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THE HAUSDORFF DIMENSION OF THE REGION OF MULTIPLICITY ONE OF OVERLAPPING ITERATED FUNCTION SYSTEMS OF THE INTERVAL

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1. Introduction

We consider iterated function systems on the unit interval $I = [0,1]$ generated by two contractive similarity transformations

$$f_0(x) = ax, \qquad f_1(x) = ax + (1-a) \tag{1}$$

with similarity ratio $0 < a < 1$. If a is grater than $1/2$, the limit set is the interval itself, and we say such an iterated function system is overlapping. We study the subset $J_1(a)$ of points of the overlapping limit set which have unique addresses. Fig.1 shows $J_1(a)$ for values of a between $1/2$ and the golden ratio $g = (\sqrt{5}-1)/2$. Note that $J_1(a) = \{0,1\}$ for $a > g$.

We explicitly determine the Hausdorff dimension of $J_1(a)$ for values of a described below. For $k = 1, 2, \ldots$, let b_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^k f_0(1) = 1 - a \qquad \text{or} \qquad a^{k+2} - a^{k+1} + 2a - 1 = 0. \tag{2}$$

Likewise, let c_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^{k+1}(0) = 1 - a \qquad \text{or} \qquad -a^{k+2} + 2a - 1 = 0. \tag{3}$$

It is easy to check that $\frac{1}{2} < \cdots < b_2 < c_2 < b_1 < c_1$ and that the sequences $\{b_k\}$ and $\{c_k\}$ converge to $1/2$ as k increases.

Theorem 1.1. *For any a with $b_k \le a \le c_k$ $(k \ge 2)$, the Hausdorff dimension of $J_1(a)$ is given by*

$$\dim_H J_1(a) = -\frac{\log \lambda_k}{\log a},$$

where λ_k is the largest eigenvalue of the matrix A_k given in section 3.

Table 1 shows the values of b_k, c_k and λ_k for k up to 10. To prove the theorem, we define and apply the theory of graph directed Markov systems. The matrix A_k is its incidence matrix.

2. Preliminary

2.1. **Multiplicity function.** Let (X,d) be a compact metric space which is a subset of $\mathbf{R}^n$ and Σ is a finite set. A conformal iterated function system is based on similarity transformations $\{f_i : X \to X \ : \ i \in \Sigma\}$. Each of similarity ratio is smaller than 1. Let $S = \{f_i : I \to I \ : \ i \in \Sigma\}$ be a conformal iterated function system of the unit interval. The code map

$$\pi : \Sigma^\infty \to I$$

k	b_k	c_k	λ_k
2	0.535744398	0.543762703	1.618033989
3	0.517336494	0.518851124	1.839286755
4	0.508398418	0.508716082	1.927561975
5	0.504119567	0.504191307	1.965948237
6	0.502051673	0.502068716	1.983582843
7	0.501040878	0.501045087	1.991964197
8	0.500542535	0.500543615	1.996031180
9	0.500295441	0.500295733	1.998029470
10	0.500172491	0.500172576	1.999018633

TABLE 1. b_k,c_k and λ_k

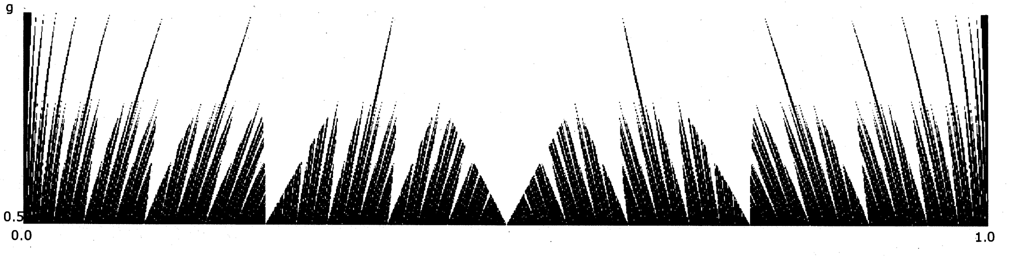

FIGURE 1. $J_1(a)$ for a between 1/2 and the golden ratio g

is defined by

$$\pi(\omega) = \bigcap_{n=0}^{\infty} f_{\omega_0} \circ \cdots \circ f_{\omega_n}(I), \quad (\omega = \omega_0\omega_1 \cdots \in \Sigma^\infty).$$

Its image $J = \pi(\Sigma^\infty)$ is the limit set. When an iterated function system satisfies $f_i(J) \cap f_j(J) = \emptyset$ for any i, j with $i \neq j$, We say the iterated function system is totally disconnected. If not, We say the iterated function system is overlapping. If the iterated function system S is totally disconnected, the code map π is one-to-one and every point $x \in J$ has a unique address $\pi^{-1}(x)$. But in case of overlapping function system, π is not one-to-one and some limit points $x \in J$ have more than one address. The multiplicity function

$$m : I \to \mathbf{N}$$

is given by

$$m(x) = \sharp\{\omega \in \Sigma^\infty \mid \pi(\omega) = x\} \quad (x \in I).$$

For $k = 0, 1, \ldots$, we define $J_k(S)$ by

$$J_k(S) = \{x \in I \mid m(x) = k\}.$$

Then the limit set decomposes into a disjoint union as

$$J = J_1(S) \cup J_2(S) \cup \cdots \cup J_\infty(S).$$

For totally disconnected iterated function systems we have $J_1(S) = J$. Here we are interested in $J_1(S)$ for overlapping iterated function systems.

Now let us consider iterated function system of the unit interval I generated by $f_0(x) = ax$ and $f_1(x) = ax + (1-a)$, where $0 < a < 1$. If $a < 1/2$, the system is totally disconnected. The limit set $J = J_1(a)$ is the Cantor set, and its Hausdorff dimension is given by the Hutchinson's theorem ([3]). When $a > 1/2$, the Hausdorff dimension of $J_1(a)$ is generally difficult to determine. But in the cases described in Theorem 1.1 we can determine the Hausdorff dimension.

2.2. **Graph directed Markov systems.** Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, E, A, i, t). The multigraph consists of a finite set V of vertices and a finite set of directed edges E. Also, a function $A : E \times E \to \{0, 1\}$ is given, called an incidence matrix. It determines which edges may follow a given edge. For each edge e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. So, the matrix has the property that if $A_{uv} = 1$, then $t(u) = i(v)$. We will consider finite and infinite code space with the vertex set consistent with the incidence matrix. We define the set of infinite code space by

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i \omega_{i+1}} = 1 \textit{ for all } i \geq 1\}.$$

By E_A^n we denote the space of codes of length $n \geq 1$,

$$E_A^n = \{\omega \in E^n : A_{\omega_i \omega_{i+1}} = 1 \textit{ for all } i \geq 1\}.$$

And by E_A^* we denote the space of codes with finite length, $E_A^* = \bigcup_{n=1}^\infty E_A^n$. A is irreducible if for all $a, b \in E$, there exists $\omega \in E_A^*$ such that $a\omega b \in E_A^*$.

A conformal graph directed Markov system (CGDMS) consists of a directed multigraph and an incidence matrix together with a set of nonempty compact spaces $\{X_v \subset \mathbf{R}^d\}_{v \in V}$, a number s, $0 < s < 1$ and for every $e \in E$, a one-to-one similar transformation $f_e : X_{i(e)} \to X_{t(e)}$ with a Lipschitz constant s. Briefly the set

$$S = \{f_e : X_{i(e)} \to X_{t(e)} \ : \ e \in E\}$$

is called a CGDMS. When V is a singleton, S is nothing but an conformal iterated function system. For every $\omega \in E_A^\infty$, the word consisting of the first n letters of ω is denoted by

$$\omega|_n = w_1 \cdots w_n \in \Sigma^n.$$

Then we can define the code map of the CGDMS.

Definition 2.1. *The code map* $\pi : E_A^\infty \to \bigcup_{v \in V} X_v$ *is defined by*

$$\pi(\omega) = \bigcap_{n=1}^\infty f_{\omega|_n}(X_{i(\omega_n)}) = \bigcap_{n=1}^\infty f_{\omega_1} \circ \cdots \circ f_{\omega_n}(X_{i(\omega_n)}).$$

We define the limit set J of the conformal graph directed Markov system by the image of the code map.

$$J = \bigcup_{\omega \in E_A^\infty} \pi(\omega)$$

With respect to the product topology, the code space E_A^∞ is compact and the code map π is continuous. Hence, the limit set J is compact.

2.3. **The Hausdorff dimension.** We need a couple of conditions when we calculate the Hausdorff dimension of the limit set. The first is the open set condition.

Definition 2.2. *We say that* $S = \{f_e : X_{i(e)} \to X_{t(e)} \ : \ e \in E\}$ *satisfies the open set condition if there exists a nonempty open set* $U \subset \bigcup_{v \in V} X_v$ *such that for all* $e, e' \in E$ $(e \neq e')$,

$$f_e(U \cap X_{i(e)}) \cap f_{e'}(U \cap X_{i(e')}) = \emptyset \qquad \text{and} \qquad \bigcup_{e \in E} f_e(U \cap X_{i(e)}) \subset U.$$

The second condition is the bounded distortion property.

Definition 2.3. *A CGDMS* $S = \{f_e : X_{i(e)} \to X_{t(e)} \ : \ e \in E\}$ *satisfies the bounded distortion property if there exists* $K \geq 1$ *such that for all* $n \in \mathbf{N}$, $\omega \in E_A^n$ *and* $x, y \in X_{i(\omega|_n)}$, $\frac{|f'_\omega(x)|}{|f'_\omega(y)|} \leq K$.

We can obtain the Hausdorff dimension of the limit set if the CGDMS satisfies these conditions and the incidence matrix is irreducible. The following theorem compiles the theorems of the Mauldin and Urbański book [2].

Theorem 2.4 (Mauldin and Urbański, 2003). *Suppose that CGDMS* $S = \{f_e : X_{i(e)} \to X_{t(e)} \ : \ e \in E\}$ *satisfies the open set condition and the bounded distCortion property and that the incidence matrix* A *is irreducible. Let* $P(t)$ *be*

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} ||f'_\omega||^t.$$

Then the Hausdorff dimension of the limit set $\dim_H J$ *is given by*

$$\dim_H J = \sup\{t > 0 \mid P(t) > 0\} = \inf\{t > 0 \mid P(t) < 0\}.$$

When all the transformations are self-similar, we can also obtain the Hausdorff dimension by a method similar to the proof of Hutchinson's theorem ([3]).

3. Cantor set

Now we consider the iterated function system $S = \{f_0, f_1\}$ defined by (1). Recall that we define b_k by (2) and c_k by (3). Assume that $a > 1/2$. We denote that $F = f_0(I) \cap f_1(I) = [1-a, a]$. For $\omega = \omega_1\omega_2 \cdots \omega_n \in \{0,1\}^n$ $(n = 1, 2, \ldots)$, we define $n+1$ level overlapping area by $F_\omega = f_{\omega_1} f_{\omega_2} \cdots f_{\omega_n}(F)$. Note that every point in F and its descendents F_ω, has multiplicity 2 or more. If $a > g$, the interval $f_0^i(F)$ overlaps with $f_0^{i+1}(F)$ for all $i = 0, 1, \ldots$, and their union

$$\bigcup_{i=0}^{\infty} f_0^i(F) = (0, a]$$

contains only points of multiplicity 2 or more. Likewise, we have $f_1^i(F) \cap f_1^{i+1}(F) \neq \phi$ for $i = 0, 1, \ldots$, and the union

$$\bigcup_{i=0}^{\infty} f_1^i(F) = [1-a, 1)$$

contains only points of multiplicity 2 or more.

Proposition 3.1. *For any* $a \geq b_1$*, we have* $\dim_H J_1(a) = 0$.

Corollary 3.2. *For any* $a \geq b_1$*, we have* $\dim_H \bigcup_{i=2}^{\infty} J_i(a) = 1$.

When $a < b_1$, it is generally difficult to determine the Hausdorff dimension of $J_1(a)$. However, we can obtain the Hausdorff dimension when $b_2 \leq a \leq c_2$.

Theorem 3.3. *Suppose that* a *satisfies* $b_2 \leq a \leq c_2$*. Then*

$$\dim_H J_1(a) = -\frac{\log \frac{1+\sqrt{5}}{2}}{\log a}.$$

Proof. Note in case when $a = b_2$, F_{011} and F_{100} are just touching with F (See Fig.2).

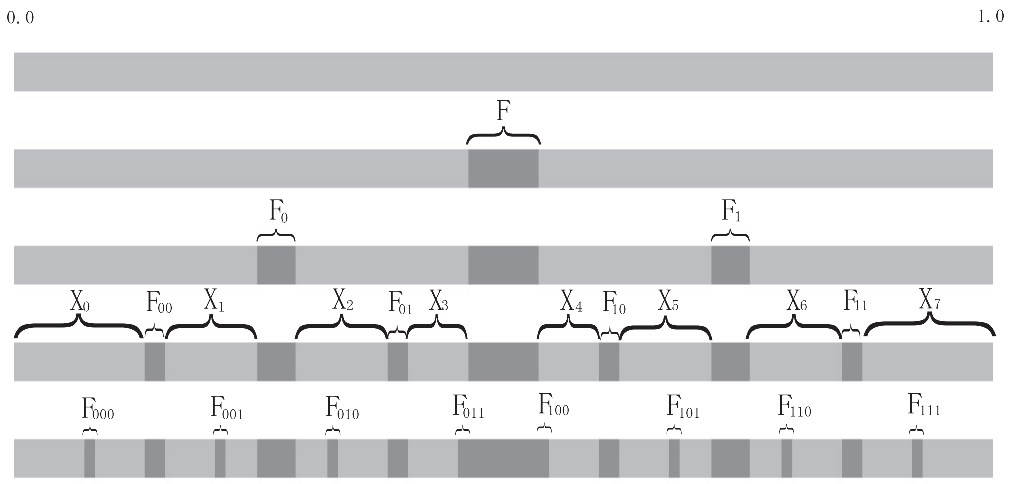

FIGURE 2. F_ω in case of $a = b_2$

Let us consider the union of all the overlapping areas for $n \leq 2$,

$$U = \bigcup_{|\omega| \leq 2} F_\omega.$$

The complement of the interior of U is a disjoint union of closed intervals. Name them $X_0, X_1, \ldots$ from left to right. We now define a multigraph (V_2, E_2, A_2, i, t) by

$$V_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$E_2 = \{(0,0), (1,0), (2,1), (3,1), (4,2), (5,2), (6,3), (1,4), (2,5), (3,5), (4,6), (5,6), (6,7), (7,7)\}$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can define a CGDMS from this multigraph naturally. The CGDMS satisfies the open set condition. It also satisfies the bounded distortion property since the contractivity of all the maps is equal to a. The limit set of the CGDMS (V_2, E_2, A_2, i, t) is $J_1(a)$. Although the incidence matrix A_2 is not irreducible, we can modify the CGDMS slightly and apply Theorem 2.4. We define the modified multigraph by

$$V_2' = \{1, 2, 3, 4, 5, 6\}$$

$$E_2' = \{(2,1), (3,1), (4,2), (5,2), (6,3), (1,4), (2,5), (3,5), (4,6), (5,6)\}$$

$$A_2' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let us denote the limit set of this modified CGDMS by $J_1'(a)$. We have $\dim_H J_1(a) \geq \dim_H J_1'(a)$. It is easy to check that A_2' is irreducible and we can apply Theorem 2.4. The Hausdorff dimension of the limit set $J_1(a)$ is the zero point of the topological pressure function

$$\begin{aligned} P(t) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} ||f_\omega'||^t \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sharp E_A^n + t \log a. \end{aligned}$$

It is well-known that the first term equals $\log \lambda$ where λ is the largest eigenvalue of A_2'. So the maximum eigenvalue of A_2' is equal to the largest eigenvalue of A_2, and is $(1+\sqrt{5})/2$. Since we can evaluate an upper bound by using maximum eigenvalue, The following is hold:

$$\dim_H J_1'(a) \leq \dim_H J_1(a) \leq -\frac{\log \lambda}{\log a} = \dim_H J_1'(a)$$

□

4. Proof of Theorem 1.1

We can generalize Theorem 3.3 and prove Theorem 1.1. First we define a multigraph (V_n, E_n, A_n, i, t) by

$$V_n = \{0, \dots, 2^{n+1} - 1\}$$
$$E_n = \{(i, \phi_0(i)) \mid 0 \leq i \leq 2^{n+1} - 2\} \cup \{(i, \phi_1(i)) \mid 1 \leq i \leq 2^{n+1} - 1\}.$$

The map $\phi_j : \{0, \dots, 2^{n+1} - 1\} \to \{0, \dots, 2^{n+1} - 1\}$ is defined by

$$\phi_0(i) = \left\lfloor \frac{i}{2} \right\rfloor,$$
$$\phi_1(i) = \left\lfloor \frac{i}{2} \right\rfloor + 2^n,$$

where $\lfloor \frac{i}{2} \rfloor$ is the maximum integer not greater than $i/2$. In terms of the binary notation $i = \omega_n \cdots \omega_1 \omega_0(2)$ ($\omega_i \in \{0, 1\}$), the map ϕ_0 (resp. ϕ_1) shifts the digits to the right and append 0 (resp. 1) to the left:

$$\phi_0(\omega_n \omega_{n-1} \dots \omega_1 \omega_0(2)) = 0\omega_n \omega_{n-1} \dots \omega_1(2)$$
$$\phi_1(\omega_n \omega_{n-1} \dots \omega_1 \omega_0(2)) = 1\omega_n \omega_{n-1} \dots \omega_1(2)$$

The incidence matrix $A_n : E_n \times E_n \to \{0, 1\}$ is defined as follows.

$$A_n((i, \phi_k(i)), (j, \phi_l(j))) = \begin{cases} 1 & (\phi_k(i) = j) \\ 0 & (\text{otherwise}) \end{cases}$$

The limit set of this CGDMS is $J_1(a)$. The CGDMS satisfies the open set condition if we consider the open set and all the function is either f_0 or f_1, it also satisfies the bounded distortion property, but the incidence matrix A_n is not irreducible. We modify the CGDMS by removing the vertices 0 and $2^{n+1} - 1$, and restricting the incidence matrix to the edges not involving the vertices $0, 2^{n+1} - 1$. The limit set $J_1'(a)$ of the modified CGDMS has the same Hausdorff dimension as $J_1(a)$ since largest eigenvalue of the modified incidence matrix A_n' is the same as that of A_n. We know that from the proof of the theorem 3.3. To see that the modified incidence matrix A_n' is irreducible, we show for $p, q \in V \setminus \{0, 2^{n+1} - 1\}$ there exists a path from p to q within $V \setminus \{0, 2^{n+1} - 1\}$. Denote p and q in binary notations as

$$p = p_n \cdots p_1 p_0(2),$$
$$q = q_n \cdots q_1 q_0(2).$$

Let l be the largest integer such that

$$q_l = q_{l-1} = \dots = q_0 = p_n = p_{n-1} = \dots = p_{n-l}.$$

Then we have $\phi_{q_n} \cdots \phi_{q_{l+1}}(p) = q$ and for all $k \in \{l+1, \dots, n\}$, we have $\phi_{q_n} \cdots \phi_{q_k}(p) \in V \setminus \{0, 2^{n+1} - 1\}$. This shows that A_n' is irreducible, and we can apply Theorem 2.4 and compute the Hausdorff dimension of the CGDMS in the same way as in the proof of Theorem 3.3.

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