AN EXTENSION OF VECTOR-VALUED METRIC SPACES AND PEROV'S FIXED POINT THEOREM

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Abstract: In this article, we have introduced the notion of Czerwik vector-valued metric spaces. By using this structure, we proved the extended results of the Perov's fixed point theorem. To illustrate our result we have provided an example and application.

Keywords: Czerwik vector-valued metric spaces, Perov's fixed point theorem, generalized metric space.

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1 Introduction

The Banach contraction principle is a fundamental result of metric fixed point theory. This result has many applications in different branches of mathematics like differential and integral equations, optimization and variational analysis, etc. The simplicity and applicability of this result attracted many researchers, thats why, this result has many generalizations in different settings. One of the worthwhile generalization of this result was given by Perov [11] in 1964. In [11], Perov extended the Banach contraction principle to a space with vector-valued metric [11]. This result helps to study the existence of solution for different types of differential and integral equations. Some interesting contributions to the development of fixed point theory and its applications in this context are obtained by Bica-Muresan [4], Bucur-Guran-Petruşel [7], Filip-Petruşel [9], O'Regan-Shahzad-Agarwal [10], Rus [12], Turinici [14] and Ali-Tchierb-Vetro [2].

Let X be a nonempty set. Throughout the paper by $\mathbb{R}_+$ we denote the set of all nonnegative real numbers and by $\mathbb{R}_m$ the set of all $m \times 1$ real matrices. Let $\alpha, \beta \in \mathbb{R}_m$, that is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$. Then by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for each $i \in \{1, 2, \ldots, m\}$. A mapping $d \colon X \times X \to \mathbb{R}_m$ is called a vector-valued metric on X [1], if the following properties are satisfied:

(d_1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$ then $x = y$, and viceversa;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Thus, a nonempty set X with a vector-valued metric d is called a generalized metric space, say (X, d). Notice that the convergence sequence and Cauchy sequence in generalized metric spaces are defined in a similar manner as in a usual metric space.

Also, in this article we denote the set of all $m \times m$ matrices with nonnegative real elements by $M_{m,m}(\mathbb{R}_+)$, the zero $m \times m$ matrix by $\bar{0}$ and the identity $m \times m$ matrix by I. Note that $A^0 = I$. Let $A \in M_{m,m}(\mathbb{R}_+)$. Then A is said to be convergent to zero if and only if $A^n \to \bar{0}$ as $n \to \infty$ (see Varga [15]). It is easy to see that the following matrices are convergent to zero.

$$A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}_+ \text{ and } a + b < 1;$$

$$B := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \text{ where } a, b, c \in \mathbb{R}_+ \text{ and } \max\{a, c\} < 1.$$

From Filip-Petruşel [9], we discuss some equivalent properties of convergent matrices to zero.

Theorem 1.1. *[9] Let $A \in M_{m,m}(\mathbb{R}_+)$. Then the following conditions are equivalent:*

(i) *A is convergent to zero;*

(ii) *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;*

(iii) *The matrix $I - A$ is nonsingular (that is, its determinant is nonzero) and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$.*

Perov [11] extended the Banach contraction principle [3] to a space endowed with generalized metric in the following way:

Theorem 1.2. *[11] Let (X, d) be a complete generalized metric space and $f : X \to X$ be a mapping for which there exists a matrix $A \in M_{m,m}(R_+)$ such that $d(fx, fy) \leq Ad(x, y)$ for all $x, y \in X$. If A is a convergent matric to zero, then*

(i) *$Fix(f) = \{x^*\}$, where $Fix(f) = \{x \in X : x = fx\}$;*

(ii) *the sequence of successive approximations $\{x_n\}$ such that $x_n = f^n x_0$ is convergent and admits the limit x^*, for all $x_0 \in X$.*

2 Main Result

We begin this section by extending the definition of generalized metric space in sense of Czerwik [8]. In the following definition, $S = (s_{ij})$ is $m \times m$ matrix such that

$$s_{ij} = \begin{cases} s, & i = j \\ 0, & i \neq j \end{cases}$$

where $s \geq 1$.

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Definition 2.1. *A mapping $d: X \times X \to \mathbb{R}_m$ is called Czerwik vector-valued metric on X, if there exists a matrix $S \in M_{m,m}(R_+)$ such that for each $x, y, z \in X$, the following conditions are satisfied:*

(d_1) *$d(x, y) \geq 0$; if $d(x, y) = 0$ then $x = y$, and viceversa;*

(d_2) *$d(x, y) = d(y, x)$;*

(d_3) *$d(x, y) \leq S[d(x, z) + d(z, y)]$.*

Then, a nonempty set X equipped with Czerwik vector-valued metric d is called Czerwik generalized metric space, denoted by (X, d, S).

Note that after some simplification the above definition can be reduced to [5, Definition 2.1].

Example 2.2. *Let $X = \mathbb{R}^2$. Then the mapping $d : X \times X \to \mathbb{R}_2$ defined by*

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = \begin{pmatrix} (x_1 - y_1)^2 \\ (x_2 - y_2)^2 \end{pmatrix} \text{ for each } x, y \in X$$

is a Czerwik generalized metric with matrix $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Note that the convergence sequence and Cauchy sequence in Czerwik generalized metric spaces are defined in a similar manner as in b-metric space/metric space.

Throughout this section, (X, d, S) is a Czerwik generalized metric space and $G = (V, E)$ is a directed graph such that the set V of its vertices coincides with X and the set E of its edges contains all loops, that is, $E \supseteq \{(x, x) : x \in V\}$. Also we denote by $CL(X)$ the set of nonempty closed subsets of X.

Theorem 2.3. *Let (X, d, S) be a complete Czerwik generalized metric space endowed with the graph G. Let $T: X \to CL(X)$ be a multi-valued mapping such that for each $(x, y) \in E$ and $u \in Tx$, there exists $v \in Ty$ satisfying the following inequality:*

$$d(u, v) \leq Ad(x, y) + Bd(y, u) \tag{2.1}$$

where $A, B \in M_{m,m}(\mathbb{R}_+)$. Further, assume that the following conditions hold:

  (i) *the matrix SA converges to zero;*

  (ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;*

  (iii) *for each $u \in Tx$ and $v \in Ty$ with $d(u, v) \leq Ad(x, y)$, we have $(u, v) \in E$ whenever $(x, y) \in E$;*

  (iv) *for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By using hypothesis (ii), we get $x_0 \in X$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E$. From (2.1), for $(x_0, x_1) \in E$, we have $x_2 \in Tx_1$ satisfying the following inequality:

$$
\begin{aligned}
d(x_1, x_2) &\leq Ad(x_0, x_1) + Bd(x_1, x_1) \\
&= Ad(x_0, x_1).
\end{aligned}
\tag{2.2}
$$

By using hypothesis (iii) and (2.2), we get $(x_1, x_2) \in E$. Again from (2.1) and (2.2), for $(x_1, x_2) \in E$ and $x_2 \in Tx_1$, we have $x_3 \in Tx_2$ such that

$$
\begin{aligned}
d(x_2, x_3) &\leq Ad(x_1, x_2) + Bd(x_2, x_2) \\
&\leq A^2 d(x_0, x_1).
\end{aligned}
\tag{2.3}
$$

Continuing in this process, we construct a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $(x_{n-1}, x_n) \in E$ and

$$
d(x_n, x_{n+1}) \leq A^n d(x_0, x_1), \ \forall \ \in \mathbb{N}.
$$

In order to prove that $\{x_n\}$ is a Cauchy sequence, we consider arbitrary $n, m \in \mathbb{N}$ with $m > n$. By using the triangle inequality, we get

$$
\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=n}^{m-1} S^i d(x_i, x_{i+1}) \\
&\leq \sum_{i=n}^{m-1} S^i A^i d(x_0, x_1) \\
&\leq S^n A^n \left(\sum_{i=0}^{\infty} S^i A^i \right) d(x_0, x_1).
\end{aligned}
\tag{2.4}
$$

Since the nonzero elements of the diagonal matrix S are same, $S^n A^n = (SA)^n$ for each $n \in \mathbb{N} \cup \{0\}$. Therefore, from (2.4) we get

$$
\begin{aligned}
d(x_n, x_m) &\leq S^n A^n \left(\sum_{i=0}^{\infty} S^i A^i \right) d(x_0, x_1) \\
&= (SA)^n \left(\sum_{i=0}^{\infty} (SA)^i \right) d(x_0, x_1) \\
&= (SA)^n (I - SA)^{-1} d(x_0, x_1).
\end{aligned}
\tag{2.5}
$$

Since the matrix SA converges to zero, this implies that the sequence $\{x_n\}$ is Cauchy in X. From the completeness of X, there exists an $x^* \in X$ such that $x_n \to x^*$. By hypothesis (iv), we obtain $(x_n, x^*) \in E$, for each $n \in \mathbb{N}$. From (2.1), for $(x_n, x^*) \in E$ and $x_{n+1} \in Tx_n$ we have $q^* \in Tx^*$ such that

$$
d(x_{n+1}, q^*) \leq Ad(x_n, x^*) + Bd(x^*, x_{n+1}).
$$

By using the triangle inequality and the above inequality, we get

$$
\begin{aligned}
d(x^*, q^*) &\leq S[d(x^*, x_{n+1}) + d(x_{n+1}, q^*)] \\
&\leq Sd(x^*, x_{n+1}) + SAd(x_n, x^*) + SBd(x^*, x_{n+1}).
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, we get $d(x^*, q^*) = 0$, that is, $x^* = q^*$. Thus $x^* \in Tx^*$. $\qquad \square$

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Example 2.4. Let $X = \mathbb{R}^2$ be endowed with Czerwik generalized metric defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = \begin{pmatrix} (x_1 - y_1)^2 \\ (x_2 - y_2)^2 \end{pmatrix}$ for each $x, y \in X$ with a matrix $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Define a mapping $T \colon \mathbb{R}^2 \to CL(\mathbb{R}^2)$ by

$$T(x_1, x_2) = \begin{cases} \left\{\left(\frac{x_1}{3} - \frac{x_2}{6} + 1, \frac{x_2}{6} + 1\right), (0, 0)\right\} & \text{for } (x_1, x_2) \in X \text{ with } x_1, x_2 \le 3 \\ \left\{\left(\frac{x_1}{3} - \frac{x_2}{6}, \frac{-5x_1}{3} + \frac{x_2}{3}\right), (1, 0)\right\} & \text{otherwise.} \end{cases}$$

Define a directed graph $G = (V, E)$ such that $V = \mathbb{R}^2$ and $E = \{((x_1, x_2), (y_1, y_2)) : x_1, x_2, y_1, y_2 \in [0, 3]\} \cup \{(z, z) : z \in \mathbb{R}\}$. Thus for each $((x_1, x_2), (y_1, y_2)) \in E$ and $(u_1, u_2) \in T(x_1, x_2)$, we have $(v_1, v_2) \in T(y_1, y_2)$ such that

$$d((u_1, u_2), (v_1, v_2)) \le Ad((x_1, x_2), (y_1, y_2)),$$

where $A = \begin{pmatrix} \frac{2}{9} & \frac{2}{36} \\ 0 & \frac{2}{36} \end{pmatrix}$. Further, it is easy to see that the matrix SA converges to zero and for $(1, 1) \in X$, we have $(0, 0) \in T(1, 1)$ such that $((1, 1), (0, 0)) \in E$. Also for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$. Thus, Theorem 2.3 implies that T has fixed point. Note that $(0, 0)$ and $(\frac{6}{5}, \frac{6}{5})$ are two fixed points of T.

In case of single-valued mappings, Theorem 2.3 reduces to the following corollary:

Corollary 2.5. *Let (X, d, S) be a complete Czerwik generalized metric space with the graph G. Let $T \colon X \to X$ be a mapping such that for each $(x, y) \in E$ we have*

$$d(Tx, Ty) \le Ad(x, y) + Bd(y, Tx),$$

where $A, B \in M_{m,m}(\mathbb{R}_+)$. Further, assume that the following conditions hold:

(i) *the matrix SA converges to zero;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$;*

(iii) *for each $(x, y) \in E$, we have $(Tx, Ty) \in E$, provided $d(Tx, Ty) \le Ad(x, y)$;*

(iv) *for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

By considering the graph $G = (V, E)$ as $V = X$ and $E = X \times X$, Corollary 2.5 reduces to the following result.

Corollary 2.6. *Let (X, d, S) be a complete Czerwik generalized metric space. Let $T \colon X \to X$ be a mapping such that for each $x, y \in X$ we have*

$$d(Tx, Ty) \le Ad(x, y) + Bd(y, Tx),$$

where $A, B \in M_{m,m}(\mathbb{R}_+)$. Also assume that the matrix SA converges to zero. Then T has a fixed point.

Next, we extend Theorem 2.3 in the setting of two Czerwik generalized metrics.

Theorem 2.7. *Let (X, d, S) be a complete Czerwik generalized metric space with the graph G and ρ be an another Czerwik generalized metric on X with the same constant matrix S. Let $T \colon X \to CL(X)$ be a multi-valued mapping such that for each $(x, y) \in E$ and $u \in Tx$, there exists $v \in Ty$ satisfying the following inequality*

$$\rho(u, v) \leq A\rho(x, y) + B\rho(y, u), \tag{2.6}$$

where $A, B \in M_{n,n}(\mathbb{R}_+)$. Further, assume that the following conditions hold:

(i) *the matrix SA converges to zero;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;*

(iii) *for each $(x, y) \in E$, we have $(u, v) \in E$ provided $\rho(u, v) \leq A\rho(x, y)$, where $u \in Tx$ and $v \in Ty$;*

(iv) *there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $d(x, y) \leq C\rho(x, y)$, whenever, there exists a path between x and y, that is, we have a sequence $\{x_i\}_{i=0}^{p}$ such that $(x_i, x_{i+1}) \in E$ for each $i \in \{0, 1, \cdots, p-1\}$ with $x_0 = x$ and $x_p = y$;*

(v) *Functional $Graph(T) = \{(x, y) \in X \times X : y \in Tx\}$ is G-closed with respect to d, that is, if a sequence $\{x_n\}$ is such that $(x_n, x_{n+1}) \in E$, $(x_n, x_{n+1}) \in Functional\ Graph(T)$ and $x_n \to x^*$, then $(x^*, x^*) \in Functional\ Graph(T)$.*

Then T has a fixed point.

Proof. By using hypothesis (ii), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (2.6), for $(x_0, x_1) \in E$ and $x_1 \in Tx_0$, we have $x_2 \in Tx_1$ such that

$$\begin{aligned} \rho(x_1, x_2) &\leq A\rho(x_0, x_1) + B\rho(x_1, x_1) \\ &= A\rho(x_0, x_1). \end{aligned}$$

By using hypothesis (iii) and the above inequality, we conclude that $(x_1, x_2) \in E$. Again from (2.6) for $(x_1, x_2) \in E$ and $x_2 \in Tx_1$, we have $x_3 \in Tx_2$ such that

$$\begin{aligned} \rho(x_2, x_3) &\leq A\rho(x_1, x_2) + B\rho(x_2, x_2) \\ &\leq A^2\rho(x_0, x_1). \end{aligned}$$

Continuing in this process, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $(x_{n-1}, x_n) \in E$ and

$$\rho(x_n, x_{n+1}) \leq A^n\rho(x_0, x_1) \text{ for each } n \in \mathbb{N}.$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in both (X, d, S) and (X, ρ, S). Consider

arbitrary $n, m \in \mathbb{N}$ and by using the triangle inequality, we get the following

$$
\begin{aligned}
\rho(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} S^i \rho(x_i, x_{i+1}) \\
&\leq \sum_{i=n}^{n+m-1} S^i A^i \rho(x_0, x_1) \\
&= \sum_{i=n}^{n+m-1} (SA)^i \rho(x_0, x_1) \qquad (2.7) \\
&\leq (SA)^n \left(\sum_{i=0}^{\infty} A^i \right) \rho(x_0, x_1) \\
&= (SA)^n (I - SA)^{-1} \rho(x_0, x_1).
\end{aligned}
$$

Since the matrix SA converges to zero, this implies that $\{x_n\}$ is a Cauchy sequence in (X, ρ, S). Further, note that for each $n, m \in \mathbb{N}$ there exists a path between x_n and x_{n+m}. Thus by hypothesis (iv) and (2.7), we get the following:

$$
\begin{aligned}
d(x_n, x_{n+m}) &\leq C\rho(x_n, x_{n+m}) \\
&\leq C[(SA)^n (I - SA)^{-1} \rho(x_0, x_1)].
\end{aligned}
$$

Thus, $\{x_n\}$ is also a Cauchy sequence in (X, d, S). Since (X, d, S) is complete, there exists $x^* \in X$, such that $x_n \to x^*$. Since the *Functional Graph(T)* is G-closed. Thus $x^* \in Tx^*$, that is, T has a fixed point. $\qquad \square$

3 Application

As an application of our result, we shall prove the existence theorem for the following system of integral equations:

$$
\begin{aligned}
x(t) &= f(t) + \int_a^b k_1(t, s, x(s), y(s)) ds \\
y(t) &= f(t) + \int_a^b k_2(t, s, x(s), y(s)) ds, \qquad (3.1)
\end{aligned}
$$

for each $t, s \in I = [a, b]$, where $f : I \to \mathbb{R}$ and $k_i : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$, are continuous functions. We denote by $(C[a, b], \mathbb{R})$ the space of all continuous real-valued functions defined on $[a, b]$.

Theorem 3.1. *Let $X = (C[a, b], \mathbb{R})$ and let the operator $T_i : X \times X \to X$ bes defined by*

$$
T_i(x(t), y(t)) = f(t) + \int_a^b k_i(t, s, x(s), y(s)) ds,
$$

for each $i = 1, 2$, where $f : I \to \mathbb{R}$ and $k_i : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$, are continuous functions. Also assume that for each $t, s \in [a, b]$ and $x, y, u, v \in X$, we have

$$
|k_i(t, s, x(s), y(s)) - k_i(t, s, u(s), v(s))| \leq a_{i1}|x(s) - u(s)| + a_{i2}|y(s) - v(s)|,
$$

for $i = 1, 2$ and the matrix $4(b-a)^2 \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{pmatrix}$ converges to zero. Then the system of integral equations (3.1) *has at least one solution.*

Proof. For each $t, s \in [a, b]$ and $x, y, u, v \in X$, by using the hypothesis, we get the following inequality:

$$
\begin{aligned}
|T_i(x(t), y(t)) - T_i(u(t), v(t))|^2 &\leq \left(\int_a^b |k_i(t, s, x(s), y(s)) - k_i(t, s, u(s), v(s))| ds \right)^2 \\
&\leq \left(\int_a^b [a_{i1}|x(s) - u(s)| + a_{i2}|y(s) - v(s)|] ds \right)^2 \\
&\leq 2(b-a)^2 [a_{i1}^2 \sup_{s \in I} |x(s) - u(s)|^2 + a_{i2}^2 \sup_{s \in I} |y(s) - v(s)|^2]
\end{aligned}
$$

for $i = 1, 2$. Define an operator $T : \mathbb{X} = X \times X \to \mathbb{X} = X \times X$ by

$$
T(\overline{x}) = T(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2)),
$$

for each $\overline{x} = (x_1, x_2) \in \mathbb{X}$ and the Czerwik generalized metric $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_2$ by

$$
d(\overline{x}(t), \overline{y}(t)) = d((x_1, x_2), (y_1, y_2)) = \begin{pmatrix} \sup_{t \in I} |x_1(t) - y_1(t)|^2 \\ \sup_{t \in I} |x_2(t) - y_2(t)|^2 \end{pmatrix} \text{ with } S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
$$

Thus we conclude that

$$
d(T\overline{x}, T\overline{y}) \leq A d(\overline{x}, \overline{y}),
$$

for each $\overline{x}, \overline{y} \in \mathbb{X}$, where

$$
A = 2(b-a)^2 \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{pmatrix}.
$$

Therefore by Corollary 2.6, there exists $\overline{v} = (v_1, v_2) \in \mathbb{X}$ such that $T\overline{v} = \overline{v}$. This implies that $v_1 = T_1(v_1, v_2)$ and $v_2 = T_2(v_1, v_2)$, that is, system of integral equations (3.1) has at least one solution. $\square$

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