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Asymptotic periodicity of Markov operator for random Nagumo-Sato model

Fumihiko NAKAMURA*
Department of Mathematics, Hokkaido University

1 Introduction

In this paper, we consider a perturbed dynamical system in which noise is applied to the Nagumo-Sato (NS) model [11], that is,

$$x_{t+1} = S_{\alpha,\beta}(x_t) + \xi_t \pmod 1, \tag{1.1}$$

$$S_{\alpha,\beta}(x) = \alpha x + \beta \pmod 1 \quad \text{for} \quad (\alpha,\beta) \in (0,1)^2, \tag{1.2}$$

where $\{\xi_t\}$ are independent random variables each having same density g satisfying $\mathrm{supp}\{g\} = [0,\theta]$ with $\theta \in [0,1]$. The piecewise linear map $S_{\alpha,\beta}$ is called Nagumo-Sato model which corresponds to a special case of Caianiello's model [4], and it describes the simplified dynamics of a single neuron. It is known that the system (1.2) shows periodic behavior of the trajectory for almost every (α,β). The transformation has one discontinuous point when $\alpha+\beta>1$, and it leads to a complicated structure for a periodicity on the parameter space. This structure is presented graphically in Fig.1(pp.6) which shows regions in which $S_{\alpha,\beta}$ has a periodic point. An important feature of the structure is that there exists a region in which $S_{\alpha,\beta}$ has a periodic point with period $(m+n)$ between the region with period m and n. In the paper [12], we stated the definition of the structure for the NS model as a Farey structure, and gives a detailed analysis of these regions. Indeed, considering the properties of a rational characteristic sequence which is one of our mathematical techniques, we calculate boundaries of each region on the parameter space in which the NS model has a periodic point for any period. Then we succeeded to find explicit parameter regions in which $S_{\alpha,\beta}$ has rational rotation number l/n for all l/n in $(0,1)$. In 1987, Ding and Hemmer [5] studied similar piecewise linear maps and these regions, and our result gives more detailed analysis of their works.

We discuss two important asymptotic properties for the Markov operator [10] corresponding to the model (1.1). It is well known that the Markov operator describes asymptotic behaviors of a trajectory. Especially, we focus on the properties of **asymptotic periodicity** and **asymptotic stability** which are introduced in section 2. These asymptotic behaviors for the NS model are also observed and discussed in [6, 10]. The asymptotic periodicity with period 1 is equivalent to the asymptotic stability, and it is important problem to classify the system as the case with period 1 or more than 1 since these two cases give different mixing properties of the system.

The main result in this paper (Theorem 5.1) shows that the sufficient condition for which the Markov operator corresponding to the perturbed NS system has either asymptotic periodicity (period >1) or

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asymptotic stability (period $=1$). More precisely, for almost all $(\alpha,\beta)\in(0,1)^2$, there exists a critical value $\theta_*(\alpha,\beta)$ such that the Markov operator generated by the system (1.1) displays asymptotic periodicity if θ is less than $\theta_*(\alpha,\beta)$. On the other hand, if θ is greater than $\theta_*(\alpha,\beta)$, the Markov operator shows asymptotic stability. In 1991, Provatas and Mackey [14] have already showed the same result as Theorem 5.1 in the case that rotation numbers is $1/n$. Thus, our theorem extends their result to all cases of rotation numbers l/n by using Theorem 4.3.

2 Asymptotic behavior of Markov operator

In this section, we prepare some mathematical tools, Markov operator, Perron-Frobenius operator and their asymptotic properties, to state our main results (Theorem 5.1). The Markov operator is an important tool which describes a density evolution generated by a system, and its convergence gives an existence of an invariant measure for the system. The topics in the section are mainly based on [10].

Let $(X,\mathcal{A},\mu)$ be a measure space, that is, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{A}$. Moreover, assume that measure space $(X,\mathcal{A},\mu)$ is finite, $\mu(X)<\infty$. We first introduce the definition of Markov operator and its basic properties. Let $L^1(X)=L^1(X,\mathcal{A},\mu)$ be a space of all integrable functions on X, i.e. $\|f\|_{L^1}:=\int_X|f(x)|d\mu(x)<\infty$.

Definition 2.1. A linear operator $P:L^1(X)\to L^1(X)$ is called a **Markov operator** if P satisfies $Pf\geq 0$ and $\|Pf\|_{L^1}=\|f\|_{L^1}$ for $f\in L^1$ with $f\geq 0$.

Definition 2.2. Let the set $D(X,\mathcal{A},\mu)$ be defined by $D(X,\mathcal{A},\mu)=\{f\in L^1(X,\mathcal{A},\mu)\mid f\geq 0,\|f\|_{L^1}=1\}$. Any function $f\in D(X,\mathcal{A},\mu)$ is called a **density**. The set $D(X,\mathcal{A},\mu)$ is sometimes denoted by D simply.

Definition 2.3. A measurable transformation $S:X\to X$ is **nonsingular** if $\mu(S^{-1}(A))=0$ for all $A\in\mathcal{A}$ such that $\mu(A)=0$.

Under these definitions, a Perron-Frobenius operator corresponding to a nonsingular transformation can be defined as follows, which plays a role to consider the evolution of density functions generated by a deterministic dynamical system.

Definition 2.4. If $S:X\to X$ is nonsingular transformation, the operator $P:L^1\to L^1$ defined by

$$\int_A Pf(x)d\mu(x)=\int_{S^{-1}(A)}f(x)d\mu(x),\quad \text{for}\quad A\in\mathcal{A}, \tag{2.1}$$

is called the **Perron-Frobenius operator** corresponding to S.

For some interval maps on $[a,b]$, corresponding Perron-Frobenius operator P allows us to obtain an explicit form Pf. If one takes the interval $[a,x]$ as A in the equation (2.1) and by differentiating, then we have

$$Pf(x)=\frac{d}{dx}\int_{S^{-1}([a,x])}f(s)ds.$$

Next, we explain a relation between important properties in an ergodic theory, ergodicity, mixing and exactness, and the convergence of $\{P^nf\}$.

Definition 2.5. Let $(X,\mathcal{A},\mu)$ be a measure space and $S:X\to X$ be a nonsingular transformation. Then S is called **ergodic** if either $\mu(A)=0$ or $\mu(X\backslash A)=0$ holds for every invariant set $A\in\mathcal{A}$, $S^{-1}(A)=A$.

Definition 2.6. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ be a measure preserving. Then S is called **mixing** if

$$\lim_{n\to\infty} \mu(A \cap S^{-n}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{A}.$$

Definition 2.7. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ be a measure preserving transformation such that $S(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. Then S is called **exact** if

$$\lim_{n\to\infty} \mu(S^n(A)) = 1 \quad \text{for every } A \in \mathcal{A},\ \mu(A) > 0.$$

Proposition 2.8. *([10], Theorem 4.4.1) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \to X$ a measure preserving transformation, and P the Perron-Frobenius operator corresponding to S. Then*

(a) *S is ergodic if and only if the sequence $\{P^t f\}$ is Cesàro convergent to 1 for all $f \in D$.*

(b) *S is mixing if and only if the sequence $\{P^t f\}$ is weakly convergent to 1 for all $f \in D$.*

(c) *S is exact if and only if the sequence $\{P^t f\}$ is strongly convergent to 1 for all $f \in D$.*

Therefore, one can classify ergodicity, mixing and exactness by using the concepts of a convergence for Perron-Frobenius operator corresponding to the system.

Next, we define two important properties of Markov operator and collect sufficient conditions for satisfying these two properties.

Definition 2.9. $\{P^t\}$ is said to be **asymptotically periodic** if there exists an integer r, two sequences of nonnegative functions $g_i \in D$ and $h_i \in L^\infty(X)$, $i = 1, \cdots, r$, and an operator $Q : L^1(X) \to L^1(X)$ such that for every $f \in L^1(X)$, Pf can be written in the form

$$Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf(x),$$

where

$$\lambda_i(f) = \int_X f(x) h_i(x) \mu(dx).$$

Moreover functions g_i and operator Q satisfy the following properties;

(i) $g_i(x) g_j(x) = 0$ for all $i \neq j$;

(ii) There exists a permutation ρ of $\{1, \cdots, r\}$ such that $Pg_i = g_{\rho(i)}$.

(iii) $||P^t Q f||_{L^1} \to 0$ as $t \to \infty$ for every $f \in L^1(X)$.

Remark 2.10. When $\{P^t\}$ is an asymptotically periodic Markov operator, then P has a stationary density f_*

$$f_*(x) = \frac{1}{r} \sum_{i=1}^{r} g_i(x).$$

Definition 2.11. $\{P^t\}$ is said to be **asymptotically stable** if there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and $\lim_{t\to\infty} ||P^t f - f_*||_{L^1} = 0$ for every $f \in D$.

Proposition 2.12. *$\{P^t\}$ is asymptotically stable if and only if $\{P^t\}$ is asymptotically periodic with $r = 1$.*

Example 2.13. Consider the generalized tent map defined by, for a parameter $a \in (1, 2]$,

$$S(x) = \begin{cases} ax & \text{for} \quad x \in [0, 1/2] \\ a(1-x) & \text{for} \quad x \in [1/2, 1]. \end{cases}$$

This map was considered in [15] and they showed that $\{P^t\}$ is asymptotically periodic for the Perron-Frobenius operator P corresponding to this map S. More precisely, when the parameter a satisfies

$$2^{1/2^{n+1}} < a \leq 2^{1/2^n} \quad \text{for} \quad n = 0, 1, 2, \cdots,$$

then $\{P^t\}$ is asymptotically periodic with the period 2^n. □

Next, we introduce a sufficient condition for the asymptotic periodicity.

Definition 2.14. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mu(X) < \infty$. A Markov operator P is called **constrictive** if there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $t_0(f)$ for which

$$\int_E P^t f(x) \mu(dx) \leq \kappa \quad \text{for all } t \geq t_0(f) \quad \text{and } E \text{ with } \mu(E) \leq \delta.$$

Proposition 2.15. *([10], Theorem 5.3.1) If P is a constrictive Markov operator, then $\{P^t\}$ is asymptotically periodic.*

The next proposition gives a sufficient condition for an asymptotic stability which plays an important role for the proof of Theorem 5.1(ii).

Proposition 2.16. *([10], Theorem 5.6.1) Let P be a constrictive Markov operator. Assume there is a set $A \subset X$ of nonzero measure, $\mu(A) > 0$, with the property that for every $f \in D$ there is an integer $t_0(f)$ such that $P^t f(x) > 0$ for almost all $x \in A$ and all $t > t_0(f)$. Then $\{P^t\}$ is asymptotically stable.*

Finally, we mention relations between asymptotically periodic and some ergodic properties as follows.

Proposition 2.17. *([10], Theorem 5.5.1) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. Then P is ergodic if and only if the permutation ρ of $\{1, \cdots, r\}$ (see Definition 2.9) is cyclical, that is, ρ has no invariant subset.*

Proposition 2.18. *([10], Theorem 5.5.2 and Theorem 5.5.3) Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. Then following are equivalent.*

$$r = 1 \iff P \text{ is mixing} \iff P \text{ is exact},$$

where r is defined in Definition 2.9.

Interestingly, the mixing and exactness are equivalent in the class of a constrictive Markov operator from the above proposition. These propositions suggest that it is important to classify the case $r > 1$ or $r = 1$ of the asymptotic periodicity.

3 Rational characteristic sequence

The rational characteristic sequence is known as mechanical words, rotation words or Christoffel words [3] and good sequence in [12], and if l/n is replaced by an irrational number, then it is known as Sturmian words or characteristic sequence [2]. Two series of functions $A_i(\alpha)$ and $F_{n,l}(i)$ (see Eq.(3.3) and (3.4)) generated by the sequence are important tools to prove Farey structure on a parameter space of NS model (Proposition 4.3), and consequently leads our main result (Theorem 5.1). We first introduce the definition and some useful properties of the rational characteristic sequence. Let $Pr(n)$ be a set of numbers $l < n$ satisfying l and n are mutually prime, $Pr(n) := \{l < n \mid GCD(n, l) = 1\}$ for each $n \in \mathbb{N}$.

Definition 3.1. For $n \in \mathbb{N}$ and $l \in Pr(n)$, we define a sequence $\{k_i\}_{i\in\mathbb{Z}}$ with $k_i \in \{0, 1\}$ for all $i \in \mathbb{Z}$ by

$$k_i := \left[\frac{(i+1)l}{n}\right] - \left[\frac{il}{n}\right] \quad \text{for} \quad i \in \mathbb{Z}, \tag{3.1}$$

where $[x]$ is the integer part of x. The sequence $\{k_i\}_{i\in\mathbb{Z}}$ is called a **Rational characteristic sequence** with respect to (n, l). Obviously, $k_0 = 0$ and $k_{n-1} = 1$ always hold.

In this paper, if we write (n, l), then n and l always satisfy $n \in \mathbb{N}_{\geq 2}$ and $l \in Pr(n)$.

Proposition 3.2. *([12], Proposition 2.2) Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n, l). We then have the following properties.*

(i) $k_{i\pm n} = k_i$ for $i \in \mathbb{Z}$,

(ii) $k_{n-1-i} = k_i$ for $i \in \mathbb{Z},\ i \notin n\mathbb{Z}, n\mathbb{Z} - 1$,

(iii) $k_{i-\hat{l}} = k_i$ for $i \in \mathbb{Z},\ i \notin n\mathbb{Z}, n\mathbb{Z} - 1$,

where $\hat{l} = \min\{t \in \mathbb{N} \mid tl = 1 \pmod{n}\}$.

Proposition 3.3. *([12], Proposition 2.3) Let $\{k_i\}_{i\in\mathbb{Z}}$ be a rational characteristic sequence with respect to (n, l) and $\{k'_i\}_{i\in\mathbb{Z}}$ be another rational characteristic sequence with respect to (n', l'). If $\frac{l}{n} < \frac{l'}{n'}$ and $nl' - n'l = 1$, then the sequence $\{\hat{k}_i\}_{i\in\mathbb{Z}}$ defined by*

$$\hat{k}_i := \begin{cases} k_i & \text{for} \quad i = 0, \cdots, n-1 \\ k'_{i-n} & \text{for} \quad i = n, \cdots, n+n'-1 \end{cases} \tag{3.2}$$

and $\hat{k}_m := \hat{k}_i$ if $m = i + t(n + n')$ with $i = 0, \cdots, n + n' - 1$ and $t \in \mathbb{Z}\backslash\{0\}$, is the rational characteristic sequence with respect to $(n + n', l + l')$.

Example 3.4. The rational characteristic sequence for (n, l) with $n = 2, 3, 4, 5$ are followings. Here we write only $k_0, \cdots, k_{n-1}$ with the bracket $\langle\cdot\rangle$ since their 01 words are repeated.

$(2,1)$: $\langle 01\rangle$, $(3,1)$: $\langle 001\rangle$, $(3,2)$: $\langle 011\rangle$, $(4,1)$: $\langle 0001\rangle$, $(4,3)$: $\langle 0111\rangle$,
$(5,1)$: $\langle 00001\rangle$, $(5,2)$: $\langle 00101\rangle$, $(5,3)$: $\langle 01011\rangle$, $(5,4)$: $\langle 01111\rangle$,

One can immediately see that the sequence for $(5,2)$, $\langle 00101\rangle$, can be made the sequence of $(3,1)$ and $(2,1)$, that is, $\langle 00101\rangle = \langle 001\rangle + \langle 01\rangle$. Similarly, we have following examples.

$(7,3) = (5,2) + (2,1)$: $\langle 00101\rangle + \langle 01\rangle = \langle 0010101\rangle$
$(12,5) = (5,2) + (3,7)$: $\langle 00101\rangle + \langle 0010101\rangle = \langle 001010010101\rangle$ □

We next define two sequences $\{A_i(\alpha)\}_{i=0}^{n-1}$ of length n and $\{F_{n,l}(i)\}_{i=2}^{n}$ as follows, and prepare a few lemmas which give some properties of the sequences.

$$A_i(\alpha) = \frac{1}{1-\alpha^n}\left(\sum_{m=0}^{i-1} k_m \alpha^{i-m-1} + \sum_{m=i}^{n-1} k_m \alpha^{n+i-1-m}\right) \tag{3.3}$$

$$F_{n,l}(i) = \frac{1}{1-\alpha^i}\sum_{m=1}^{i-1} k_m \alpha^m, \quad \alpha \in (0,1), \quad i = 2, \cdots, n. \tag{3.4}$$

Lemma 3.5. *([12], Lemma 2.6) We can write*

$$\min_{\{i|k_i=0\}} A_i(\alpha) = \frac{1}{1-\alpha^n}\sum_{i=1}^{n-1} k_i\alpha^{i-1} \quad \text{and} \tag{3.5}$$

$$\max_{\{i|k_i=1\}} A_i(\alpha) = \frac{1}{1-\alpha^n}\left(\sum_{i=1}^{n-1} k_i\alpha^{i-1} - \alpha^{n-2} + \alpha^{n-1}\right). \tag{3.6}$$

Lemma 3.6. *([13], Lemma 2.3) For any (n,l), the inequality*

$$\frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n} < F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1-\alpha^i} \tag{3.7}$$

holds for $\alpha \in (0,1)$ and $i = 2, \cdots, n-1$.

We give the proof of Lemma 3.6 by using a special type of induction based on the Farey series as follows;

Step(1) The inequality holds for $(n,1)$ and $(n,n-1)$ for $n \in \mathbb{N}_{\geq 2}$.

Step(2) Assume that the inequality holds for (n,l) and (n',l') with $nl' - n'l = 1$. Then, the inequality holds for $(n+n', l+l')$.

By the definition of the Farey series [1], it is obvious that the above induction shows the inequality holds for all (n,l).

4 Deterministic Nagumo-Sato Model

In this section, we show that the system (1.2) possesses the Farey structure in the parameter space which is a layered structure (see Fig.1) and gives the regions of parameter space in which $S_{\alpha,\beta}$ has a periodic point. After that, we show a property for a preimage of zero for deterministic NS model (Proposition 4.7).

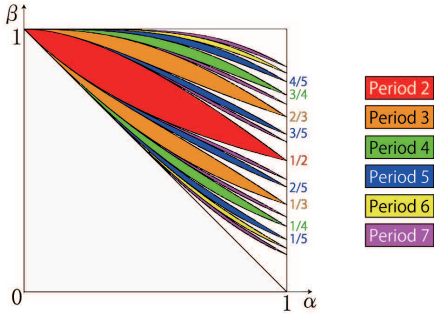

Figure 1: The region of the parameter space (α,β) in which $S_{\alpha,\beta}$ has a periodic point with period $n = 2, \cdots, 7$.

Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be two projections with $\pi_1(\alpha,\beta) = \alpha$ and $\pi_2(\alpha,\beta) = \beta$, and define a Farey structure as follows. Let E be a bounded subset of $\mathbb{R}^2$ and let $\{D_{n,l}\}_{n\in\mathbb{N}, l\in Pr(n)}$ be a family of subsets of

E satisfying the following properties. For each $\alpha \in \pi_1(E)$, there exist real numbers $B^U_{n,l}(\alpha)$ and $B^L_{n,l}(\alpha)$ such that

$$\pi_2\left(D_{n,l} \cap \pi_1^{-1}\{\alpha\}\right) = \left[B^L_{n,l}(\alpha), B^U_{n,l}(\alpha)\right). \tag{4.1}$$

We denote $D_{n,l} \prec D_{n',l'}$ if $B^U_{n,l}(\alpha) < B^L_{n',l'}(\alpha)$ holds for any $\alpha \in \pi_1(E)$. We then consider a two parameter family of transformations of $[0,1)$, $\{T_{\alpha,\beta} : [0,1) \to [0,1)\}_{(\alpha,\beta)\in E}$.

Definition 4.1. $\{T_{\alpha,\beta}\}_{(\alpha,\beta)\in E}$ possesses a **Farey structure** in a parameter subspace $E \subset \mathbb{R}^2$ if there exists $\{D_{n,l}\}_{(n,l)} \subset E$ satisfying the property (4.1) such that

(i) $Leb(D_{n,l}) > 0$ for all (n,l),

(ii) $T_{\alpha,\beta}$ with $(\alpha,\beta) \in D_{n,l}$ has a periodic point with period n for each (n,l),

(iii) $D_{n+1,1} \prec D_{n,1}$ and $D_{n,n-1} \prec D_{n+1,n}$ hold for every $n \in \mathbb{N}$. If (n,l) and (n',l') satisfying $nl'-n'l = 1$ and $D_{n,l} \prec D_{n',l'}$, then $D_{n,l} \prec D_{n+n',l+l'} \prec D_{n',l'}$.

To state the next Theorem 4.3, which shows that $\{S_{\alpha,\beta}\}$ has this Farey structure, we define two functions $B^U_{n,l}(\alpha)$ and $B^L_{n,l}(\alpha)$ and sets $\{D_{n,l}\}_{(n,l)}$ as follows;

$$B^U_{n,l}(\alpha) = (1-\alpha)\left(\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m + 1\right), \tag{4.2}$$

$$B^L_{n,l}(\alpha) = (1-\alpha)\left(\frac{1}{1-\alpha^n}\sum_{m=1}^{n-1} k_m\alpha^m + 1 - \frac{\alpha^{n-1}-\alpha^n}{1-\alpha^n}\right), \tag{4.3}$$

and

$$D_{n,l} = \{(\alpha,\beta) \in (0,1)^2 \mid B^L_{n,l}(\alpha) \le \beta < B^U_{n,l}(\alpha)\}, \tag{4.4}$$

where the sequence $\{k_\imath\}_{\imath\in\mathbb{Z}}$ is a rational characteristic sequence (3.1) with respect to (n,l). Note that these functions can be rewritten as follows by using the equations (3.5) and (3.6);

$$B^U_{n,l}(\alpha) = (1-\alpha)\left(\alpha \min_{\{\imath|k_\imath=0\}} A_\imath(\alpha) + 1\right), \tag{4.5}$$

$$B^L_{n,l}(\alpha) = (1-\alpha)\left(\alpha \max_{\{\imath|k_\imath=1\}} A_\imath(\alpha) + 1\right). \tag{4.6}$$

The parameter regions in Fig.1 correspond to above sets $\{D_{n,l}\}$ for $n = 2,3,\cdots,7$. The next proposition derives the property of $\{D_{n,l}\}$ defined above which implies (iii) of Farey structure.

Proposition 4.2. *([12], Lemma 3.2) Let $D_{n,l}$ be defined by (4.2), (4.3) and (4.4). Then, for every $n \in \mathbb{N}_{\ge 2}$, relations $D_{n+1,1} \prec D_{n,1}$ and $D_{n,n-1} \prec D_{n+1,n}$ hold. Moreover, if $D_{n,l} \prec D_{n',l'}$ and satisfy $nl' - n'l = 1$, then there exists a region $D_{n+n',l+l'}$ such that $D_{n,l} \prec D_{n+n',l+l'} \prec D_{n',l'}$.*

Proposition 4.3. *([12], Theorem 4.1) $\{S_{\alpha,\beta}\}_{(\alpha,\beta)\in E}$ possesses the Farey structure in $E = \{(\alpha,\beta) \in (0,1)^2 \mid \alpha+\beta > 1\}$ with $\{D_{n,l}\}_{n\in\mathbb{N},l\in Pr(n)}$ defined by (4.4).*

The next corollary leads a zero Lebesgue measure of parameter sets for which $S_{\alpha,\beta}$ has no periodic point.

Corollary 4.4. *([12], Corollary 4.2) For $Leb-a.e.$ $(\alpha,\beta) \in E$, $S_{\alpha,\beta}$ has a periodic point.*

Remark 4.5. The Proposition 4.2 show that the following relations hold for rational characteristic sequences $\{k_m\}$, $\{k'_m\}$ and $\{\hat{k}_m\}$ with respect to (n,l), (n',l') and $(n+n',l+l')$, respectively, with $nl'-n'l=1$;

$$\begin{aligned} B^L_{n',l'}(\alpha) &> B^U_{n,l}(\alpha), \\ B^L_{n',l'}(\alpha) &> B^U_{n+n',l+l'}(\alpha), \qquad \text{for } \alpha \in (0,1). \\ B^L_{n+n',l+l'}(\alpha) &> B^U_{n,l}(\alpha), \end{aligned}$$

By using explicit formulas (4.2), (4.3) and (3.4), these can be rewritten as follows respectively; for $\alpha \in (0,1)$,

$$F_{n',l'}(n') - F_{n,l}(n) > \frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}, \tag{4.7}$$

$$F_{n',l'}(n') - F_{n+n',l+l'}(n+n') > \frac{\alpha^{n'-1}-\alpha^{n'}}{1-\alpha^{n'}}, \tag{4.8}$$

$$F_{n+n',l+l'}(n+n') - F_{n,l}(n) > \frac{\alpha^{n+n'-1}-\alpha^{n+n'}}{1-\alpha^{n+n'}}. \tag{4.9}$$

These inequalities will be useful for proving the Lemma 3.6 whose proof are written in appendix. Note that these inequalities (4.7),(4.8) and (4.9) are the properties of a rational characteristic sequence, which is not necessary to the NS model.

Remark 4.6. The system $S_{\alpha,\beta}$ has a periodic point with period n when $(\alpha,\beta) \in D_{n,l}$. The set of these periodic points with period n is given by

$$\mathrm{Per}_n(S_{\alpha,\beta}) = \left\{ \frac{\beta}{1-\alpha} - A_i(\alpha) \;\middle|\; i = 0,\cdots,n-1 \right\}. \tag{4.10}$$

In [9], Keener showed that if the set of preimages of a discontinuous point is finite, then the map has a periodic solution. The next proposition gives a new property of the preimage of zero for the NS model, which concludes that the set of preimages is finite for any parameter $(\alpha,\beta) \in D_{n,l}$. Remark that zero is a preimage of the discontinuity point of NS model. This result is used for the proof of Lemma 5.3.

Proposition 4.7. *([13], Proposition 4) Assume that $(\alpha,\beta) \in D_{n,l}$, then*

$$S^{-i}_{\alpha,\beta}(0) = \sum_{m=1}^{i} \frac{k_{n-i+m-1}-\beta}{\alpha^m} \in [0,1], \qquad (i = 1,\cdots,n-1), \tag{4.11}$$

where $\{k_m\}$ is a rational characteristic sequence with respect to (n,l).
Moreover, for $i=n$, $S^{-n}_{\alpha,\beta}(0)$ is not in $[0,1]$.

5 Perturbed Nagumo-Sato Model

In this section, we introduce our main theorem which states that a Markov operator generated by the system (1.1) has one of two different asymptotic properties depending on the maximum value θ of the noise. We come to consider our random dynamical system (1.1). From [7], we have already known that the Markov operator $\overline{P} : L^1([0,1]) \to L^1([0,1])$ defined by

$$\overline{P}f(x) = \int_{[0,1]} f(y) \sum_{i=0}^{1} g(x - T(y) + i)dy \quad \text{for} \quad f \in L^1. \tag{5.1}$$

which is generated by (1.1) is asymptotically periodic. Therefore, the following main theorem gives a sufficient condition for $r > 1$ (asymptotic periodicity) and for $r = 1$ (asymptotic stability).

Theorem 5.1. *([13], Theorem 3.6) Let $\overline{P}$ be the Markov operator corresponding to system (1.1). Fix $n \in \mathbb{N}$ and $l \in Pr(n)$. Assume that $(\alpha, \beta) \in D_{n,l}$. Then there exists $\theta_*(\alpha, \beta) = \theta_*(\alpha, \beta, n, l) \in [0,1]$ such that*

(i) *if $\theta \leq \theta_*(\alpha, \beta)$, then $r = n > 1$, and $\{\overline{P}^t\}$ is asymptotically periodic with period n,*

(ii) *if $\theta > \theta_*(\alpha, \beta)$, then $r = 1$, and $\{\overline{P}^t\}$ is asymptotically stable,*

where r is the number defined in Definition 2.9, and

$$\theta_*(\alpha, \beta) = \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n} - \beta + B^L_{n,l}(\alpha). \tag{5.2}$$

Remark 5.2. Note that the inequality $\theta \leq \theta_*(\alpha, \beta)$ means $(\alpha, \beta + \xi) \in D_{n,l}$ with arbitrary $\xi \in [0, \theta]$.

To prove Theorem 5.1(i), we prepare the following key lemma.

Lemma 5.3. *([13], Lemma 3.7) Assume that $(\alpha, \beta), (\alpha, \beta + \theta) \in D_{n,l}$ for $n \in \mathbb{N}$ and $l \in Pr(n)$. For $i = 0, \cdots, n-1$, let G_i be an interval defined by*

$$G_i = \left[\frac{\beta}{1-\alpha} - A_i(\alpha),\ \frac{\beta+\theta}{1-\alpha} - A_i(\alpha)\right], \tag{5.3}$$

where $A_i(\alpha)$ is defined by (3.3). Then, for $i = 0, 1, \cdots, n-2$,

$$x_{t+1} \in G_{i+1} \text{ if } x_t \in G_i \quad \textit{and} \quad x_{t+1} \in G_0 \text{ if } x_t \in G_{n-1}, \tag{5.4}$$

where x_{t+1} is determined by the system (1.1). Moreover, there exists a number $N \in \mathbb{N}$ such that $x_{t+1} \in \bigcup_{i=0}^{n-1} G_i$ for $t > N$ and a.e.$x_0 \in [0,1] \backslash \bigcup_{i=0}^{n-1} G_i$.

Let c be the discontinuity point of NS model, i.e. $c = \frac{1-\beta}{\alpha}$. Then we have the following corollary of Lemma 5.3.

Corollary 5.4. *([13], Corollary 1) Assume that $(\alpha, \beta) \in D_{n,l}$ and $\theta \leq \theta_*(\alpha, \beta)$. Then the rotation number of the perturbed NS model (1.1) is given by*

$$\rho = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} 1_{[c,1)}(x_i) = \frac{l}{n}, \tag{5.5}$$

for μ-a.e.x_0 and almost every realization of the system.

In addition to Theorem 5.1, the argument used in the proof of Theorem 5.1 (ii) plays a role to obtain the following result which shows an asymptotic behavior for the parameter satisfying $\beta = B^U_{n,l}(\alpha)$.

Theorem 5.5. *([13], Theorem 3.8) Let $\overline{P}$ be the Markov operator corresponding to system (1.1) and give a parameter $(\alpha, \beta) \in [0,1]^2$ satisfying $\beta = B^U_{n,l}(\alpha)$. Then, $\{\overline{P}^t\}$ is asymptotically stable for any $\theta > 0$.*

Remark 5.6. The condition $\beta = B^U_{n,l}(\alpha)$ implies that $S_{\alpha,\beta}$ does not have periodic point. The case $\alpha = 1/2$, $\beta = B^U_{4,1}(1/2) = 17/30$, $\theta = 1/15$ of behavior was observed numerically in [10, 12]. Although these observations showed us a periodic behavior with period 3, Theorem 5.5 indicates the asymptotic stability for the case. And recently, Kaijser [8] showed that it displays asymptotic stability in this special case $\alpha = 1/2$, $\beta = 17/30$ and $\theta = 1/15$.

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Department of Mathematics
Hokkaido University
Sapporo 060-0810
JAPAN
E-mail address: nfumihiko@math.sci.hokudai.ac.jp

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