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CLT for random walks on nilpotent covering graphs with weak asymmetry

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1 Introduction

To investigate long time asymptotics of random walks on graphs is one of the most central topics in harmonic analysis, geometry, graph theory, to say nothing of probability theory. In particular, central limit theorems (CLTs) has been studied intensively and extensively in various settings. The main concern of this article is CLTs for non-symmetric random walks on a Γ-*nilpotent covering graph* X, that is, X is a covering graph of a finite graph X_0 whose covering transformation group Γ is a finitely generated and nilpotent group. If Γ is abelian, then X is called a Γ-*crystal lattice* (see Figure 1 for typical examples of crystal lattices).

In a series of papers [Kot02, KS00-1, KS00-2, KS06], the authors studied long time asymptotics of symmetric random walks on a crystal lattice X by employing the theory of *discrete geometric analysis*, which has been developed by themselves. Note that the name of the theory was given by Sunada (see [Sun13] for more details). Especially, in [KS00-2], the authors introduced the notion of *standard realization*, which is a discrete harmonic map Φ_0 from a crystal lattice X into the Euclidean space $\Gamma \otimes \mathbb{R}$ equipped with the *Albanese metric*, to characterize an equilibrium configuration of crystals. In [KS00-1], the authors proved the CLT by applying a homogenization method through the standard realization Φ_0. As the scaling limit, they captured a homogenized Laplacian on $\Gamma \otimes \mathbb{R}$. In [Ish03], the author discussed a similar problem to [Kot02, KS00-1] for symmetric random walks on a Γ-nilpotent covering graph X. It is known that X is properly realized into a nilpotent Lie group G such that Γ is isomorphic to a cocompact lattice of G (cf. [Mal51]), so that we define a realization of X by

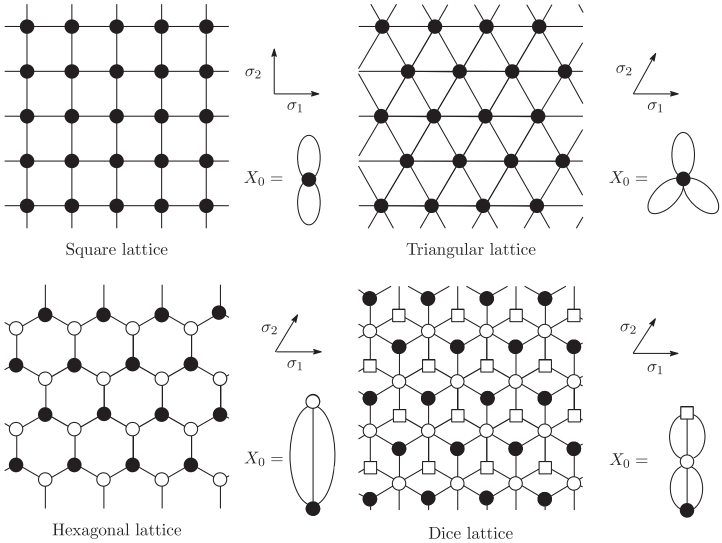

Figure 1: Crystal lattices with the covering transformation group $\Gamma = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2$

a Γ-equivariant map $\Phi : X \longrightarrow G$. By extending the notion of harmonic realizations to the nilpotent case, he established a CLT for symmetric random walks on X.

If we consider non-symmetric cases, the above method cannot be applied directly since the diverging drift term arising from the non-symmetry of the given random walk does not vanish. To overcome these difficulties in the case of crystal lattices, the authors introduced in [IKK17] two schemes.

Scheme 1: Replace the usual transition operator by the *transition-shift operator* to "delete" the diverging drift term. By combining this scheme with a modification of the harmonicity of the realization $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R}$, they proved that

$$\Big(\frac{1}{\sqrt{n}}\big\{\Phi_0(w_{[nt]}) - [nt]\rho_{\mathbb{R}}(\gamma_p)\big\}\Big)_{0\le t\le 1} \longrightarrow (B_t)_{0\le t\le 1} \quad \text{in law}$$

as $n \to \infty$, where $(B_t)_{0\le t\le 1}$ is a $\Gamma\otimes\mathbb{R}$-valued standard Brownian motion. Here $\rho_{\mathbb{R}}(\gamma_p) \in \Gamma\otimes\mathbb{R}$ is the so-called *asymptotic direction* which appears in the law of large numbers for the random walk $\{\Phi_0(w_n)\}_{n=0}^{\infty}$ on $\Gamma \otimes \mathbb{R}$.

Scheme 2: Introduce a one-parameter family of $\Gamma\otimes\mathbb{R}$-valued random walks $(\xi^{(\varepsilon)})_{0\le\varepsilon\le 1}$ which "weakens" the diverging drift term, where this family interpolates the original non-symmetric random walk $\xi_n^{(1)} := \Phi_0(w_n)$ $(n = 0, 1, 2, \ldots)$ and the symmetrized one $\xi^{(0)}$. Putting $\varepsilon = n^{-1/2}$ and letting $n \to \infty$, we have

$$\Big(\frac{1}{\sqrt{n}}\xi_{[nt]}^{(n^{-1/2})}\Big)_{0\le t\le 1} \longrightarrow \big(B_t + \rho_{\mathbb{R}}(\gamma_p)t\big)_{0\le t\le 1} \quad \text{in law}$$

as $n \to \infty$. We emphasize that this scheme is well-known in the study of the hydrodynamic limit of *weakly asymmetric* exclusion processes. See e.g., Kipnis–Landim [KL99], Tanaka [Tan12] and references therein.

In [IKN18-1], we proved a functional CLT (i.e., Donsker-type invariance principle) for a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on the Γ-nilpotent covering graph X. by applying **Scheme 1** to the nilpotent setting. To establish it, we generalize the notion of the harmonic realization to the non-symmetric case, which is called the *modified harmonic realization* $\Phi_0 : X \longrightarrow G$ (see Section 2 for the definition). Let $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$ be the graded Lie algebra of G, where r is the step number of G. Note that $\mathfrak{g}^{(1)}$ is the generating part of $\mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i-1)}]$ for $i = 2, 3, \dots, r$. As the CLT-scaling limit, we captured a diffusion process on G generated by a homogenized sub-Laplacian with a non-trivial $\mathfrak{g}^{(2)}$-valued drift $\beta(\Phi_0)$ arising from the non-symmetry of the given random walk. The main purpose of this article is to give a rough sketch of the proof of a functional CLT for a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on the Γ-nilpotent covering graph X by applying **Scheme 2** to the nilpotent setting. As will be seen later, we will capture a different diffusion process on G generated by a homogenized sub-Laplacian with the constant drift of the ($\mathfrak{g}^{(1)}$-)asymptotic direction. We refer to our recent paper [IKN18-2] for more details and complete proofs of main results.

2 Notations

Denote Γ by a finitely generated, torsion free and nilpotent group of step r. Let $X = (V, E)$ be a Γ-nilpotent covering graph, where V is the set of its vertices and E is the set of all edges. For an oriented edge $e \in E$, we denote by $o(e)$, $t(e)$ and $\overline{e}$ the origin, the terminus and the inverse edge of e, respectively. We write $E_x = \{e \in E \mid o(e) = x\}$ for $x \in V$. We denote by $\Omega_{x,n}(X)$ the set of all paths of length $n \in \mathbb{N} \cup \{\infty\}$ starting from $x \in V$. For simplicity, we write $\Omega_x(X) := \Omega_{x,\infty}(X)$.

By Malcév's theorem (cf. [Mal51]), we find a connected and simply connected nilpotent Lie group $(G, \cdot)$ of step r such that Γ is isomorphic to a cocompact lattice in G. Let $\mathfrak{g}$ be the corresponding Lie algebra of G. By replacing the product $\cdot$ by a certain deformed one $*$, the nilpotent Lie algebra $\mathfrak{g}$ admits the direct sum decomposition $\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{g}^{(k)}$ satisfying $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}$ for $i + j \leq r$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i-1)}]$ for $i = 2, 3, \dots, r$. The nilpotent Lie group $(G, *)$ is called a *limit group* of G. Moreover, the dilation operator $\tau_\varepsilon : G \longrightarrow G$ $(\varepsilon \geq 0)$ becomes not only a diffeomorphism but also a group homomorphism. See e.g., [IKN18-1, Section 2] for details.

Let $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ be the time-homogeneous Markov chain on X induced by a non-negative Γ-invariant transition probability $p : E \longrightarrow [0, 1)$ satisfying

$$\sum_{e \in E_x} p(e) = 1 \qquad (x \in V) \quad \text{and} \quad p(e) + p(\overline{e}) > 0 \qquad (e \in E).$$

Through the covering map $\pi : X \longrightarrow X_0 := \Gamma\backslash X$, we may also consider a random walk $\{w_n = \pi(w_n)\}_{n=0}^{\infty}$ with values in X_0, which is associated with the transition probability $p : E_0 \longrightarrow [0, 1)$ by abuse of notation. Let $m : V_0 \longrightarrow (0, 1]$ be the normalized invariant measure on X_0 and we also write $m : V \longrightarrow (0, 1]$ be a Γ-invariant lift of m to X. Then the random walk on X_0 is said to be $(m\text{-})$*symmetric* if $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$ for $e \in E_0$. Otherwise, it is said to be $(m\text{-})$*non-symmetric*.

We define the *homological direction* of X_0 by

$$\gamma_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0, \mathbb{R}),$$

where $\mathrm{H}_1(X_0, \mathbb{R})$ is the first homology group of X_0. We note that a random walk on X_0 is $(m\text{-})$symmetric if and only if $\gamma_p = 0$. By employing the discrete analogue of Hodge–Kodaira theorem (cf. [KS06, Lemma 5.2]), we equip the first cohomology group $\mathrm{H}^1(X_0, \mathbb{R})$ with the inner product

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)\omega_1(e)\omega_2(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle \qquad \big(\omega_1, \omega_2 \in \mathrm{H}^1(X_0, \mathbb{R})\big)$$

associated with the transition probability p. Let $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ be the canonical surjective linear map induced by the canonical surjective homomorphism $\rho : \pi_1(X_0) \longrightarrow \Gamma$, where $\pi_1(X_0)$ is the fundamental group of X_0. We call $\rho_{\mathbb{R}}(\gamma_p)$ the $(\mathfrak{g}^{(1)}\text{-})$*asymptotic direction* of X_0. It should be noted that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$ though the converse does not hold in general. Then, through the transpose map ${}^t\rho_{\mathbb{R}}$, a flat metric g_0 on $\mathfrak{g}^{(1)}$ is induced from $\langle\!\langle \cdot, \cdot \rangle\!\rangle_p$ as in the diagram below.

$$\begin{tikzcd}
(\mathfrak{g}^{(1)}, g_0) \arrow[d, leftrightarrow, "\text{dual}"] & \mathrm{H}_1(X_0, \mathbb{R}) \arrow[l, twoheadrightarrow, "\rho_{\mathbb{R}}"'] \arrow[d, leftrightarrow, "\text{dual}"] \\
\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \arrow[r, hookrightarrow, "{}^t\rho_{\mathbb{R}}"'] & (\mathrm{H}^1(X_0, \mathbb{R}), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p).
\end{tikzcd}$$

We call the metric g_0 the *Albanese metric*.

A map $\Phi : X \longrightarrow G$ is said to be a Γ-*equivariant realization* of X when it satisfies $\Phi(\gamma x) = \gamma \cdot \Phi(x)$ for $\gamma \in \Gamma$ and $x \in X$. A Γ-equivariant realization $\Phi_0 : X \longrightarrow G$ is said to be *modified harmonic* if it holds that

$$\sum_{e \in E_x} p(e) \Big\{ \log\big(\Phi_0\big(t(e)\big)\big)\Big|_{\mathfrak{g}^{(1)}} - \log\big(\Phi_0\big(o(e)\big)\big)\Big|_{\mathfrak{g}^{(1)}} \Big\} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V),$$

where $\log : G \longrightarrow \mathfrak{g}$ means the inverse map of the usual exponential map $\exp : \mathfrak{g} \longrightarrow G$. Note that such Φ_0 is uniquely determined up to $\mathfrak{g}^{(1)}$-translation, however, it has the ambiguity in the components corresponding to $\mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$.

3 A one-parameter family of modified harmonic realizations

The aim of this section is to introduce a one-parameter family of non-symmetric transition probabilities and discuss several properties of the corresponding family of modified harmonic realizations, which play a crucial role in **Scheme 2**. For the given transition probability p, define a family of Γ-invariant transition probabilities by $(p_\varepsilon)_{0\le\varepsilon\le 1}$ on X by

$$p_\varepsilon(e) := p_0(e) + \varepsilon q(e) \qquad (e \in E), \tag{3.1}$$

where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big) \quad \text{and} \quad q(e) := \frac{1}{2}\Big(p(e) - \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big).$$

Needless to say, the family $(p_\varepsilon)_{0\le\varepsilon\le 1}$ is given by the linear interpolation between the transition probability $p = p_1$ and the m-symmetric probability p_0. Moreover, We easily see that $\gamma_{p_\varepsilon} = \varepsilon\gamma_p$ for $0 \le \varepsilon \le 1$ and the normalized invariant measure associated with p_ε coincides with m for $0 \le \varepsilon \le 1$ (cf. [KS06, Proposition 2.3]).

Let $L_{(\varepsilon)}$ be the transition operator associated with p_ε for $0 \le \varepsilon \le 1$. We also denote by $g_0^{(\varepsilon)}$ the Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε. We write $G_{(\varepsilon)}$ for the nilpotent Lie group of step r whose Lie algebra is $\mathfrak{g} = (\mathfrak{g}^{(1)}, g_0^{(\varepsilon)}) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$. We equip $G_{(0)}$ with the Carnot–Carathéodory metric d_{CC} defined by

$$d_{\mathrm{CC}}(g,h) := \inf\Big\{ \int_0^1 \|\dot{c}(t)\|_{g_0^{(0)}}\,dt \,:\, c(0) = g, c(1) = h, \dot{c}(t) \in \mathfrak{g}^{(1)}_{c(t)} \Big\}$$

for $g, h \in G_{(0)}$, where $\mathfrak{g}^{(1)}_{c(t)}$ denotes the evaluation of $\mathfrak{g}^{(1)}$ at $c(t)$. We note that $(G_{(0)}, d_{\mathrm{CC}})$ is not only a complete metric space but also a geodesic space so that we can consider the geodesic interpolation of $G_{(0)}$-valued random walks. Let $\Phi_0^{(\varepsilon)} : X \longrightarrow G$ be the (p_ε-)modified harmonic realization for $0 \le \varepsilon \le 1$, that is,

$$\sum_{e \in E_x} p_\varepsilon(e)\Big\{ \log\big(\Phi_0^{(\varepsilon)}\big(t(e)\big)\big)\big|_{\mathfrak{g}^{(1)}} - \log\big(\Phi_0^{(\varepsilon)}\big(o(e)\big)\big)\big|_{\mathfrak{g}^{(1)}}\Big\} = \varepsilon\rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V). \tag{3.2}$$

We now impose the following assumption on $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$.

(A1): *For every* $0 \le \varepsilon \le 1$,

$$\sum_{x \in \mathcal{F}} m(x) \log\big(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x)\big)\big|_{\mathfrak{g}^{(1)}} = 0, \tag{3.3}$$

where $\mathcal{F}$ *denotes a fundamental domain of* X.

Since the modified harmonic realizations $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$ are uniquely determined up to $\mathfrak{g}^{(1)}$-translation, it is always possible to take $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$ satisfying **(A1)**.

We are interested in the quantity defined by

$$\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}) := \sum_{e \in E_0} p_\varepsilon(e) m\big(o(e)\big) \log\big(\Phi_0^{(\varepsilon)}(o(\widetilde{e}))^{-1} \cdot \Phi_0^{(\varepsilon)}(t(\widetilde{e}))\big)\big|_{\mathfrak{g}^{(2)}} \qquad (0 \le \varepsilon \le 1).$$

Note that, if the transition probability p_0 is m-symmetric, then $\beta_{(0)}(\Phi_0^{(0)}) = \mathbf{0}$. In particular, we need to know the short time behavior of $\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})$ as $\varepsilon \searrow 0$ for later use. Intuitively, it is difficult to know the behavior since $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$ has the ambiguity in $\mathfrak{g}^{(2)}$-components. However, we can show the following by imposing only **(A1)**.

Proposition 3.1 *Under* **(A1)**, *we have*

$$\lim_{\varepsilon \searrow 0} \beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}) = \beta_{(0)}(\Phi_0^{(0)}) = \mathbf{0}.$$

This proposition will be used in the proof of Lemma 4.1.

4 Main results

We state our main results in this section. We define an *approximation operator* $P_\varepsilon : C_\infty(G_{(0)}) \longrightarrow C_\infty(X)$ by $P_\varepsilon f(x) := f\big(\tau_\varepsilon \Phi_0^{(\varepsilon)}(x)\big)$ for $0 \le \varepsilon \le 1$ and $x \in V$. We extend each element in $\mathfrak{g}$ to a left invariant vector field on $(G, *)$. The following lemma plays a key role to establish the first main result.

Lemma 4.1 *For any* $f \in C_0^\infty(G_{(0)})$, *as* $N \to \infty$, $\varepsilon \searrow 0$ *and* $N^2\varepsilon \searrow 0$, *we have*

$$\Big\| \frac{1}{N\varepsilon^2}(I - L_{(\varepsilon)}^N) P_\varepsilon f - P_\varepsilon \mathcal{A} f \Big\|_\infty \longrightarrow 0,$$

where $\mathcal{A}$ *is the sub-elliptic operator on* $C_0^\infty(G_{(0)})$ *defined by*

$$\mathcal{A} = -\frac{1}{2}\sum_{i=1}^{d_1} V_i^2 - \rho_{\mathbb{R}}(\gamma_p). \tag{4.1}$$

Here, $\{V_1, V_2, \dots, V_{d_1}\}$ *stands for an orthonormal basis of* $(\mathfrak{g}^{(1)}, g_0^{(0)})$.

Outline of the proof. To show Lemma 4.1, we need to apply the Taylor expansion formula to $(I - L_{(\varepsilon)}^N)P_\varepsilon f$ in ε. Then, the first order terms give rise to the constant drift of $\rho_{\mathbb{R}}(\gamma_p)$ due to the modified harmonicity of $\Phi_0^{(\varepsilon)}$ so that we formally have, for $x \in V$,

$$\frac{1}{N\varepsilon^2}(I - L_{(\varepsilon)}^N)P_\varepsilon f(x) = P_\varepsilon\Big(-\frac{1}{2}\sum_{i=1}^{d_1} V_i^2 - \rho_{\mathbb{R}}(\gamma_p) - \beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})\Big) f(x) + O\Big(\frac{1}{N}\Big) + O(N^2\varepsilon)$$

as $N \to \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$. Then, use Lemma 3.1 to verify the assertion of Lemma 4.1. This completes the proof. ∎

Now combine the *Trotter approximation theorem* (cf. [Tro58]) with Lemma 4.1 and we arrive at the following first main result.

Theorem 4.1 (1) *For* $0 \le s \le t$ *and* $f \in C_\infty(G_{(0)})$, *we have*

$$\lim_{n\to\infty} \Big\| L_{(n^{-1/2})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_\infty = 0, \tag{4.2}$$

where $(\mathrm{e}^{-t\mathcal{A}})_{t\ge 0}$ *is the* C^0*-semigroup whose infinitesimal generator* $\mathcal{A}$ *is given by* (4.1).

(2) *Let μ be a Haar measure on $G_{(0)}$. Then, for any $f \in C_\infty(G_{(0)})$ and for any sequence $\{x_n\}_{n=1}^\infty \subset V$ satisfying* $\lim_{n\to\infty} \tau_{n^{-1/2}}\big(\Phi_0^{(n^{-1/2})}(x_n)\big) =: g \in G_{(0)}$, *we have*

$$\lim_{n\to\infty} L_{(n^{-1/2})}^{[nt]} P_{n^{-1/2}} f(x_n) = \mathrm{e}^{-t\mathcal{A}} f(g) := \int_{G_{(0)}} \mathcal{H}_t(h^{-1} * g) f(h)\,\mu(dh) \qquad (t \geq 0), \tag{4.3}$$

where $\mathcal{H}_t(g)$ is a fundamental solution to the heat equation

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) u(t,g) = 0 \qquad (t > 0,\ g \in G_{(0)}).$$

We now fix a reference point $x_* \in V$ such that $\Phi_0^{(0)}(x_*) = \mathbf{1}_G$ and put

$$\xi_n^{(\varepsilon)}(c) := \Phi_0^{(\varepsilon)}\big(w_n(c)\big) \qquad \big(0 \leq \varepsilon \leq 1,\ n = 0,1,2,\dots,\ c \in \Omega_{x_*}(X)\big).$$

Note that **(A1)** does not imply that $\Phi_0^{(\varepsilon)}(x_*) = \mathbf{1}_G$ for $0 < \varepsilon \leq 1$ in general. We then obtain a G-valued random walk $(\Omega_{x_*}(X), \mathbb{P}_{x_*}^{(\varepsilon)}, \{\xi_n^{(\varepsilon)}\}_{n=0}^\infty)$ associated with the transition probability p_ε. For $t \geq 0$, $n = 1,2,\dots$ and $0 \leq \varepsilon \leq 1$, let $\mathcal{X}_t^{(\varepsilon,n)}$ be a map from $\Omega_{x_*}(X)$ to G given by

$$\mathcal{X}_t^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\big(\xi_{[nt]}^{(\varepsilon)}(c)\big) \qquad \big(c \in \Omega_{x_*}(X)\big).$$

We write $\mathcal{D}_n$ for the partition $\{t_k = k/n \,|\, k = 0,1,2,\dots,n\}$ of the time interval $[0,1]$ for $n \in \mathbb{N}$. We define

$$\mathcal{Y}_{t_k}^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\big(\xi_{nt_k}^{(\varepsilon)}(c)\big) = \tau_{n^{-1/2}}\big(\Phi_0^{(\varepsilon)}(w_k(c))\big) \qquad \big(t_k \in \mathcal{D}_n,\ c \in \Omega_{x_*}(X)\big)$$

and consider a G-valued continuous stochastic process $(\mathcal{Y}_t^{(\varepsilon,n)})_{0\leq t\leq 1}$ defined by the d_{CC}-geodesic interpolation of $\{\mathcal{Y}_{t_k}^{(\varepsilon,n)}\}_{k=0}^n$. We consider a stochastic differential equation

$$dY_t = \sum_{i=1}^{d_1} V_i^{(0)}(Y_t) \circ dB_t^i + \rho_{\mathbb{R}}(\gamma_p)(Y_t)\,dt, \qquad Y_0 = \mathbf{1}_G, \tag{4.4}$$

where $(B_t)_{0\leq t\leq 1} = (B_t^1, B_t^2, \dots, B_t^{d_1})_{0\leq t\leq 1}$ is a standard Brownian motion with values in $\mathbb{R}^{d_1}$ starting from $B_0 = \mathbf{0}$. We know that the infinitesimal generator of (4.4) coincides with $-\mathcal{A}$ defined by (4.1). Let $(Y_t)_{0\leq t\leq 1}$ be the $G_{(0)}$-valued diffusion process which is the solution to (4.4). We write $\mathrm{Lip}([0,1]; G_{(0)})$ for the set of all Lipschitz continuous paths taking values in $G_{(0)}$. We define a Polish space by

$$C^{0,\alpha}([0,1]; G_{(0)}) := \overline{\mathrm{Lip}([0,1]; G_{(0)})}^{\rho_\alpha} \qquad (\alpha < 1/2),$$

where ρ_α is an α-Hölder distance on $C([0,1]; G_{(0)})$ given by

$$\rho_\alpha(w^1, w^2) := \sup_{0\leq s<t\leq 1} \frac{d_{\mathrm{CC}}(u_s, u_t)}{|t-s|^\alpha} + d_{\mathrm{CC}}(\mathbf{1}_G, u_0), \qquad u_t := (w_t^1)^{-1} \cdot w_t^2 \quad (0 \leq t \leq 1).$$

To establish a functional CLT for the family of non-symmetric random walks $\{\xi_n^{(\varepsilon)}\}_{n=0}^{\infty}$, we need to impose an additional assumption on $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$.

(A2): *There exists a positive constant C such that, for $k=2,3,\dots,r$,*

$$\sup_{0\le\varepsilon\le 1}\max_{x\in\mathcal{F}}\Big\|\log\big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(0)}(x)\big)\Big|_{\mathfrak{g}^{(k)}}\Big\|_{\mathfrak{g}^{(k)}}\le C, \tag{4.5}$$

where $\|\cdot\|_{\mathfrak{g}^{(k)}}$ denotes a Euclidean norm on $\mathfrak{g}^{(k)}\cong\mathbb{R}^{d_k}$ for $k=2,3,\dots,r$.

Intuitively speaking, the situations that the distance between $\Phi_0^{(\varepsilon)}$ and $\Phi_0^{(0)}$ tends to be too big as $\varepsilon\searrow 0$ are removed under **(A2)**. By setting

$$\log\big(\Phi_0^{(\varepsilon)}(x)\big)\Big|_{\mathfrak{g}^{(k)}}=\log\big(\Phi_0^{(0)}(x)\big)\Big|_{\mathfrak{g}^{(k)}}\qquad(x\in\mathcal{F},\ k=2,3,\dots,r)$$

for $\Phi_0^{(\varepsilon)}:X\longrightarrow G$ with (3.3), the family $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$ satisfies **(A2)**. This means that it is always possible to take a family $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$ satisfying **(A2)** as well as **(A1)**.

Then the second main result is now stated as follows:

Theorem 4.2 *We assume* **(A1)** *and* **(A2)**. *Then the sequence $(\mathcal{Y}_t^{(n^{-1/2},n)})_{0\le t\le 1}$ converges in law to the diffusion process $(Y_t)_{0\le t\le 1}$ in $C^{0,\alpha}([0,1];G_{(0)})$ as $n\to\infty$ for all $\alpha<1/2$.*

Outline of the proof. It is known that we need to show the convergence of the finite dimensional distribution of $\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=0}^{\infty}$ and the tightness of the family of probability measures $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ induced by$\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=0}^{\infty}$ to establish Theorem 4.2. The latter part is most technical in the proof. Namely, we concentrate on the proof of the following.

Lemma 4.2 $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ *is tight in $C^{0,\alpha}([0,1];G_{(0)})$, where $\alpha<1/2$.*

We denote by $G_{(0)}^{(k)}$ the connected and simply connected nilpotent Lie group of step k whose Lie algebra is $(\mathfrak{g}^{(1)},g_0^{(0)})\oplus\mathfrak{g}^{(2)}\oplus\cdots\oplus\mathfrak{g}^{(k)}$. Let $\{\mathcal{Y}^{(n^{-1/2},\,n,\,k)}\}_{n=1}^{\infty}$ be the family of truncated process of $\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ up to step k and write $\{\mathbf{P}^{(n^{-1/2},\,n,\,2)}\}_{n=1}^{\infty}$ for the corresponding family of image probability measures, where $k=1,2,\dots,r$.

Step 1. As a first step, we show the following.

Lemma 4.3 $\{\mathbf{P}^{(n^{-1/2},\,n,\,2)}\}_{n=1}^{\infty}$ *is tight in $C^{0,\alpha}([0,1];G_{(0)}^{(2)})$, where $\alpha<1/2$.*

To show Lemma 4.3, it is sufficient to deduce that there exists a constant $C>0$ independent of $n\in\mathbb{N}$ such that

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n^{-1/2},\,n,\,2)},\mathcal{Y}_t^{(n^{-1/2},\,n,\,2)}\big)^{4m}\Big]\le C(t-s)^{2m}$$

for $m\in\mathbb{N}$ and $0\le s\le t\le 1$. We use several martingale inequalities (e.g., Birkholder–Davis–Gundy inequality) in order to establish the desired moment estimate above.

Step 2. We next show the following.

Lemma 4.4 *For $m, n \in \mathbb{N}$ and $k = 1, 2, \dots, r$, there exist a measurable set $\Omega_k^{(n)} \subset \Omega_{x_*}(X)$, a non-negative random variable $\mathcal{K}_k^{(n)} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*}^{(n^{-1/2})})$ and a Hölder exponent $\alpha < \frac{2m-1}{4m}$ such that $\mathbb{P}_{x_*}^{(n^{-1/2})}(\Omega_k^{(n)}) = 1$ and*

$$d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n^{-1/2},n;\,k)}(c), \mathcal{Y}_t^{(n^{-1/2},n;\,k)}(c)\big) \leq \mathcal{K}_k^{(n)}(c)(t-s)^{\alpha} \quad (c \in \Omega_k^{(n)},\ 0 \leq s \leq t \leq 1). \tag{4.6}$$

We prove Lemma 4.4 by induction on the step number $k = 1, 2, \dots, r$. By virtue of the Kolmogorov–Chentsov criterion and Lemma 4.3, the base cases ($k = 1, 2$) are immediately obtained. Now suppose that (4.6) is true up to k. Then we can construct a measurable set $\Omega_{k+1}^{(n)}$ and a non-negative random variable $\mathcal{K}_{k+1}^{(n)} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*}^{(n^{-1/2})})$ in terms of $\{\Omega_i^{(n)}\}_{i=1}^k$ and $\{\mathcal{K}_i^{(n)}\}_{i=1}^k$. We emphasize that a part of the proof is much inspired by Lyons' original proof (cf. [Lyo98, Theorem 2.2.1]) for the *extension theorem* in the context of rough path theory. We extend his technique on a *free* nilpotent Lie group of step r to the case where the nilpotent Lie group is not always free, which plays a crucial role in the construction of $\Omega_{k+1}^{(n)}$ and $\mathcal{K}_{k+1}^{(n)}$.

Step 3. Finally, we come back to the proof of Lemma 4.2. By (4.6) for $k = r$, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n^{-1/2},n;\,r)}, \mathcal{Y}_t^{(n^{-1/2},n;\,r)}\big)^{4m}\Big] &\leq \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\big[\big(\mathcal{K}_r^{(n)}\big)^{4m}\big](t-s)^{4m\alpha} \\
&\leq \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\big[\big(\mathcal{K}_r^{(n)}\big)^{4m}\big](t-s)^{2m-1} \\
&\leq C(t-s)^{2m-1}
\end{aligned}$$

for some constant $C > 0$ independent of $n \in \mathbb{N}$, where we used $\alpha < \frac{2m-1}{4m}$ and the L^{4m}-integrability of $\mathcal{K}_r^{(n)}$. By applying the Kolmogorov tightness criterion, we have proved that the family $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ is tight in $C^{0,\alpha}([0,1]; G_{(0)})$ for $\alpha < 1/2$. ∎

Remark 4.1 By applying the *corrector method* in the context of stochastic homogenization theory, our CLTs (Theorems 4.1 and 4.2) can be generalized to the case where the family of realizations $(\Phi^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}$ does not necessarily satisfy the condition (3.2). See [IKN18-2] for more details.

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