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Indifferent fixed points of position dependent random maps and invariant measures

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1 Intorduction

For a deterministic one dimensional map, if the absolute value of the derivative of a fixed point is 1, the fixed point is called an indifferent fixed point.

In this article, first we remind the relation between an indifferent fixed point and an absolutely continuous invariant measure for a deterministic map. Next, we consider what is the canonical definition of an indifferent fixed point for a position dependent random map in relation to the invariant measure.

2 Indifferent fixed points for deterministic maps

In this section we make clear the definition of an indifferent fixed point for a deterministic one-dimensional map. And, we consider an example and the absolutely continuous invariant measure of it.

Let $T : [0,1] \to [0,1]$ be a piecewise C^1 map. A fixed point q $(T(q) = q)$ is called an indifferent fixed point of T if $|T'(q)| = 1$.

We consider the following simple example.

Example 2.1. Let $T : [0,1] \to [0,1]$. Define $T(x) = x + cx^d \mod 1$, wehere $c > 0$ and $d > 1$ are constants.

In this example, 0 is an indifferent fixed point. It is well known (for example [I1], [I2],

[T]) that T has an absolutely continuous σ-finite invariant measure, say μ, which satisfies the following:

(i) If $d \geq 2$, then $\mu([0, \varepsilon)) = \infty$.

(ii) If $1 < d < 2$, then $\mu([0, \varepsilon)) < \infty$ and $\lim_{x \to 0} h(x) = \infty$, where h is a density of μ.

We would like to define an indifferent fixed point for a random map as the invariant measure has a similar property.

Off course, the concept of an indifferent fixed point is based on the local structure of a map. On the other hand, the concept of an invariant measure is based on the global structure of a map. So, we consider the maps with an absolutely continuous σ-finite invariant measure and we do not make big change outside a neighborhood of the fixed point.

3 Position dependent one-dimensional random maps

In this section we define one-dimensional random maps and define invariant measures for random maps. (There is a detailed explanation in [I3].)

Let $(W, \mathcal{B}, \nu)$ be a σ-finite measure space. We use W as a parameter space. Let $([0,1], \mathcal{A}, m)$ be the measure space with Lebesgue measure m. Let $\tau_t : [0,1] \to [0,1]$ $(t \in W)$ be a nonsingular transformation, which means that $m(\tau_t^{-1}A) = 0$ if $m(A) = 0$ for any $A \in \mathcal{A}$. Assume that $\tau_t(x)$ is a measurable function of $t \in W$ and $x \in [0,1]$.

Let $p : W \times [0,1] \to [0, \infty)$ be a measurable function which is a probability density function of $t \in W$ for each $x \in [0,1]$, that is, $\int_W p(t,x)\nu(dt) = 1$ for each $x \in [0,1]$.

The random map $T = \{\tau_t; p(t,x) : t \in W\}$ is defined as a Markov process with the following transition function:

$$\mathbf{P}(x, D) := \int_W p(t,x) 1_D(\tau_t(x)) \nu(dt) \quad \text{for any} \quad D \in \mathcal{A},$$

where 1_D is the indicator function for D. The transition function $\mathbf{P}$ induces an operator $\mathbf{P}_*$ on measures on $[0,1]$ defined by

$$\mathbf{P}_*\mu(D) := \int_{[0,1]} \mathbf{P}(x, D)\mu(dx) = \int_{[0,1]} \int_W p(t,x)\ 1_D(\tau_t(x))\ \nu(dt)\ \mu(dx)$$

for any measure μ on $[0,1]$ and any $D \in \mathcal{A}$.

If $\mathbf{P}_*\mu = \mu$, μ is called an *invariant measure* for the random map $T = \{\tau_t; p(t,x) : t \in W\}$. This definition of invariant measures for random maps is an extension of it for deterministic maps.

If μ has a density g, then $\mathbf{P}_*\mu$ also has a density, which we denote $\mathcal{L}_T g$. In other words, $\mathcal{L}_T : L^1(m) \to L^1(m)$ is the operator satisfying

$$\int_D \mathcal{L}_T g(x) m(dx) = \int_X \int_W p(t,x)\ 1_D(\tau_t(x))\ \nu(dt)\ g(x) m(dx)$$

for any $D \in \mathcal{A}$. We call $\mathcal{L}_T$ the Perron-Frobenius operator corresponding to the random map T. If you would like to know more the Perron-Frobenius operator for a deterministic map, see [Bo-G] or [L-M].

4 Some options of definitions of an indifferent fixed point for a random map

We consider some options of definitions of an indifferent fixed point for a random map T. Let τ_t and W be as in the previous section. First, we define a fixed point of a random map. A point $q \in [0,1]$ is called a *fixed point* of the random map T if $\tau_t(q) = q$ for any $t \in W$. The following option is a simple extension of the definition of an indifferent fixed point for a deterministic one-dimensional map.

Option 1 of Definition. A fixed point q is called an indifferent fixed point if $|\tau_t'(q)| = 1$ for any $t \in W$.

We consider the following example.

Example 4.1. Let $\tau_t : [0,1] \to [0,1]$. Define $\tau_t(x) = x + c_t x^{d_t} \mod 1$, where $c_t > 0$ and $d_t > 1$ are bounded constants for each $t \in W$. Assume that $p(t,x)$ is given.

In this example, 0 is an indifferent fixed point. Each individual map τ_t is considered in Example 2.1 and it has an absolutely continuous σ-finite invariant measure. Moreover, using the result of Inoue [I4], T has an absolutely continuous σ-finite invariant measure, say μ, which satisfies the following:

(i) If $d_t \geq 2$ for all $t \in W$, then $\mu([0,\varepsilon)) = \infty$.

(ii) If there exists a measurable set W_1 such that

$$1 < d_t < 2 \ \text{ for all } \ t \in W_1 \quad \text{and} \quad \inf_{x \in [0,\varepsilon)} \int_{W_1} p(t,x)\nu(dt) > 0,$$

then, $\mu([0,\varepsilon)) < \infty$ and $\lim_{x \to 0} h(x) = \infty$, where h is a density of μ.

Option 1 can be generalized. Here, we consider some options of definitions of an indifferent fixed point for a position dependent random map.

Option 2 of Definition. A fixed point q is called an indifferent fixed point of T, if there exists a measurable set W_1 such that

$$|\tau_t'(q)| = 1 \text{ for any } t \in W_1 \text{ and } \int_{W_1} p(t,q)\nu(dt) = 1.$$

We consider the following example.

Example 4.2. Let $W = \{1, 2\}$ and let $\nu(\{1\}) = \nu(\{2\}) = 1$. Let $\tau_t : [0, 1] \to [0, 1]$ $(t \in W)$. Define

$$\tau_1(x) = 2x \mod 1, \quad p(1, x) = x^\alpha,$$
$$\tau_2(x) = x, \quad p(2, x) = 1 - x^\alpha,$$

where $\alpha > 0$ is a constant.

Put $W_1 = \{2\}$. Then the fixed point 0 satisfies the definition (Option 2) of an indifferent fixed point of T.

By the result of [I5], this random map has an absolutely continuous σ-finite invariant measure μ with

$$\mu(D) = \int_D \frac{1}{x^\alpha} m(dx).$$

So, $\mu([0, \varepsilon)) = \infty$ if $\alpha \geq 1$.

Option 2 can be more generalized.

Option 3 of Definition. A fixed point q is called an indifferent fixed point of T, if

$$\int_W p(t,q) \log\left(|\tau_t'(q)|\right) \nu(dt) = 0.$$

We consider the following example, which does not satisfy Option 2 of Definition.

Example 4.3. Let $W = \{1, 2\}$ and let $\nu(\{1\}) = \nu(\{2\}) = 1$. Let $\tau_t : [0, 1] \to [0, 1]$ $(t \in W)$. Define

$$\tau_1(x) = 2x \mod 1, \quad p(1, x) = \frac{1}{2},$$
$$\tau_2(x) = \frac{1}{2}x, \quad p(2, x) = \frac{1}{2}.$$

Then, the fixed point 0 satisfies the definition (Option 3) of an indifferent fixed point of T. Modifying the result of section 4 in [P], this random map has an absolutely continuous σ-finite invariant measure μ with $\mu([0,\varepsilon)) = \infty$.

In Examples 2.1, 4.1 and 4.2, there are two cases such that $\mu([0,\varepsilon)) = \infty$ and $\mu([0,\varepsilon)) < \infty$ depending on the parameter or the probability. However, there are not any parameters in Example 4.3. But, if we modify Example 4.3 a little, we can get an example of a random map with a parameter. Let us consider the following example.

Example 4.4. Let α be a constant with $\frac{1}{3} \le \alpha \le \frac{1}{2}$. Let $W = \{1,2\}$ and let $\nu(\{1\}) = \nu(\{2\}) = 1$. Let $\tau_t : [0,1] \to [0,1]$ $(t \in W)$. Define

$$\begin{aligned} \tau_1(x) &= 2x \mod 1, \quad p(1,x) = 1-\alpha, \\ \tau_2(x) &= \frac{1}{2}x, \quad p(2,x) = \alpha. \end{aligned}$$

If we set $\alpha = \frac{1}{2}$ in Example 4.4, then we have Example 4.3. So, Example 4.4 is a generalization of Example 4.3. If $\frac{1}{3} \le \alpha < \frac{1}{2}$, then the fixed point 0 does not satisfy Option 3 of Definition. On the other hand, in this case the random map T has an absolutely continuous invariant probability measure μ such that $\lim_{x\to 0} h(x) = \infty$, where h is a density of μ ([P]). Furthermore, if $\alpha < \frac{1}{3}$, then we have

$$\sup_{x\in[0,1]} \int_W \frac{p(t,x)}{|\tau_t'(x)|}\nu(dt) < 1.$$

Hence, it follows from the result in [I3] that T has an absolutely continuous invariant probability measure μ whose density is bounded. (Off course, this fact is already known by [P].)

We reach to the following option of the definition of an indifferent fixed point.

Option 4 of Definition. A fixed point q is called an indifferent fixed point of T, if q with $\tau_t(q) = q$ for any $t \in W$ satisfies the following two conditions:

(i) $\displaystyle\int_W p(t,q)\log\left(|\tau_t'(q)|\right)\nu(dt) \ge 0.$

(ii) $\displaystyle\int_W \frac{p(t,q)}{|\tau_t'(q)|}\nu(dt) \ge 1.$

In Example 4.4, 0 satisfies the definition (Option 4) of an indifferent fixed point of T.

We give questions to readers and close this note.

Questions to Readers

Which option of the definition of an indifferent fixed point is better?

Do you have another better option of it?

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