数理解析研究所講究録 第2114巻 2019年 103-110

APPROXIMATION OF ATTRACTIVE POINTS OF NONLINEAR MAPPINGS

—COMMON ATTRACTIVE POINTS, COMMON ACUTE POINTS, COMMON FIXED POINTS AND CONVERGENCE THEOREMS —

山梨大学大学院教育学研究科 厚芝幸子
SACHIKO ATSUSHIBA
DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF EDUCATION, UNIVERSITY OF YAMANASHI

ABSTRACT. In this paper, we study the concept of attractive points of nonlinear mappings. We also study the concept of acute points of nonlinear mappings, and the concept of common acute points of the families of nonlinear mappings. We prove weak and strong convergence theorems for a semigroup generated by k-pseudo-contractive mappings by using these concepts.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [14] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1975, Baillon [5] proved the following first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Fixed point, , acute point, iteration, nonexpansive mapping, nonexpansive semigroup, strong convergence.

itself. Then, for any $x \in C$, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T (see also [12]).

Recently, Kocourek, Takahashi and Yao [7] introduced a wide class of nonlinear mappings called generalized hybrid which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem. Motivated by Baillon [5], and Kocourek, Takahashi and Yao [7], Takahashi and Takeuchi [14] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for a generalized hybrid mapping.

In this paper, we study the concept of attractive points of nonlinear mappings. We also study the concept of acute points of nonlinear mappings, and the concept of common acute points of the families of nonlinear mappings. We prove weak and strong convergence theorems for a semigroup generated by k-pseudo-contractive mappings by using these concepts.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|$$

for all $y \in C$. The mapping P_C is called the metric projection of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \ge 0$$

for all $y \in C$. See [12] for more details. The following result is well-known; see also [12].

lem 2.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a continuous representation of S as mappings on C if it satisfies the following conditions:

(i) $s \mapsto T(s)x$ is continuous;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$.

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

3. Acute points and convergence theorems

In this section, we prove convergence theorems by using the concept of k-acute points of a mapping for $k \in [0, 1]$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [14] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

Let C be a subset of a Hilbert space H and let T be a mapping of C into H. A mapping T is said to be L-Lipschitzian if $\|Tx - Ty\| \leq L\|x - y\|$ for any $x, y \in C$, where $L \in [0, \infty)$. Usually, T is said to be quasi-nonexpansive if

$$(1)\, F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\| \leq \|x - v\| \quad \text{for} \quad x \in C, v \in F(T).$$

Let I be the identity mapping on C. T is said to be k-pseudo-contractive if, for any $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2.$$

where $k \in (0, 1)$. Usually, T is said to be hemi–contractive if
(1) $F(T) \neq \emptyset$, (2) $\|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2$ for $x \in C$, $v \in$

$F(T)$. These concepts depend on the condition $F(T) \neq \emptyset$. Usually, T is said to be k-demi-contractive if

$$(1)F(T) \neq \emptyset,\ (2)\|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \text{ for } x \in C, v \in F(T)$$

We also call T a demi–contraction if T is a k-demi–contraction for some $k \in [0, 1)$. Assume $F(T) \neq \emptyset$.

Let $k \in [0, 1]$. We define the set of k–acute points $\mathcal{A}_k(T)$ of T by

$$\mathcal{A}_k(T) = \{\ v \in H : \ \|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for all } x \in C\ \}.$$

We denote $\mathcal{A}_0(T)$ by $A(T)$ because $\mathcal{A}_0(T)$ and attractive points set of T are the same. We denote $\mathcal{A}_1(T)$ by $\mathcal{A}(T)$, that is,

$$\mathcal{A}(T) = \{\ v \in H : \ \|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for all } x \in C\ \}.$$

Now, we get the following convergence theorems [4]. We consider weak convergence theorems in the case $A(S) \neq \emptyset$ and $F(S) \subset \mathcal{A}(S)$. To have the following results, we have to assume demiclosedness at 0 of $I - S$.

Theorem 3.1 ([4]). *Let $a, b \in (0, 1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a, b]$. Let C be a weakly closed subset of a Hilbert space H. Let S be a self–mapping on C such that $F(S) \subset \mathcal{A}(S)$, $A(S) \neq \emptyset$, and $I - S$ is demiclosed at 0. Suppose there is a sequence $\{u_n\}$ in C such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(S)$.

Theorem 3.2 ([4]). *Let $a, b \in (0, 1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a, b]$. Let C be a weakly closed subset of a Hilbert space H and T be a self-mapping on C such that $I - T$ is demiclosed at 0. Assume that one of the followings hold.*

(1) T is hemi-contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.
(2) T is k-demi–contractive. S is the mapping defined by $S = kI + (1 - k)T$.
(3) T is quasi–nonexpansive. S is the mapping defined by $S = T$.

Suppose S is a self–mapping on C and there is a sequence $\{u_n\}$ in C such that

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Theorem 3.3 ([4]). *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a compact subset of a Hilbert space H and T be a continuous self–mapping on C. Assume that one of the followings holds.*

(1) T is hemi–contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.
(2) T is k–demi–contractive. S is the mapping defined by $S = kI + (1-k)T$.
(3) T is quasi–nonexpansive. S is the mapping defined by $S = T$.

Suppose S is a self–mapping on C and there is a sequence $\{u_n\}$ in C such that

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in N.$$

Then, $\{u_n\}$ converges strongly to some $u \in F(T)$.

Now, we get a nonlinear mean ergodic theorem (see also [5]).

Theorem 3.4 ([4]). *Let $k \in [0,1)$. Let C be a bounded subset of a Hilbert space H. Let T be a k–strictly pseudo–contractive self–mapping on C. Let S be the mapping defined by $Sx = (kI + (1-k)T)x$ for $x \in C$. Assume that S is a self–mapping on C. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and*

$$v_{n+1} = Sv_n, \quad b_n = \frac{1}{n}\sum_{t=1}^{n} v_t \qquad \text{for} \quad n \in \mathbb{N}.$$

Then the followings hold.

(1) $\mathcal{A}_k(T)$ is non-empty, closed and convex.
(2) $\{b_n\}$ converges weakly to some $u \in \mathcal{A}_k(T)$.

Furthermore, if C is closed and convex then the followings hold.

(3) $F(T)$ is non-empty, closed and convex.
(4) $\{b_n\}$ converges weakly to $u \in F(T)$.

4. Main theorems

Using the ideas in Section 3, We prove a nonlinear mean ergodic theorem for a semigroup generated by pseudo-contractive mappings on C (see also [6, 2, 12]). We also weak and strong convergence theorems for the semigroups.

Theorem 4.1 ([2]). *Let C be a nonempty bounded closed subset of a Hilbert space H and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a semigroup generated by k-pseudo-contractive mappings with weak Nagumo condition on C. Let $x \in C$. Then, $\{\frac{1}{t}\int_0^t T(t)xdt\}$ converges weakly to $z_0 \in \mathcal{A}_k(\mathcal{S})$. Futher, if C is closed and convex, then $\{\frac{1}{t}\int_0^t T(t)xdt\}$ converges weakly to $z_0 \in F(\mathcal{S})$, where $z_0 = \lim_{t\in\infty} P_{F(\mathcal{S})}T(t)x$.*

Theorem 4.2 ([2]). *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a weakly closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a semigroup generated by k-pseudo-contractive mappings with weak Nagumo condition on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $\mathbb{R}^+$ with $t_n \to \infty$. Let $\{u_n\}$ be a sequence in C defined by $u_1 \in C$ and*

$$u_{n+1} = a_n u_n + (1-a_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \qquad \textit{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

We say that a mapping T of C into itself is asymptotically regular if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all $x \in C$ (see also [12]). We also say that a mapping T of C into itself is uniformly asymptotically regular if for every bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$$

holds. We say that a semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is asymptotically regular if for each $h \in \mathbb{R}^+$,

$$\lim_{t\to\infty} \|T(h+t)x - T(t)x\| = 0$$

holds for all $x \in C$ (see also [12]). We also say that a semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is uniformly asymptotically regular if for $h \in \mathbb{R}^+$ and bounded subset K of C,

$$\lim_{t\to\infty} \sup_{x\in K} \|T(h+t)x - T(t)x\| = 0$$

holds.

Theorem 4.3 ([2]). *Let H be a Hilbert space and let C be a closed convex subset of H. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a semigroup generated by k-pseudo-contractive mappings with weak Nagumo condition on C such that $F(\mathcal{S}) \neq \emptyset$. Assum that $\mathcal{S}$ is uniformly asymptotically regular. Let $z \in C$. Let $\{s_n\}$ be a sequence in $\mathbb{R}^+$ with $s_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)T(s_n)x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{S})}z$, where $P_{F(\mathcal{S}))}$ is the metric projection from H onto $F(\mathcal{S})$.

References

1. S. Atsushiba, *Strong convergence to common attractive points of uniformly asymptotically regular nonexpansive semigroups*, J. Nonlinear Convex Anal. **16** (2015), 69-78.
2. S. Atsushiba, *Weak and strong convergence theorems for some semigroups of nonlinear mappings*, submitted.
3. S. Atsushiba, and W. Takahashi, *Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 209-219.
4. S. Atsushiba, S. Iemoto, R. Kubota and Y. Takeuchi *Convergence theorems for some classes of nonlinear mappings in Hilbert spaces*, Linear and Nonlinear Analysis, **2** (2016), 125-153.
5. J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sei. Paris Ser. A-B **280** (1975), 1511 - 1514.
6. A. Hester, C. H. Morales, *Semigroups generated by pseudo-contractive map- pings under the Nagumo condition*, J. Differential Equations 245 (2008), 994–1013.
7. P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
8. P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.
9. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

10. G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172–178.
11. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space,* Proc. Amer. Math. Soc. **81** (1981), 253-256.
12. W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
13. W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal., **11** (2010), 79-88.
14. W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

(S. Atsushiba) DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF EDUCATION, UNIVERSITY OF YAMANASHI, 4-4-37, TAKEDA KOFU, YAMANASHI 400-8510, JAPAN

E-mail address: asachiko@yamanashi.ac.jp