m-1}(e^{-itT};C)C(e^{itT})^{m-1}C$ are nonnegative. If $T$ is a skew $m$-complex symmetric operator with the conjugation $C$ where $m$ is even, then $T$ is skew $(m-1)$-complex symmetric and $e^{itT}$ is an $(m-1, C)$-isometric operator for all $t \in \mathbb{R}$.

**Corollary 3.9** If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric with a conjugation $C$, $m$ is even, and $[T, C] = 0$, then $T$ is skew $(m-1)$-complex symmetric.

4 On an operator $T$ commuting with $CTC$

In this section, we focus on an operator $T$ commuting with $CTC$. Given $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$ on $\mathcal{H}$, let

$$C_{C}(T) := \{S \in \mathcal{L}(\mathcal{H}) \mid [CTC, S] = 0\}$$

where $[R, S] := RS - SR$. In this section, we study the case when

$$T \in C_{C}(T), \text{ that is, } [CTC, T] = 0.$$

We observe that $C_{C}(T)$ need not contain complex symmetric operators.

**Example 4.1** Let $\mathcal{H} = \ell^{2}$, let $\{e_{n}\}$ be an orthonormal basis of $\mathcal{H}$ and let $C : \mathcal{H} \to \mathcal{H}$ be the conjugation given by $C(\sum_{n=0}^{\infty}x_{n}e_{n}) = \sum_{n=0}^{\infty}\overline{x}_{n}e_{n}$ where $\{x_{n}\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=0}^{\infty}|x_{n}|^{2} < \infty$ and $Ce_{n} = e_{n}$ for all $n$. If $W \in \mathcal{L}(\mathcal{H})$ is the weighted shift given by $We_{n} = \alpha_{n}e_{n+1}$ for all $n \geq 1$, then it is easy to compute $WCe_{n} = WCWe_{n}$ for all $n$. Hence $W \in C_{C}(W)$. In particular, if $\alpha_{n} = 1$ for all $n$, then $W = S$ is the unilateral shift and so $S \in C_{C}(S)$. However, $S$ is not complex symmetric.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^{*}T = TT^{*}$ and binormal if $T^{*}T$ and $TT^{*}$ commute where $T^{*}$ is the adjoint of $T$. Note that every normal operator is binormal.

**Example 4.2** Let $\mathcal{H} = \mathbb{C}^{2}$ and let $C$ be a conjugation on $\mathcal{H}$ given by $C(x, y) = (\overline{y}, \overline{x})$. Assume that $R = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ on $\mathcal{H}$. Then $CRC = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = R$. Hence $R \in C_{C}(R)$. However, $R$ is not normal, but binormal.
Example 4.3 Let $C$ and $J$ be conjugations on $\mathcal{H}$. Assume that $T = \begin{pmatrix} 0 & CJ \\ I & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Then $JTTJ = TTTJ = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Hence $T \in \mathcal{C}_J(T)$ is normal.

In the next example, we know that there exists $T$ such that $T \notin \mathcal{C}_C(T)$, in general.

Example 4.4 Let $\mathcal{H} = \mathbb{C}^n$ and $C(z_1, z_2, z_3, \cdots, z_n) = (\overline{z_n}, \cdots, \overline{z_3}, \overline{z_2}, \overline{z_1})$. If $T = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & 0 \\ \cdots & \cdots & 0 & \lambda_{n-1} & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{n-1} & \cdots & 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$ for all $\lambda_j \neq 0$, then $0 = (CTC)Te_1 \neq T(CTC)e_1 = \lambda_1 \cdot \overline{\lambda_{n-1}} \cdot e_1$. Hence $T \notin \mathcal{C}_C(T)$. But, it is clear that $T$ is binormal.

Theorem 4.5 If $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, then $T \in \mathcal{C}_C(T)$ for some conjugation $C$.

Note that every normal operator is complex symmetric (see [11]).

Proposition 4.6 Let $T \in \mathcal{C}_C(T)$ for some conjugation $C$. Then the following statements hold.

(i) $T^* \in \mathcal{C}_C(T^*)$.
(ii) $p(T) \in \mathcal{C}_C(p(T))$ for every polynomial $p$.
(iii) If $T$ is invertible, then $T^{-1} \in \mathcal{C}_C(T^{-1})$.
(iv) If $X \in \mathcal{L}(\mathcal{H})$ is invertible with $[X, C] = 0$, then $X^{-1}TX \in \mathcal{C}_C(X^{-1}TX)$.
(v) If $R \in \mathcal{L}(\mathcal{H})$ is unitarily equivalent to $T$, i.e., $R = UTU^*$, then $R \in \mathcal{C}_D(R)$ for a conjugation $D = UCU^*$.
(vi) $[T^m, CT^nC] = 0$ for all $n, m \in \mathbb{N}$.
(vii) The class of operators which satisfy $T \in \mathcal{C}_C(T)$ is norm closed.

Proposition 4.7 Let $C, C_1, C_2$ be conjugations on $\mathcal{H}$. Then the following statements hold.

(i) If $T_i \in \mathcal{L}(\mathcal{H}_i)$ be such that $T_i \in \mathcal{C}(T_i)$ for conjugations $C_i$ with $i = 1, 2$, respectively, then $T_1 \oplus T_2 \in \mathcal{C}_{C_1 \oplus C_2}(T_1 \oplus T_2)$ for a conjugation $C_1 \oplus C_2$.
(ii) Let $T \in \mathcal{C}_C(T)$ and $S \in \mathcal{C}_C(S)$. If $[T, S] = 0$ and $[CTC, S] = 0$, then $T+S \in \mathcal{C}_C(T+S)$ and $TS \in \mathcal{C}_C(TS)$ for a conjugation $C$.
(iii) If $T \in \mathcal{C}_C(T)$ and $S \in \mathcal{C}_C(S)$ for conjugations $C_1$ and $C_2$, respectively, then $T \otimes S \in \mathcal{C}_{C_1 \otimes C_2}(T \otimes S)$ for a conjugation $C_1 \otimes C_2$. 
In [11], if $T$ is complex symmetric, then $ReT$ and $ImT$ are complex symmetric.

**Proposition 4.8** Let $T \in C_C(T)$. Then the following statements hold:

(i) Let $R = \frac{T + CTC}{2}$ and $S = \frac{T - CTC}{2i}$. Then $R$ and $S$ belong to $C_C(T)$ such that $T = R + iS$ and $[R, S] = 0$, $[R, C] = 0$, and $[S, C] = 0$ hold.

(ii) If $T$ is normal, then $Re\, T \in C_C(Re\, T)$ and $Im\, T \in C_C(Im\, T)$.

**Lemma 4.9** ([17]) Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Then $\sigma(CTC) = \sigma(T)^*$ and $\sigma_a(CTC) = \sigma_a(T)^*$.

Therefore, if $T$ satisfies $[T, C] = 0$, then $\sigma(T) = \sigma(T)^*$, that is, $\sigma(T)$ is a symmetric set with the real line. For a commuting pair $(T, S) \in \mathcal{L}(\mathcal{H})^2$, $\sigma_T(T, S)$ and $\sigma_{ja}(T, S)$ denote the Taylor spectrum and the joint approximate point spectrum of $(T, S)$, respectively (see [2] and [19] for more details).

**Corollary 4.10** Let $T \in C_C(T)$. Then there exist commuting operators $R$ and $S$ such that the following statements hold:

(i) $T = R + iS$ and $(T, R, S)$ is a commuting 3-tuple.

(ii) $\sigma(R)$ and $\sigma(S)$ are symmetric sets with the real line.

(iii) If $\lambda \in \sigma(T)$, then there exist $\alpha \in \sigma(R)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.

(iv) If $\alpha \in \sigma(R)$, then there exist $\lambda \in \sigma(T)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.

(v) If $\beta \in \sigma(S)$, then there exist $\lambda \in \sigma(T)$ and $\alpha \in \sigma(R)$ such that $\lambda = \alpha + i\beta$.

Remark that the statements (iii), (iv) and (v) hold for the approximate point spectra $\sigma_a(T), \sigma_a(R)$ and $\sigma_a(S)$. Please see [2] for the spectral mapping theorem for the joint approximate point spectrum.

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\alpha_m(T; C)$ by

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C \cdot T^j.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-symmetric operator if $\alpha_m(T; C) = 0$ (see [5]).

**Theorem 4.11** If $T \in C_C(T)$ is an $[m, C]$-symmetric operator, then $CTC - T$ is $m$-nilpotent, i.e., $(CTC - T)^m = 0$.

**Corollary 4.12** If $T \in C_C(T)$ is an $[m, C]$-symmetric operator, then

$$\sigma_T(CTC, T) = \{ (\lambda, \lambda) : \lambda \in \sigma(T) \}.$$ 

In this case, it holds $\sigma(CTC) = \sigma(T) = \sigma(T)^*$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{ (\lambda, \lambda) : \lambda \in \sigma_a(T) \}.$
For an operator $T \in \mathcal{L}(\mathcal{H})$, $T$ is said to be \textit{normaloid} if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of $T$.

**Corollary 4.13** Let $T \in C_C(T)$ be an $[m, C]$-symmetric operator. If $CTC - T$ is normaloid, then $CTC - T = 0$.

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C \cdot T^{m-j}.$$ 

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-isometric operator if $\lambda_m(T; C) = 0$. See [4] for properties of $[m, C]$-isometric operators.

**Theorem 4.14** If $T \in C_C(T)$ is an $[m, C]$-isometric operator, then $CTC - I$ is $m$-nilpotent, i.e., $(CTC - I)^m = 0$.

**Corollary 4.15** If $T \in C_C(T)$ is an $[m, C]$-isometric operator, then $\sigma_T(CTC, T) = \{\left(\frac{1}{\lambda}, \lambda\right) : \lambda \in \sigma(T)\}$. In this case, it holds $\sigma(CTC) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{\left(\frac{1}{\lambda}, \lambda\right) : \lambda \in \sigma_a(T)\}$.

**Theorem 4.16** Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with a conjugation $C$. Suppose that $T = U|T|$ is the polar decomposition of $T$ where $U = CJ$ and $J$ is a partial conjugation supported on $\text{ran}(|T|)$, which commutes with $|T|$. Then the following statements are equivalent.

(i) $T$ is binormal.
(ii) $|T| \in C_C(|T|)$.
(iii) $[|\tilde{T}^D|, |T|] = 0$ where $\tilde{T}^D := |T|U$ is the Duggal transform of $T$.

**Corollary 4.17** Let $T \in \mathcal{L}(\mathcal{H})$ be such that $T^2$ is normal. Then $|T| \in C_C(|T|)$.

**Example 4.18** Let $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$. Then $T$ is complex symmetric with the conjugation $C$ defined by $C(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$ for $z_1, z_2 \in \mathbb{C}$. Since $|T| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, it follows that

$$C|T|C|T| = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } |T|C|T|C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$ 

Hence $T$ is not binormal by Theorem 4.16.
Example 4.19 Let \( \mathcal{H} = \ell^2 \) and let \( C \) be the canonical conjugation given by \( C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x}_n e_n \) with \( C e_n = e_n \) for all \( n \). Assume that \( T = \begin{pmatrix} S^* & I \\ 0 & S \end{pmatrix} \) on \( \mathcal{H} \oplus \mathcal{H} \), where \( S \in \mathcal{L}(\mathcal{H}) \) is the unilateral shift. Then \( S \) and \( S^* \) commute with the conjugation \( C \). Denote the conjugation \( C \) given by \( C = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \). Then we obtain that

\[
CT^* - TC = \begin{pmatrix} C & CS^* \\ CS & 0 \end{pmatrix} - \begin{pmatrix} C & S^*C \\ SC & 0 \end{pmatrix} = 0.
\]

Hence \( T \) is a complex symmetric operator (cf.[14]). Moreover, since \( T = \begin{pmatrix} S^* & I \\ 0 & S \end{pmatrix} \), it follows that \( T^*T = \begin{pmatrix} SS^* & S \\ S^* & 2I \end{pmatrix} \) and \( TT^* = \begin{pmatrix} 2I & S^* \\ S & SS^* \end{pmatrix} \). So, we have \( TT^*T^*T = \begin{pmatrix} 2SS^* + S^*2 & 2S + 2S^* \\ S^2S^* + SS^*2 & S^2 + 2SS^* \end{pmatrix} \) and \( T^*TT^*T^* = \begin{pmatrix} S^2 + 2SS^* & SS^*2 + S^2S^* \\ 2S + 2S^* & S^2 + 2SS^* \end{pmatrix} \). Hence \( T \) is not binormal. On the other hand, if \( S \) is the unilateral shift on \( \mathcal{H} \), then \( T = S^* \oplus S \) is binormal and complex symmetric.

References


[9] —, On \((m, C)\)-isometric operators, Complex Analysis and Operator Theory, 10(8), (2016), 1679-1694.


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