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GENERAL ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce two general iterative algorithms (one implicit algorithm and other explicit algorithm) for nonexpansive mappings in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Strong convergence theorems for the sequences generated by the proposed algorithms are established.

1. INTRODUCTION

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be the dual space of E. Let J denote the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pair between E and E^*. Let C be a nonempty closed convex subset of E. For the mapping $T : C \to C$, we denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : Tx = x\}$. Recall that the mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,\ y \in C.$$

In a Banach space E having a single-valued normalized duality mapping J, we say that an operator A is *strongly positive* on E if there exists a $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \ge \overline{\gamma}\|x\|^2 \tag{1.1}$$

and

$$\|aI - bA\| = \sup_{\|x\|\le 1} |\langle (aI - bA)x, J(x)\rangle|, \quad a \in [0, 1], \quad b \in [-1.1],$$

for all $x \in E$, where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.1) reduce to

$$\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

One classical way to study nonexpansive mappings it to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in E,$$

where $u \in E$ is an arbitrarily chosen point. Banach's contraction mapping principle guarantees that T_t has unique a fixed point x_t in E, which uniquely solves the following fixed point equation:

$$x_t = tu + (1 - t)Tx_t,$$

(Such a path $\{x_t\}$ is said to be an approximating fixed point of T since it posesesses the property that if $\{x_t\}$ is bounded, then $\lim_{t\to 0} \|Tx_t - x_t\| = 0$). It is unclear, in general, what

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The results presented in this lecture are collected mainly from the work [8] by the author of this report.

is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [11] extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from E onto $Fix(T)$. Xu [17] proved Reich's results hold in reflexive Banach space having a weakly continuous duality mapping.

In a real Hilbert space H, in 2000, Moudafi [10] introduced the following viscosity approximation methods for nonexpansive mapping T on C in an implicit way and an explicit way, respectively:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$; and $f : C \to C$ is a contractive mapping (i.e., there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x,\ y \in H$).

In 2006, Marino and Xu [9] considered the following general iterative algorithm for nonexpansive mapping T on H in an implicit way:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \quad \forall t \in (0, \min\{1, \|A\|^{-1}\}), \tag{1.3}$$

where $A : H \to H$ is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$; $f : H \to H$ is a contractive mapping; and $\gamma > 0$. In 2011, Wangkeeree *et al.* [14] extended the result of Marino and Xu [9] to a reflexive Banach space having a weakly continuous duality mapping. The results of Marino and Xu [9] and Wangkeeree *et al.* [14] improved upon the corresponding results of Browder [3], Moudafi [10], Reich [11] and Xu [17] to a general approximating fixed point $\{x_t\}$ defined by (1.3). Combining the Moudafi's method (1.2) with Xu's method [16], Marino and Xu [9] also considered the following general iterative algorithm for a nonexpansive mapping T in an explicit way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.4}$$

where f is a contractive mapping on H; and $\gamma > 0$. They proved that if the sequence $\{\alpha_n\}$ in $(0, 1)$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of a certain variational inequality related to A.

In this paper, as a continuation of study in this direction, we present new general iterative algorithms for the nonexpansive mapping in a reflexive Banach space with a uniformly Gâteaux differentiable norm. First, we introduce a general implicit iterative algorithm. Consequently, by discretizing the continuous implicit method, we provide a general explicit iterative algorithm for finding a fixed point of the nonexpansive mapping. Under some control conditions, we establish the strong convergence of the proposed explicit algorithm to a fixed point of the mapping, which solves a ceratin variational inequality.

1. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual.

A Banach space E is called *strictly convex* if its unit sphere $U = \{x \in E : \|x\| = 1\}$ does not contain any linear segment. For every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x + y}{2}\| : \|x\| \leq 1, \ \ \|y\| \leq 1, \ \ \|x - y\| \geq \varepsilon\}.$$

E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then E is reflexive and strictly convex.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U=\{x\in E:\|x\|=1\}$. It is said to be *uniformly Gâteaux differentiable* if for each $y\in U$, this limit is attained uniformly for $x\in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x,y)\in U\times U$. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [5].

Let J be the normalized duality mapping from E into 2^{E^*}. It is well-known that J is single valued if and only if E is smooth, and that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*. For these facts, see [5, 13].

Let LIM be a linear continuous functional on ℓ^∞. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ for every $a=\{a_n\}\in\ell^\infty$. LIM is called a *Banach limit* if $\|LIM\|=LIM(1)=1$ and $LIM_n(a_{n+1})=LIM_n(a_n)$ for every $a=\{a_n\}\in\ell^\infty$.

Recall that a closed convex subset C of E is said to have the *fixed point property* for nonexpansive self-mappings (FPP for short) if every nonexpansive mapping $T:C\to C$ has a fixed point, that is, there is a point $p\in C$ such that $Tp=p$. It is well-known that every bounded closed convex subset of a uniformly smooth Banach space has the FPP ([7, p. 45]).

The mapping $T:C\to C$ is said to be *pseudocontractive* if there exists $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle\le\|x-y\|^2,\quad \forall x,\ y\in C,$$

and T is said to be *strongly pseudocontractive* it there exists a constant $k\in(0,1)$ and $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle\le k\|x-y\|^2,\quad \forall x,\ y\in C.$$

We need the following lemmas for the proof of our main results.

Lemma 2.1. ([5]) *Let E be a Banach space, let C be a nonempty closed convex subset of E, and let $T:C\to C$ be a continuous strongly pseudocontractive mapping. Then T has a fixed point in C.*

Lemma 2.2 ([4]) *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma}>0$ and $0<\rho<\|A\|^{-1}$. Then $\|I-\rho A\|\le 1-\rho\overline{\gamma}$.*

Lemma 2.3 ([15]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1}\le(1-\lambda_n)s_n+\lambda_n\delta_n+\omega_n,\quad \forall n\ge 1,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and ω_m satisfy the following conditions:

(i) $\{\lambda_n\}\subset[0,1]$ *and* $\sum_{n=1}^\infty\lambda_n=\infty$ *or, equivalently,* $\prod_{n=1}^\infty(1-\lambda_n)=0$;
(ii) $\limsup_{n\to\infty}\delta_n\le 0$ *or* $\sum_{n=1}^\infty\lambda_n|\delta_n|<\infty$;
(iii) $\omega_n\ge 0$ *and* $\sum_{n=1}^\infty\omega_n<\infty$.

Then $\lim_{n\to\infty}s_n=0$.

Lemma 2.4. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1}=\lambda_nx_n+(1-\lambda_n)y_n,\quad \forall n\ge 0,$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$ such that

$$0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<1.$$

Assume that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.5. ([1, 2]) *Let C be a closed convex of a reflexive and strictly convex Banach space E. Then $C^o=\{x\in C:\|x\|=\inf\{\|y\|:y\in C\}\}$ is a singleton.*

Lemma 2.6. *Let E be a smooth Banach space. Then there holds*

$$\|x+y\|^2\le\|x\|^2+2\langle y,J(x+y)\rangle,\quad \forall x,\ y\in E.$$

2. Main results

Throughout the rest of this paper, we always assume the following:

- E is a real smooth Banach space;
- C is a nonempty closed subspace of E;
- $A:C\to C$ is a strongly positive linear bounded operator with a constant $\overline{\gamma}>0$;
- $h:C\to C$ is a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k\in(0,1)$;
- The constant $\gamma>0$ satisfies $0<\gamma<\frac{\overline{\gamma}}{k}$;
- $T:C\to C$ is a nonexpansive mapping with $Fix(T)\neq\emptyset$.

In this section, first, we introduce the following general iterative algorithm that generates a net $\{x_t\}$, $t\in(0,\min\{1,\|A\|^{-1}\})$ in an implicit way:

$$x_t=t\gamma h(x_t)+(I-tA)Tx_t, \tag{3.1}$$

Now, for $t\in(0,\min\{1,\|A\|^{-1}\})$, consider the mapping $G_t:C\to C$ defined by

$$G_t(x):=t\gamma h(x)+(I-tA)Tx,\quad x\in C.$$

Then G_t is a continuous strongly pseudocontractive mapping with a pseudocontractive coefficient $1-t(\overline{\gamma}-\gamma k)\in(0,1)$. Indeed, from Lemma 2.2 we have for each $x,\ y\in C$,

$$\begin{aligned}
&\langle G_tx-G_ty,J(x-y)\rangle\\
&=t\gamma\langle h(x)-h(y),J(x-y)\rangle+\langle(I-tA)(Tx-Ty),J(x-y)\rangle\\
&\le t\gamma k\|x-y\|^2+\|I-tA\|\|Tx-Ty\|\|x-y\|\\
&\le t\gamma k\|x-y\|^2+(1-t\overline{\gamma})\|x-y\|^2\\
&=(1-t(\overline{\gamma}-\gamma k))\|x-y\|^2.
\end{aligned}$$

Thus, by Lemma 2.1, G_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Let $\{x_t\}$ be defined via* (3.1)*. Then the following hold:*

(a) *x_t is a unique path $t\mapsto x_t\in C$, $t\in(0,\min\{1,\|A\|^{-1}\})$.*

(b) *If v is a fixed point of T, then for each $t\in(0,\min\{1,\|A\|^{-1}\})$*

$$\langle(A-\gamma h)x_t,J(x_t-v)\rangle\le\langle A(I-T)x_t,J(x_t-v)\rangle.$$

(c) *If T has a fixed point in C, then the path $\{x_t\}$ is bounded and $\|x_t-Tx_t\|\to 0$ as $t\to 0$.*

Using Proposition 3.1, we establish strong convergence of $\{x_t\}$.

Theorem 3.2. *Let E be a a reflexive Banach space with a uniformly Gâteaux differentiable norm. Assume that every weakly compact convex subset of E has the FPP for nonexpansive mappings. Let $\{x_t\}$ be defined via* (3.1)*. Then, as $t \to 0$, $\{x_t\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality*

$$\langle (A - \gamma h)p, J(p - q)\rangle \leq 0, \quad \forall q \in Fix(T). \tag{3.2}$$

Next, we substitute the fixed point property assumption, mentioned in Theorem 3.2, by assuming that the space E is strict convex.

Theorem 3.3. *Let E be a a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $\{x_t\}$ be defined via* (3.1)*. Then, as $t \to 0$, $\{x_t\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality* (3.2)*.*

Now, we propose the following general iterative algorithm which generates a sequence in an explicit way:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Using Theorem 3.2 and Theorem 3.3, we obtain strong convergence of the sequence $\{x_n\}$ generated by (3.3).

Theorem 3.4. *Let $\{x_n\}$ be a sequence generated by the explicit algorithm* (3.3)*. Let $\{\alpha_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$.

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of E has the FPP for nonexpansive mappings;*

(H2) *E is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,*

then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality (3.2)*.*

Corollary 3.5. *Let E be a uniformly smooth Banach space. Let $\{x_n\}$ be a sequence generated by the explicit algorithm* (3.3)*. Let $\{\alpha_n\}$ satisfy the conditions* (C1) *and* (C2) *in Theorem 3.4. Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality* (3.2)*.*

Removing the condition $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ on the sequence $\{\alpha_n\}$ in Theorem 3.4, we have the following result.

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by the following explicit algorithm :*

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n \gamma h(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Tx_n, \quad n \geq 1, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of E has the FPP for nonexpansive mappings;*

(H2) *E is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,*

then $\{x_n\}$ *converges strongly to a fixed point* p *of* T*, which is the unique solution in* $Fix(T)$ *of the variational inequality* (3.2).

Proof. By conditions (C1) and (C2), we may assume, without loss of generality, that $\frac{\alpha_n}{1-\beta_n} < \|A\|^{-1}$ for all $n \geq 1$. By Lemma 2.2, we have $\|(1-\beta_n)I - \alpha_n A\| \leq (1-\beta_n - \alpha_n\overline{\gamma})$.

Step 1. We show that $\{x_n\}$, $\{h(x_n)\}$, $\{Tx_n\}$ and $\{ATx_n\}$ are bounded. Indeed, pick any $p \in Fix(T)$ to obtain

$$\|x_{n+1} - p\| \leq \ \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma h(p) - Ap\| + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\overline{\gamma})\|x_n - p\|$$

It follows from induction that $\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma h(p) - Ap\|}{\overline{\gamma} - \gamma k}\right\}$, $\forall n \geq 1$. Hence $\{x_n\}$ is bounded. Moreover, since h is a bounded mapping, $\{h(x_n)\}$ is bounded. Also, $\{Tx_n\}$ and $\{ATx_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this end, define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$ so that

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n. \tag{3.5}$$

We now observe that

$$\begin{aligned} &z_{n+1} - z_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma h(x_{n+1}) - ATx_{n+1}) + Tx_{n+1} - Tx_n + \frac{\alpha_n}{1-\beta_n}(ATx_n) - \gamma h(x_n)). \end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma h(x_{n+1})\| + \|ATx_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma h(x_n)\| + \|ATx_n\|). \end{aligned} \tag{3.7}$$

By conditions (C1), (C2) and (3.7), we obtain that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.4, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.8}$$

It then follows from condition (C2), (3.5) and (3.8) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In fact, from (3.4) it follows that

$$\|Tx_n - x_n\| \leq\leq \ \|\alpha_n\gamma h(x_n) - \alpha_n ATx_n\| + \beta_n\|x_n - Tx_n\| + \|x_{n+1} - x_n\|$$

This implies that

$$(1-\beta_n)\|Tx_n - x_n\| \leq \alpha_n(\gamma\|h(x_n)\| + \|ATx_n\|) + \|x_{n+1} - x_n\|.$$

Thus, by conditions (C1) and (C2) and Step 2, we have $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Step 4. We show that $\limsup_{n\to\infty}\langle\gamma h(p) - Ap, J(x_n - p)\rangle \leq 0$, where $p = \lim_{t\to 0} x_t$ and x_t is defined by (3.1). In fact, let $x_t = t\gamma h(x_t) + (I - tA)Tx_t$. Then, it follows from Theorem 3.2 or Theorem 3.3 that $\{x_t\}$ converges strongly to $p \in Fix(T)$ which is the unique solution of the variational inequality (3.2). Noting that

$$x_t - x_n = t(\gamma h(x_t) - Ax_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t^2 A(\gamma h(x_t) - ATx_t),$$

we have

$$\begin{aligned}\|x_t - x_n\|^2 \le\ & t\langle \gamma h(x_t) - Ax_t, J(x_t - x_n)\rangle + \|x_t - x_n\|^2 \\ & + \|Tx_n - x_n\|\|x_t - x_n\| + t^2\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\|,\end{aligned}$$

which implies that

$$\langle \gamma h(x_t) - Ax_t, J(x_n - x_t)\rangle \le \frac{\|Tx_n - x_n\|}{t}M + tL, \tag{3.9}$$

where $M = \sup\{\|x_t - x_n\| : n \ge 1 \text{ and } t \in (0, \min\{1, \|A\|^{-1}\})\}$ and $L = \sup\{\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\| : n \ge 1 \text{ and } t \in (0, \min\{1, \|A\|^{-1}\})\}$. Since $x_n - Tx_n \to 0$ by Step 3 , taking the upper limit as $n \to \infty$ in (3.9), we derive

$$\limsup_{n\to\infty} \langle \gamma h(x_t) - Ax_t, J(x_n - x_t)\rangle \le tL, \tag{3.10}$$

Taking the lim sup as $t \to 0$ in (3.10) and noticing that the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we obtain

$$\limsup_{n\to\infty} \langle \gamma h(p) - Ap, J(x_n - p)\rangle \le 0.$$

Step 5. We show that $\lim_{n\to\infty} x_n = p$, where $p = \lim_{t\to 0} x_t \in Fix(T)$, x_t being defined by (3.1), which is the unique solution of the variational inequality (3.2). Indeed, from (3.4), observe that

$$x_{n+1} - p = \alpha_n(\gamma h(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(Tx_n - p).$$

By Lemma 2.2 and Lemma 2.6, we derive

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + \alpha_n\gamma k(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\ & + 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & \left(1 - \frac{2\alpha_n(\overline{\gamma} - \gamma k)}{1 - \alpha_n\gamma k}\right)\|x_n - p\|^2 + \frac{2\alpha_n(\overline{\gamma} - \gamma k)}{1 - \alpha_n\gamma k}\cdot\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \gamma k)}K \\ & + \frac{2\alpha_n(\overline{\gamma} - \gamma k)}{1 - \alpha_n\gamma k}\cdot\frac{1}{\overline{\gamma} - \gamma k}\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle,\end{aligned} \tag{3.11}$$

where $K = \sup\{\|x_n - p\| : n \ge 1\}$. Put $\lambda_n = \frac{2\alpha_n(\overline{\gamma} - \gamma k)}{1 - \alpha_n\gamma k}$ and

$$\delta_n = \frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \gamma k)}L + \frac{1}{\overline{\gamma} - \gamma k}\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle.$$

Then it follows from the condition (C1) and Step 4 that $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$. (3.11) reduces to

$$\|x_{n+1} - p\|^2 \le (1 - \lambda_n)\|x_n - p\|^2 + \lambda_n\delta_n. \tag{3.11}$$

Thus, applying Lemma 2.3 together with $\omega_n = 0$ to (3.11), we conclude that $\lim_{n\to\infty} x_n = p$. This completes the proof. □

Remark Our results in this paper extend, improve and develop the corresponding results in [9, 10, 11, 14] and the references therein.

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