Semi-bounded quadratic functions over a sublevel set of another quadratic function

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Abstract

Before attempting to solve an optimization problem, it is important to detect whether the optimal value is finite. This is called the *boundedness* problem. A function f is called *semi-bounded* if it is either bounded from below or bounded from above. In this article, we apply the separation peoperty developed by Nguyen and Sheu [10] to study the semi-boundedness of a quadratic fucntion f over the sublevel set $\{x \in \mathbb{R}^n | \ g(x) \leq 0\}$ of another quadratic function g.

Key Words: S-lemma; S-lemma with equality; Separation property.

1 Introduction

Given a pair of quadratic functions (f, g) in n real variables $x = (x_1, x_2, \ldots, x_n)^T$ with $f(x) = x^T Ax + 2a^T x + c$ and $g(x) = x^T Bx + 2b^T x + d$, the theorem of S-lemma with equality states the necessary and sufficient conditions under which the following two statements are equivalent [i.e. $(\mathrm{E}_1) \sim (\mathrm{E}_2)$]:

(E_1) $(\forall x \in \mathbb{R}^n)\ g(x) = 0 \implies f(x) \geq 0.$

(E_2) $(\exists \lambda \in \mathbb{R})\ (\forall x \in \mathbb{R}^n)\ f(x) + \lambda g(x) \geq 0.$

The equivalence by $(\mathrm{E}_1) \sim (\mathrm{E}_2)$ means that both statements are true or false synchronously. Since (E_2) trivially implies (E_1), it is clear that $(\mathrm{E}_1) \sim (\mathrm{E}_2)$ if and only if (E_1) can imply (E_2). Geometrically, $(\mathrm{E}_1) \Leftrightarrow \{x \mid g(x) = 0\} \subseteq \{x \mid f(x) \geq 0\}$. See Figure 1 for an example. The question is, does there exist λ such that the non-zero linear

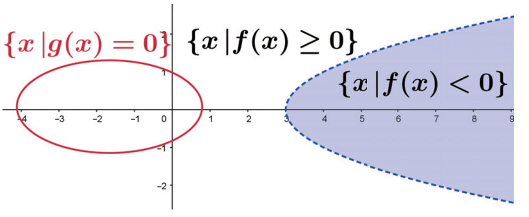

Figure 1: Example for $(\mathrm{E}_1) : \{x \mid g(x) = 0\} \subseteq \{x \mid f(x) \geq 0\}$

combination $f + \lambda g$ is non-negative on the entire $\mathbb{R}^n$?

The answer can be positive. Let $f(x, y) = -x^2 + y^2,\ g(x, y) = -x^2$. Then,

$$g(x, y) = 0 \ \Rightarrow\ x = 0 \ \Rightarrow f(0, y) = y^2 \geq 0,$$

so (E_1) holds. On the other hand, choose $\lambda = -1$. Then $f(x, y) + \lambda g(x, y) = y^2 \geq 0,\ \forall (x, y) \in \mathbb{R}^2$, so (E_2) holds as well.

The answer can be negative. Let $f(x_1, x_2) = -x_1^2 + 4x_2^2 + 2$, $g(x_1, x_2) = x_1 - x_2$. Then, (E_1) holds due to

$$g(x_1, x_2) = x_1 - x_2 = 0 \Longrightarrow f(x_1, x_2) = 3x_2^2 + 2 \geq 0.$$

But for any $\lambda \in \mathbb{R}$, we consider $(f + \lambda g)$ restricted on the x-axis:

$$(f + \lambda g)(x_1, 0) = -x_1^2 + 2 + \lambda x_1 < 0, \quad \text{as } x_1 \to \pm\infty.$$

Then, (E_2) fails for this pair of quadratics (f, g) and thus (E_1) $\not\sim$ (E_2).

The two examples indicate that to tell whether (E_1) $\sim$ (E_2) is not a simple question. It indeed was a long story. The first variant of the S-lemma with equality was proposed by Finsler [5] in 1937 where (f, g) is a pair of homogeneous quadratic forms. The second one was proposed by Fradkov and Jakubovich [6] in 1979 where they claimed that $B \neq 0$ implies that (E_1) $\sim$ (E_2). The result was unfortunately wrong with the following counter example $f(x, y) = -x^2 + y^2$, $g(x, y) = (x - y)^2$ where we notice that g does not satisfy the two-side Slater condition. The third variant was proposed by Luo, Sturm, and Zhang [8] in 2004 where g is assumed to be strictly concave (or strictly convex) satisfying the two-side Slater condition. The fourth variant was proposed by Hoang Tuy and Hoang Duong Tuan [15] in 2013 for a quadratic function f and a quadratic form g. The fifth variant was a complete necessary and sufficient condition for (E_1) $\not\sim$ (E_2) by Xia, Wang and Sheu [14] in 2016. There are other different special cases of the S-lemma with equality such as those in [3, 11] and in an early survey paper by Polik and Terlaky [12] in 2007.

Recently, Nguyen and Sheu [10] defined a notation of "separation property" and proved that it is indeed equivalent to the S-lemma with equality. Suppose the sublevel set $\{x | f(x) < 0\}$ consists of two connected components, denoted by L_- and L_+. We also define various level sets of f by $L_\alpha^*(f) = \{x | f(x) * \alpha\}$, where $* \in \{\leq, <, =, >, \geq\}$.

The hypersurface $\{x \in \mathbb{R}^n | g(x) = 0\}$ is said to separate $\{x \in \mathbb{R}^n | f(x) < 0\}$ if the two connected components L_- and L_+ of $\{x \in \mathbb{R}^n | f(x) < 0\}$ lie in the opposite sides of $\{x \in \mathbb{R}^n | g(x) = 0\}$ such that $L_- \cup L_+ = \{x \in \mathbb{R}^n | f(x) < 0\}$ and

$$g(L_-)g(L_+) = g(a^-)g(a^+) < 0, \ \forall\ a^- \in L_-;\ \forall\ a^+ \in L_+.$$

By proper coordinate change, a non-constance quadratic function $f(x)$ must adopt one of the following five canonical forms:

$$-x_1^2 - \cdots - x_k^2 + \delta(x_{k+1}^2 + \cdots + x_m^2) + \theta; \tag{1}$$

$$-x_1^2 - \cdots - x_k^2 + \delta(x_{k+1}^2 + \cdots + x_m^2) - 1; \tag{2}$$

$$-x_1^2 - \cdots - x_k^2 + \delta(x_{k+1}^2 + \cdots + x_m^2) + x_{m+1}; \tag{3}$$

$$x_1^2 + \cdots + x_m^2 + \delta x_{m+1} + c'. \tag{4}$$

$$\delta x_1 + c'. \tag{5}$$

where $k \geq 1$ is the number of negative eigenvalues of $f(x)$ and $\delta, \theta \in \{0, 1\}$. The following theorems proved in [10] will be used for studying the semi-boundedness property of the article.

Theorem 1 *[10] If $f(x)$ is a non-constance quadratic function then $L_0^<(f) = \{x \in \mathbb{R}^n |\ f(x) < 0\}$ contains exactly two connected components if and only if f has form (1) with $k = 1$. Furthermore, the two connected components are*

$$L_{0^-}^<(f) = \{x \in L_0^<(f) | x_1 < 0\} \text{ and } L_{0^+}^<(f) = \{x \in L_0^<(f) | x_1 > 0\}. \tag{6}$$

Theorem 2 *[10] Hypersurface $L_0^=(g)$ separates $L_0^<(f)$ if and only if*

(i) $f(x)$ has the form $-x_1^2 + \delta(x_2^2 + \cdots + x_m^2) + \theta,\ \delta, \theta \in \{0, 1\}$;

(ii) With the same basis and δ as in (i), $g(x)$ has the form $b_1x_1 + \delta(b_2x_2 + \cdots + b_mx_m) + b_0, b_1 \neq 0$;

(iii) $f|_{L_0^=(g)}(x) = -(\delta\frac{b_2}{b_1}x_2 + \cdots + \delta\frac{b_m}{b_1}x_m + \frac{b_0}{b_1})^2 + \delta(x_2^2 + \cdots + x_m^2) + \theta \geq 0,\ \forall (x_2, \cdots, x_n)^T \in \mathbb{R}^{n-1}$.

Theorem 3 *[10] If $g(x)$ takes both positive and negative values, then "(E_1) holds, but (E_2) fails" (S-lemma with equality fails) if and only if $L_0^=(g)$ separates $L_0^<(f)$.*

2 Main results

We first observe that $f(x)$ is bounded from below on $L_0^{\leq}(g)$ if, and only if $\exists\ \alpha_0 \in \mathbb{R}$ such that $L_{\alpha_0}^{<}(f) \cap L_0^{\leq}(g) = \emptyset$. If $g(x)$ satisfies the Slater condition, then

$$\begin{aligned} & (\exists\ \alpha_0 \in \mathbb{R})\ L_{\alpha_0}^{<}(f) \cap L_0^{\leq}(g) = \emptyset \\ \Leftrightarrow\ & (\exists\ \alpha_0 \in \mathbb{R})\ L_0^{\leq}(g) \subseteq\ L_{\alpha_0}^{\geq}(f) \\ \Leftrightarrow\ & (\exists\ \alpha_0 \in \mathbb{R})\ g(x) \leq 0 \Rightarrow\ (f - \alpha_0)(x) \geq 0 \\ \Leftrightarrow\ & (\exists\ \alpha_0 \in \mathbb{R})(\lambda \geq 0)(\forall x \in \mathbb{R}^n)\ f(x) + \lambda g(x) \geq \alpha_0, \quad \text{(By S-lemma)} \end{aligned}$$

where the last expression can be determined by solving an SDP program. Otherwise, when there is no Slater point so that $g(x) \geq 0,\ \forall x \in \mathbb{R}^n$, $L_0^{\leq}(g) = L_0^{=}(g)$ is subspace of $\mathbb{R}^n$. Whether $f(x)$ is bounded from below on $L_0^{\leq}(g)$ can be determined by restricting $f(x)$ on the subspace $L_0^{=}(g)$ to see whether the restriction is convex or not. If we want to test whether $\sup_{L_0^{\leq}(g)} f(x) < \infty$, the same can be done by verifying whether $\inf_{L_0^{\leq}(g)} -f(x) > -\infty$.

Below we apply the separation theorem for the level set of quadratic functions (Theorem 2) to combine the two tests (test for the lower bound and for the upper bound) of a semi-bounded function into a simple necessary and sufficient criterion. First, we have the following property:

Lemma 1 *Assume that $g(x) \leq 0$ satisfies the Slater condition. Then, $f(x)$ is bounded from below on $L_0^{<}(g)$ if and only if $f(x)$ is bounded from below on $L_0^{\leq}(g)$.*

Proof Proof. Let $\bar{x}$ be a boundary point of $L_0^{<}(g)$ such that $g(\bar{x}) = 0$. We claim that there exists a sequence in $L_0^{<}(g)$ converging to $\bar{x}$. By a change coordinate $s = x - \bar{x}$, $g(s) = s^T Bs + 2(\bar{x}^T B + 2b^T)s$ and $s = 0$ is a boundary point of $L_0^{<}(g)$. Therefore, we may simply assume that the boundary point $\bar{x} = 0$ and that $g(x) = x^T Bx + 2b^t x$. If $\nabla g(0) = 2Bx + 2b = 0$, then $b = 0$ and $g(x) = x^T Bx$ is a quadratic form. Since $g(x) \leq 0$ satisfies the Slater condition, B has a negative eigenvalue associated with an eigenvector

w. Then, the sequence $\{\frac{w}{n}\}_{n\in\mathbb{N}}$ converges to $\bar{x}=0$ and $g(\frac{w}{n})<0$. On the other hand, let $\nabla g(0)\neq 0$. By Gateaux differentiability of g at 0 along the direction $-\nabla g(0)$, we have

$$\lim_{h\to 0^+}\frac{g(0+h(-\nabla g(0))-g(0)}{h}=\lim_{h\to 0^+}\frac{g(h(-\nabla g(0))}{h}=\langle\nabla g(0),-\nabla g(0)\rangle<0.$$

It shows that, with h small enough, $g(-h\nabla g(0))<0$ and $\{-h\nabla g(0)\}$ converges to 0 as $h\to 0^+$. Now suppose that $f(x)$ is bounded from below on $L_0^<(g)$ such that $g(x)<0\Rightarrow f(x)\geq\alpha_0$ for some $\alpha_0\in\mathbb{R}$. Since every point $\bar{x}\in L_0^=(g)$ has a convergent sequence in $L_0^<(g)$ that approaches to it, by the continuity of f, $f(\bar{x})\geq\alpha_0$. Therefore, f is bounded from below on $L_0^{\leq}(g)$. Conversely, if $f(x)$ is bounded from below on $L_0^{\leq}(g)$ and $g(x)$ has a Slater point, it trivially implies that $f(x)$ is bounded from below on $L_0^<(g)$. □

With Lemma 1, we can pose the semi-boundedness theorem equivalently on $L_0^<(g)$.

Theorem 4 *Function f is semi-bounded on $L_0^<(g)$ if and only if*

$$\begin{cases}(\exists\ \alpha_0)\ g(x)<0\Rightarrow f(x)\neq\alpha_0\ (\textit{equivalently}, (\exists\ \alpha_0)\ f(x)=\alpha_0\Rightarrow g(x)\geq 0);\\ L_{\alpha_0}^=(f)\ \textit{does not separate}\ L_0^<(g).\end{cases}$$

Proof Proof for necessity: If, on the contrary, $\forall\ \alpha\in\mathbb{R}$, there exists some $\bar{x}\in\mathbb{R}^n$ such that $g(\bar{x})<0$ and $f(\bar{x})=\alpha$. Then, $f(x)$ is unbounded from below and also from above on $L_0^<(g)$. It cannot be not semi-bounded. Secondly, for α_0 such that $g(x)<0\Rightarrow f(x)\neq\alpha_0$, if $L_{\alpha_0}^=(f)$ separates $L_0^<(g)$, by theorem 2, there exists a basis of $\mathbb{R}^n$ such that $g(x)$ is of form (1) that $g(x)=-x_1^2+\delta(x_2^2+\cdots+x_m^2)+\theta$; and $f-\alpha_0$ is affine function with form $b_1x_1+\delta(b_2x_2+\cdots+b_mx_m)+b_0, b_1\neq 0$. Then, it is easy to see that points $(x_1,0,\cdots,0)\in L_0^<(g)$ for $|x_1|$ sufficiently large, on which $f(x)$ is unbound from below and also form above.

Proof for Sufficiency: If there exists α_0 such that $g(x)<0\Rightarrow f(x)\neq\alpha_0$ and $L_{\alpha_0}^=(f)$ does not separate $L_0^<(g)$, we need to prove that f is semi-bounded on $L_0^<(g)$. Indeed, if $f(x)$ is semi-bounded on $\mathbb{R}^n$, it is also semi-bounded on $L_0^<(g)$. If $f(x)$ is not semi-bounded on $\mathbb{R}^n$, it implies that $f(x)-\alpha_0$ takes both positive and negative values on $\mathbb{R}^n$. Due to $g(x)<0\Rightarrow f(x)\neq\alpha_0$, we have $f(x)-\alpha_0=0\Rightarrow g(x)\geq 0$ and thus

(E_1) holds for the pair $(g, f-\alpha_0)$. From the assumption that $L^{=}_{\alpha_0}(f)$ does not separate $L^{<}_0(g)$, the S-lemma with equality holds by theorem 3. It implies that $\exists\,\lambda \in \mathbb{R}$ such that $g(x)+\lambda(f(x)-\alpha_0) \geq 0, \forall\, x \in \mathbb{R}^n$.

If $\lambda \geq 0$, by (classical) S-lemma, $(\exists\ \alpha_0)\ f(x)-\alpha_0 \leq 0 \Rightarrow g(x) \geq 0$. Therefore, $L^{\leq}_{\alpha_0}(f) \cap L^{<}_0(g) = \emptyset$ and $f(x)$ is bounded from below by α_0 on $L^{<}_0(g)$. If $\lambda < 0$, $(\exists -\lambda \geq 0)\ g(x)+(-\lambda)(-f(x)+\alpha_0) \geq 0, \forall\, x \in \mathbb{R}^n$. Again by S-lemma, $-f(x)+\alpha_0 \leq 0 \Rightarrow g(x) \geq 0$. Then, $g(x) < 0 \Rightarrow f(x) < \alpha_0$ and $f(x)$ is bounded from above by α_0 on $L^{<}_0(g)$. It means that f is semi-bounded on $L^{<}_0(g)$. □

Example: Let $g(x,y) = x^2+y^2+3xy+x+y+1,\ f(x,y) = x^2+y^2-2xy-x-y$. By a change of variables $x = u+v, y = -u+v$, we find that $g(u,v) = -u^2+5v^2+2v+1$ is of form (1) so $L^{<}_0(g)$ consists of two connected components. Under the $u-v$ coordinate system, $f(u,v) = 4u^2-2v$. It is easy to see that $f(u,v) = 0 \Leftrightarrow v = 2u^2$ on which $g(u,2u^2) > 0$ so that $f = 0 \Rightarrow g \geq 0$. On the other hand, since $f(u,v) = 4u^2-2v$ has a non-vanishing quadratic term $4u^2$, by Theorem 2, $L^{=}_0(f)$ cannot separate $L^{<}_0(g)$. By Theorem 4, f is semi-bounded on $L^{<}_0(g)$.

Now let $g(x,y) = x^2+y^2+3xy+x+y+1,\ f(x,y) = x-y$. With $x = u+v, y = -u+v$, $g(u,v) = -u^2+5v^2+2v+1, f(u,v) = 2u$. Then, $f(u,v) = 0 \Leftrightarrow u = 0$ on which $g(0,v) = 5v^2+2v+1 > 0$. By Theorem 2, $L^{=}_0(f)$ separates $L^{<}_0(g)$, so f is not semi-bounded on $L^{<}_0(g)$. We can check that, with $X_n(-2n,n), Y_n(n,-2n)$, we have $g(X_n) = g(Y_n) = -n^2-n+1$, which indicates that $X_n, Y_n \in L^{<}_0(g)$ for n large enough. However, $f(X_n) = -3n \overset{n\to\infty}{\longrightarrow} -\infty$, $f(Y_n) = 3n \overset{n\to\infty}{\longrightarrow} +\infty$. The function f is not semi-bounded on $L^{<}_0(g)$. The example justifies Theorem 4.

Theorem 5 *Assume degree of g is two then function f is semi-bounded on $L^{=}_0(g)$ if and only if*

$$\begin{cases} (\exists\ \alpha_0)\ g(x) = 0 \Rightarrow f(x) \neq \alpha_0; \\ \textit{When } \operatorname{rank}(B) \geq 2 \textit{ then } L^{=}_{\alpha_0}(f) \textit{ does not separate } L^{*}_0(g) \textit{ where } * \in \{<,>\}. \end{cases}$$

Proof Proof for necessity: If, on the contrary, $\forall\ \alpha \in \mathbb{R}$, there exists some $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) = 0$ and $f(\bar{x}) = \alpha$. Then, $f(x)$ is unbounded from below and also from above on $L_0^=(g)$. It cannot be not semi-bounded.

Secondly, with $\text{rank}(B) \geq 2$, for α_0 such that $g(x) = 0 \Rightarrow f(x) \neq \alpha_0$, and consider two following cases:

Case 1: $L_{\alpha_0}^=(f)$ separates $L_0^<(g)$. Then by Theorem 2, there exists a basis of $\mathbb{R}^n$ such that $g(x)$ is of form (1) that $g(x) = -x_1^2 + \delta(x_2^2 + \cdots + x_m^2) + \theta$ where $\delta \in \{0, 1\}$; and $f - \alpha_0$ is affine function with form $b_1x_1 + \delta(b_2x_2 + \cdots + b_mx_m) + b_0, b_1 \neq 0$. By $\text{rank}(B) \geq 2$, δ must be 1 (since if $\delta = 0$ then $g(x) = -x_1^2 + \theta$, $\text{rank}(B) = 1$). If $b_2 = \cdots = b_m = 0$ then, it is easy to see that points $(\pm\sqrt{x_2^2 + \theta}, x_2, \cdots, 0) \in L_0^=(g)$ for $|x_2|$ sufficient large, on which $f(x)$ is unbound from below and also form above. If $b_i \neq 0$ for some $i \in \{2, \cdots, m\}$, without loss of generality, we assume $i = 2$. We consider a subset of $L_0^=(g)$ form $\{(x_1, x_2, 0, \cdots, 0) | g(x_1, x_2, 0, \cdots, 0) = -x_1^2 + x_2^2 + \theta = 0\}$, on which $f(x_1, x_2, 0, \cdots, 0) = b_1x_1 + b_2x_2 + b_0$. It is easy to see that $\begin{cases} b_1x_1 + b_2x_2 + b_0 = \alpha \\ -x_1^2 + x_2^2 + \theta = 0 \end{cases}$ has solution for all large enough $|\alpha|$. (Indeed, this system equations is equivalence to $\begin{cases} b_1x_1 + b_2x_2 + b_0 = \alpha \\ -(\dfrac{\alpha - b_0 - b_2x_2}{b_1})^2 + x_2^2 + \theta = 0 \end{cases}$, and it has solution if and only if $\Delta = b_1^2(\alpha - b_1)^2 + \theta b_1^2 - \theta b_2^2 \geq 0$, obviously, with $|\alpha|$ is large enough, $\Delta > 0$). It means f is unbounded from below and also from above on $L_0^=(g)$.

Case 2: $L_{\alpha_0}^=(f)$ separates $L_0^>(g)$. Note that $L_0^>(g) = L_0^<(-g)$ which implies that $L_{\alpha_0}^=(f)$ separates $L_0^<(\bar{g})$ where $\bar{g} = -g$. Apply the result in *Case 1* for pair $(\bar{g}, f)$, we therefore conclude that f is unbounded from below and also from above on $L_0^=(\bar{g}) = L_0^=(-g) = L_0^=(\bar{g})$.

Proof for the sufficiency: If there exists α_0 such that $g(x) = 0 \Rightarrow f(x) \neq \alpha_0$ and $L_{\alpha_0}^=(f)$ does not separate $L_0^<(g)$ and $L_0^>(g)$ when $\text{rank}(B) \geq 2$, we need to prove that f is semi-bounded on $L_0^=(g)$. Indeed, if $f(x)$ is semi-bounded on $\mathbb{R}^n$, it is also semi-bounded on

$L_0^{<}(g)$. If $f(x)$ is not semi-bounded on $\mathbb{R}^n$, it implies that

$$\bar{f} = f(x) - \alpha_0 \text{ takes both positive and negative values on } \mathbb{R}^n \quad (7)$$

By $L_0^{=}(\bar{f}) = L_{\alpha_0}^{=}(f)$ has no any common point with $L_0^{=}(g)$ (due to assumption $g(x) = 0 \Rightarrow f(x) \neq \alpha_0$), which implies that $\bar{f}(x) = f(x) - \alpha_0 = 0$ then $g(x) > 0$ or $-g(x) > 0$. Therefore

$$\text{The condition } (\mathrm{E}_1) \text{ holds for pair } (\mathrm{g}, \bar{\mathrm{f}}) \text{ or pair } (-\mathrm{g}, \bar{\mathrm{f}}) \quad (8)$$

When rank$(B) \geq 2$. Combine assumption that $L_{\alpha_0}^{=}(f)$ does not separate $L_0^{*}(g)$ where $* \in \{<, >\}$ with (7), (8) and by theorem 3, S-lemma with equality holds for pair $(g, \bar{f})$ or pair $(-g, \bar{f})$. It implies that $\exists\, \lambda \in \mathbb{R}$ such that $g(x) + \lambda(f(x) - \alpha_0) \geq 0\, \forall\, x \in \mathbb{R}^n$ or $-g(x) + \lambda(f(x) - \alpha_0) \geq 0\, \forall\, x \in \mathbb{R}^n$.

If $\lambda \geq 0$, By (classical) S-lemma, $(\exists\ \alpha_0)\ f(x) - \alpha_0 \leq 0 \Rightarrow g(x) \geq 0$ or $-g(x) \geq 0$. Therefore, $L_{\alpha_0-1}^{\leq}(f) \cap L_0^{\leq}(g) = \emptyset$ or $L_{\alpha_0-1}^{\leq}(f) \cap L_0^{\leq}(-g) = \emptyset$. It means $f(x)$ is bounded from below on $L_0^{\leq}(g)$ or $L_0^{\leq}(-g)$, it implies f is semi-bounded on $L_0^{=}(g)$.

If $\lambda < 0$, it means that $\exists\, -\lambda \geq 0$ such that $g(x) + (-\lambda)(-f(x) + \alpha_0) \geq 0\, \forall\, x \in \mathbb{R}^n$ or $-g(x) + (-\lambda)(-f(x) + \alpha_0) \geq 0\, \forall\, x \in \mathbb{R}^n$. Again by S-lemma, $-f(x) + \alpha_0 \leq 0 \Rightarrow g(x) \geq 0$ or $-g(x) \geq 0$. Therefore, $L_{\alpha_0+1}^{\geq}(f) \cap L_0^{\leq}(g) = \emptyset$ or $L_{\alpha_0+1}^{\geq}(f) \cap L_0^{\leq}(-g) = \emptyset$. It means $f(x)$ is bounded from above on $L_0^{\leq}(g)$ or $L_0^{\leq}(-g)$, it implies that f is semi-bounded on $L_0^{=}(g)$.

□

Note that with rank$(B) = 1$, if $L_{\alpha_0}^{=}(f)$ does not separate $L_0^{<}(g)$ and $L_0^{>}(g)$, by the same argument above, f is semi-bounded on $L_0^{=}(g)$. If $L_{\alpha_0}^{=}(f)$ separates $L_0^{<}(g)$ or $L_0^{>}(g)$, by theorem 2, there exists a basis of $\mathbb{R}^n$ such that $g(x)$ is of form (1) that $g(x) = -x_1^2 + \theta$ or $-g(x) = -x_1^2 + \theta$ (since rank$(B) = 1$); and $f - \alpha_0$ is affine function with form $b_1 x_1 + b_0, b_1 \neq 0$. It is easy to see that $f(x)$ is bound on $L_0^{=}(g)$ since on $L_0^{=}(g)$, where x_1 is constant.

3 Conclusions

This is a short article which introduces the concept of semi-boundedness of a quadratic function f over the sublevel set of another quadratic function g. By the separation property and the S-lemma with equality, we have, in Theorem 4, given necessary and sufficient condition for the semi-boundedness property. The image set of f over $\{g \leq 0\}$ is fundamentally important in mathematics. Being able to characterize the boundedness of the image set of f over $\{g \leq 0\}$ is the first step to study the problem. We hope that more results, such as the connectedness of the image set of f over $\{g \leq 0\}$, can be obtained following the separation property in the future.

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