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Fluctuation scaling limit of inverse local times of jumping-in diffusions

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1 Introduction

In this paper, we consider the fluctuation scaling limit of the inverse local times of jumping-in diffusions. For a strong Markov process X on the half line $[0,\infty)$, we call it jumping-in diffusion process if it is a natunal scale diffusion up to the first hitting time of 0 and, as soon as X hits 0, X jumps into the interior $(0,\infty)$ and starts afresh. It was shown by Feller[2] and Itô[4] that such a process can be characterized by the speed measure dm which characterizes the diffusion on the interior $(0,\infty)$ and the jumping-in measure j which characterizes the law of jumps from the boundary 0 to the interior $(0,\infty)$. Hence we denote this process by $X_{m,j}$.

Let us consider the inverse local time $\eta_{m,j}$ at 0 of a jumping-in diffusion $X_{m,j}$. We propose a condition on m and j for the existence of the following fluctuation scaling limit:

$$\frac{1}{\gamma^{1/\alpha}K(\gamma)}(\eta_{m,j}(\gamma t)-b\gamma t)\xrightarrow[\gamma\to\infty]{d}T(t)\text{ in }\mathbb{D} \tag{1.1}$$

for constants $b\geq 0$, $\alpha\in(0,2]$ and a slowly varying function $K(\gamma)$ at ∞. Here $T(t)$ denotes a α-stable process without negative jumps and $\mathbb{D}$ denotes the space of càdlàg paths from $[0,\infty)$ to $\mathbb{R}$ equipped with Skorokhod's J_1-topology.

Applying the results above, we show the fluctuation scaling limit of the occupation time of two-sided jumping-in diffusions. Two-sided jumping-in diffusions are constructed by connecting two jumping-in diffusion processes with respect to 0. Let X be such a process and define $A(t)=\int_0^t 1_{(0,\infty)}(X_s)ds$. We give conditions for the existence of the limit distribution $\frac{1}{t}A(t)$ as $t\to\infty$. Moreover, in the case where the limit degenerate, that is,

$$\frac{1}{t}A(t)\xrightarrow[t\to\infty]{P}p\in(0,1) \tag{1.2}$$

holds, we show the scaling limit of the fluctuation around the limit constant, that is, the following limit:

$$\frac{1}{\gamma^{1/\alpha}K(\gamma)}(A(\gamma t)-p\gamma t)\xrightarrow[\gamma\to\infty]{d}Z(t) \tag{1.3}$$

for a process $Z(t)$.

Let us explain the difficulty and our methods to overcome it. It was shown by Feller[2] and Itô[4] that the excursion measure of the process $X_{m,j}$ is represented as

$$n_{m,j}(A)=\int_0^\infty P_x^m(A)j(dx). \tag{1.4}$$

Here P_x^m $(x \in (0,\infty))$ is the law of the $\frac{d}{dm}\frac{d^+}{dx}$-diffusion starting from x and killed at 0 and $\frac{d^+}{dx}$ denotes the right-differentiation operator. We denote the law of $X_{m,j}$ starting from 0 by P. Then the Laplace exponent $\chi_{m,j}$ of $\eta_{m,j}$ satisfies the following:

$$\chi_{m,j}(\lambda) = -\log P[\mathrm{e}^{-\lambda \eta_{m,j}(1)}] \tag{1.5}$$

$$= \int_0^\infty (1-\mathrm{e}^{-\lambda u}) n_{m,j}(T_0 \in du) \tag{1.6}$$

$$= \int_0^\infty P_x^m[1-\mathrm{e}^{-\lambda T_0}] j(dx) \tag{1.7}$$

where T_0 denotes the first hitting time at 0. Let the function $u = g_\lambda(m;\cdot)$ is the unique, non-negative and non-increasing solution of the equation $\frac{d}{dm}\frac{d^+}{dx}u = \lambda u$ satisfying the boundary condition $u(0) = 1$ and $\lim_{x\to\infty} \frac{d^+}{dx}u(x) = 0$. It is well-known that the following holds (see e.g.[3]):

$$g_\lambda(m;x) = P_x^m[\mathrm{e}^{-\lambda T_0}]. \tag{1.8}$$

Hence we obtain the following expression:

$$\chi_{m,j}(\lambda) = \int_0^\infty (1-g_\lambda(m;x)) j(dx) \ (\lambda > 0). \tag{1.9}$$

When the boundary 0 for dm is regular, we have a unique solution $u = \varphi_\lambda$ to

$$\frac{d}{dm}\frac{d^+}{dx}u = \lambda u, \ u(0) = 1, \ u^+(0) = 0, \tag{1.10}$$

and a unique solution $u = \psi_\lambda$ to

$$\frac{d}{dm}\frac{d^+}{dx}u = \lambda u, \ u(0) = 0, \ u^+(0) = 1 \tag{1.11}$$

and we can exploit the two functions φ_λ and ψ_λ to analyze the $\frac{d}{dm}\frac{d^+}{dx}$-diffusion. When the boundary 0 for dm is exit, we still have ψ_λ but do not φ_λ. We introduce functions φ_λ^d $(d \in \mathbb{N})$ which play the role corresponding to φ_λ and satisfies the boundary condition which we call modified Neumann boundary condition. Then for a suitable constant $c_\lambda^d(m)$, we obtain the following:

$$g_\lambda(m;x) = \varphi_\lambda^d(m;x) - c_\lambda^d(m)\psi_\lambda(m;x). \tag{1.12}$$

Hence for a sequence of speed measures $\{dm_n\}_n$, jumping-in measures $\{j_n\}_n$ and constants $\{b_n\}_n$, the Laplace exponent $\widetilde{\chi}_{m_n,j_n,b_n}$ of the process $\eta_{m_n,j_n}(t) - b_n t$ is the following:

$$\widetilde{\chi}_{m_n,j_n,b_n}(\lambda) = \chi_{m_n,j_n}(\lambda) - b_n\lambda \tag{1.13}$$

$$= \left(\int_0^\infty (1-\varphi_\lambda^d(m_n;x)) j_n(dx) - b_n\lambda\right) + c_\lambda^d(m_n)\int_0^\infty \psi_\lambda(m_n;x) j_n(dx). \tag{1.14}$$

Therefore our study is reduced to the proper choice of $\{b_n\}_n$ and the analysis of the two terms in RHS of (1.14).

2 Notations

Definition 2.1. We say that $m : (0,\infty) \to \mathbb{R}$ is a string when m is strictly increasing and right-continuous and satisfies $\int_{0+} x dm(x) < \infty$. We denote the set of all strings as $\mathcal{M}$.

Definition 2.2. For $m \in \mathcal{M}$, we define as follows:

$$G(m;x) = \int_0^x m(y)dy \ (x \geq 0), \tag{2.1}$$

$$\widetilde{m}(x) = m(x) - m(1) \ (x > 0), \tag{2.2}$$

$$G^1(m;x) = \int_0^x \widetilde{m}(y)dy \ (x \geq 0). \tag{2.3}$$

$$G^k(m;x) = -\int_0^x dy \int_y^1 G^{k-1}(m;z)dm(z) \ (k \geq 2, \ x \geq 0). \tag{2.4}$$

Remark 2.3. For every $k \geq 2$, the function $G^k(m;x)$ is finite for every $x \geq 0$.

We introduce a subset of $\mathcal{M}$ as follows.

Definition 2.4. Define

$$\mathcal{M}_0 = \{m \in \mathcal{M} \mid \lim_{x \to +0} m(x) > -\infty\}. \tag{2.5}$$

We introduce hierarchy in strings.

Definition 2.5. For $m \in \mathcal{M}_0$, we define $d(m) = 0$ and for $m \in \mathcal{M} \setminus \mathcal{M}_0$

$$d(m) = \inf \left\{ k \geq 1 \;\middle|\; \int_0^1 (-1)^k G^k(m;x)dm(x) < \infty. \right\} \tag{2.6}$$

where $\inf \emptyset = \infty$.

Definition 2.6. For $\alpha \in (0,2)$ we define

$$m^{(\alpha)}(x) = \begin{cases} (1-\alpha)^{-1} x^{1/\alpha - 1}, & \text{if } \alpha \in (0,1), \\ \log x, & \text{if } \alpha = 1, \\ -(\alpha - 1)^{-1} x^{1/\alpha - 1}, & \text{if } \alpha \in (1,2). \end{cases} \tag{2.7}$$

Remark 2.7. The measure $dm^{(\alpha)}$ is the speed measure of the $(2-2\alpha)$-dimensional Bessel process under the natural scale.

3 The Krein-Kotani correspondence

For strings m with $d(m) \leq 1$, we can apply the Krein-Kotani correspondence established in Kotani[7]. It is an extension of the Krein correspondence which has been used in the studies of one-dimensional diffusions(see e.g. Kotani and Watanabe[8] or Kasahara[5]). We briefly summarize the Krein-Kotani correspondence.

If a function $w : \mathbb{R} \to [0, \infty]$ is right-continuous, non-decreasing and satisfies

$$\int_{-\infty}^{a} x^2 dw(x) < \infty \tag{3.1}$$

for some $a \in \mathbb{R}$, we call it Kotani's string. For a Kotani's string w and $\lambda > 0$, we consider the solution $u = f_\lambda$ to the following ODE:

$$\frac{d}{dw}\frac{d^+}{dx}u = \lambda u,\ u(-\infty) = 1,\ u^+(-\infty) = 0\ (x < \ell). \tag{3.2}$$

Here $\ell = \inf\{x \in \mathbb{R} \mid w(x) = \infty\}$. Then define

$$h(w; \lambda) = a + \int_{-\infty}^{a} \left(\frac{1}{f_\lambda(x)^2} - 1 \right) dx + \int_{a}^{\ell} \frac{dx}{f_\lambda(x)^2}\ (\lambda > 0). \tag{3.3}$$

for some $a \in \mathbb{R}$. Note that the value $h(w; \lambda)$ is finite for every $\lambda > 0$ and the function $h(w; \cdot)$ does not depend on the choice of a. Since $h(w; \cdot)$ is the Herglotz function, for a constant $\alpha \in \mathbb{R}$ and a Radon measure σ on $[0, \infty)$ such that $\int_0^\infty \frac{\sigma(d\xi)}{\xi^2+1} < \infty$, we have the following expression:

$$h(w; \lambda) = \alpha + \int_{0-}^{\infty} \left(\frac{1}{\xi + \lambda} - \frac{\xi}{\xi^2 + 1} \right) \sigma(d\xi). \tag{3.4}$$

We note that the measure σ in RHS of (3.4) is the spectral measure of the differential operator $-\frac{d}{dw}\frac{d^+}{dx}$. Hence we call $h(w; \cdot)$ the spectrally characteristic function of w. Let $\mathcal{H}$ be the set of functions which are expressed in the form of RHS of (3.4) for a constant $\alpha \in \mathbb{R}$ and a Radon measure σ on $[0, \infty)$ such that $\int_0^\infty \frac{\sigma(d\xi)}{\xi^2+1} < \infty$. It was proved in [7] that the map $\{$ Kotani's string $\} \ni w \mapsto h(w; \cdot) \in \mathcal{H}$ is bijective. We call this correspondence the Krein-Kotani correspondence. The following theorem shown in Kasahara and Watanabe [6] which asserts a kind of continuity of the Krein-Kotani correspondence is important.

Theorem 3.1. (Kasahara and Watanabe [6, Theorem 2.9])
Let $m_n, m \in \mathcal{M}$ with $d(m_n), d(m) \leq 1$ and $\sigma \geq 0$. Assume the following holds:

(i) $\lim_{n\to\infty} m_n(x) = m(x)$ *for every continuity point x of m,*

(ii) $\lim_{x\to+0} \limsup_{n\to\infty} |\int_0^x m_n(y)^2 dy - \sigma^2| = 0.$

Then we have

$$\lim_{n\to\infty} h_n(m_n^*; \lambda) = h(m^*; \lambda) - \sigma^2\lambda \text{ for every } \lambda > 0. \tag{3.5}$$

Kotani's strings and our strings are related as follows:

Proposition 3.2. *For $m \in \mathcal{M}$, we define its dual string*

$$m^*(x) = \inf\{y > 0 \mid m(y) > x\} \ (x \in \mathbb{R}). \tag{3.6}$$

Then the following holds:

$$d(m) \leq 1 \Rightarrow m^* \text{ is a Kotani's string.} \tag{3.7}$$

Remark 3.3. The set of strings which are dual strings of $m \in \mathcal{M}$ with $d(m) \leq 1$ is that of Kotani's strings which are continuous.

Proposition 3.4. (Kotani [7, Section 4]) *For $m \in \mathcal{M}$ with $d(m) \leq 1$, the function $\lambda h(m^*; \lambda)$ is the Laplace exponent of a Lévy process without Gaussian part and negative jumps.*

We denote the Lévy process whose Laplace exponent is $\lambda h(m^*; \lambda)$ as $T(m; t)$.

4 Representation of $c^1_\lambda(m)$

By the help of the Krein-Kotani correspondence and its continuity, we obtain the following explicit representation of $c^1_\lambda(m)$. This is an extension of a well-known result in the case the boundary 0 is regular.

Theorem 4.1. *Let $m \in \mathcal{M}$ with $d(m) \leq 1$ and $\lambda > 0$. It holds that*

$$c^1_\lambda(m) = \lambda h(m^*; \lambda) - \lambda m(1). \tag{4.1}$$

5 Convergence of $c^d_\lambda(m_n)$

For strings m with $d(m) \geq 2$, we no longer expect the explicit representation of $c^d_\lambda(m)$. However, when a sequence of strings $\{m_n\}_n$ degenerates in a good manner, we can show the degenerate of the sequence $\{c^d_\lambda(m_n)\}_n$.

Definition 5.1. For $m_n \in \mathcal{M}$, we denote $m_n \xrightarrow{G} 0$ when the following hold:

(i) $\lim_{n\to\infty} m_n(x) = 0$ for every $x > 0$,

(ii) $\lim_{n\to\infty} \int_0^1 y dm_n(y) = 0$,

(iii) $\lim_{n\to\infty} \int_0^1 G^d(m_n; x) dm_n(x) = 0$ for some integer $d \geq 1$.

Theorem 5.2. *Let $m_n \in \mathcal{M}$. Suppose $m_n \xrightarrow{G} 0$. Then there exists an integer $N \geq 0$ and for every $d \geq N$, the following holds:*

$$\lim_{n\to\infty} c^d_\lambda(m_n) = 0. \tag{5.1}$$

6 Scaling limit of inverse local times

By using the results in Section 4, 5, we can obtain the desired results on fluctuation scaling limits of inverse local times.

Theorem 6.1 ($\alpha \in (1,2)$). *Let $m \in \mathcal{M}$ with $d(m) \leq 1$, j be a Radon measure on $(0,\infty)$ and K be a slowly varying function at ∞. Suppose the following hold:*

(i) $X_{m,j}$ *exists,*

(ii) $m(x) \sim -(\alpha-1)^{-1}x^{1/\alpha-1}K(x)$ $(x \to \infty)$ *for a constant* $\alpha \in (1,2)$*,*

(iii) $\int_0^\infty xj(dx) < \infty$.

Then if we take $b = -\int_0^\infty G(m;x)j(dx)$*, we have*

$$\frac{1}{\gamma^{1/\alpha}K(\gamma)}(\eta_{m,j}(\gamma t) - b\gamma t) \xrightarrow[\gamma\to\infty]{d} T(m^{(\alpha)};\kappa t) \text{ on } \mathbb{D}. \tag{6.1}$$

Here $\kappa = \int_0^\infty xj(dx)$.

Theorem 6.2 ($\alpha = 1$). *Let $m \in \mathcal{M}$ with $d(m) \leq 1$, j be a Radon measure on $(0,\infty)$ and K be a slowly varying function at ∞ such that K and $1/K$ are locally bounded on $[0,\infty)$. Suppose the following conditions hold:*

(i) $X_{m,j}$ *exists,*

(ii) $\lim_{\gamma\to\infty} \frac{m(\gamma x)-m(\gamma)}{K(\gamma)} = \log x$ *for every* $x > 0.$*,*

(iii) $j(x,\infty) \leq Cx^{-1-\delta}$ *for constants* $C > 0$ *and* $\delta \in (0,1)$ *and every* $x \geq 1$.

Then if we take $b_\gamma = -\int_0^\infty (G(m;x) - m(\gamma)x)j(dx)$*, we have*

$$\frac{1}{\gamma K(\gamma)}(\eta_{m,j}(\gamma t) - b_\gamma \gamma t) \xrightarrow[\gamma\to\infty]{d} T(m^{(1)};\kappa t) \text{ on } \mathbb{D}. \tag{6.2}$$

Here $\kappa = \int_0^\infty xj(dx)$.

Theorem 6.3 ($\alpha = 2$). *Let $m \in \mathcal{M}$ with $d(m) \leq 1$ and j be a Radon measure on $(0,\infty)$. Suppose the following hold:*

(i) $X_{m,j}$ *exists,*

(ii) The function $K(\gamma) = \int_0^\gamma m(y)^2 dy$ *varies slowly at* ∞*,*

(iii) $-\int_0^\infty j(dx)\int_0^x dy \int_0^y G(m;z)dm(z) < \infty$*,*

(iv) $\int_1^\infty |G(m;x)|j(dx) < \infty$.

Then if we take $b = -\int_0^\infty G(m;x)j(dx)$, *we have*

$$\frac{1}{\sqrt{\gamma K(\gamma)}}(\eta_{m,j}(\gamma t) + b\gamma t) \xrightarrow[\gamma\to\infty]{d} B(2\kappa t) \text{ on } \mathbb{D}. \tag{6.3}$$

Here

$$\kappa = \int_0^\infty \left(x + \frac{1}{K(\infty)} \int_0^x dy \int_0^y G(m;z)dm(z) \right) j(dx). \tag{6.4}$$

Theorem 6.4 ($\alpha > 2$)**.** *Let* $m \in \mathcal{M}$ *with* $d(m) < \infty$ *and let* j *be a Radon measure on* $(0,\infty)$. *Suppose the following hold:*

(i) $X_{m,j}$ *exists,*

(ii) $-m(x) \le C_1 x^{1/\alpha - 1}$ *holds for constants* $C_1 > 0$ *and* $\alpha > 2$ *and every* $x \ge 1$,

(iii) $-\int_0^\infty j(dx) \int_0^x dy \int_y^\infty G(m;z)dm(z) < \infty$,

(iv) $j(x,\infty) \le C_2 x^{-\beta}$ *for constants* $C_2 > 0$ *and* $\beta > 2/\alpha$ *and every* $x \ge 1$.

Then for every $t \ge 0$ *it holds that*

$$\frac{1}{\sqrt{\gamma}} \left(\eta_{m,j}(\gamma t) + \gamma t \int_0^\infty G(m;y)j(dy) \right) \xrightarrow{d} B(2\kappa t) \ (\gamma \to \infty). \tag{6.5}$$

Here B *is the standard Brownian motion and* $\kappa = -\int_0^\infty j(dx) \int_0^x dy \int_y^\infty G(m;z)dm(z)$.

7 Limit theorems for the occupation time of two-sided jumping-in diffusions

In this section, we treat two-sided jumping-in diffusions i.e. Markov processes on $\mathbb{R}$ which behave like X_{m_+,j_+} while X is positive and like $-X_{m_-,j_-}$ while X is negative for two jumping-in diffusions X_{m_+,j_+} and X_{m_-,j_-} and, as soon as the process hit the origin they jump into $\mathbb{R} \setminus \{0\}$ according to jumping-in measure $j_+(dx) + j_-(-dx)$. We denote the process $X_{m_+,j_+;m_-j_-}$. For the precise definition, we need the excursion theory and omit here.

Define $A(t) = \int_0^t 1_{(0,\infty)}(X_{m_+,j_+;m_-,j_-}(s))ds$ for $t \ge 0$. We consider the fluctuation scaling limit of $A(t)$.

Theorem 7.1. *Assume the following hold:*

(i) $m_\pm(x) \sim -c_\pm(\alpha - 1)^{-1} x^{1/\alpha - 1} K(x) (x \to \infty)$ *for constants* $\alpha \in (1,2)$, $c_\pm \ge 0$ *and a slowly varying function* K *at* ∞, *respectively,*

(ii) $\kappa_\pm := \int_0^\infty x j(dx) < \infty$.

Then we have

$$f(\gamma)(A(\gamma t)-p\gamma t)\xrightarrow[\gamma\to\infty]{f.d.}(1-p)c_+T(m^{(\alpha)};\kappa_+t)-pc_-\widetilde{T}(m^{(\alpha)};\kappa_-t) \tag{7.1}$$

Here

$$a_\pm=-\int_0^\infty G(m_\pm;x)j_\pm(dx), \tag{7.2}$$

$$p=\frac{a_+}{a_++a_-}, \tag{7.3}$$

$$f(\gamma)=\frac{1}{\gamma^{1/\alpha}K(\gamma)}(a_++a_-)^{1/\alpha}, \tag{7.4}$$

$\widetilde{T}(m^{(\alpha)};t)\stackrel{d}{=}T(m^{(\alpha)};t)$ *and* $T(m^{(\alpha)};t)$ *and* $\widetilde{T}(m^{(\alpha)};t)$ *are independent.*

The similar results hold for $\alpha=1,2$ and $\alpha>2$ in some sense by slight modifications of the assumptions, but we omit here.

8 Related studies

8.1 On one-dimensional diffusions(without jumping-in)

Kasahara and Watanabe[6] constructed via stochastic integral the process $T(m;t)$ for speed measures dm with $\int_{0+}m(x)^2dx<\infty$, which can be regarded as an (renormalized) inverse local time at 0 of diffusions on $(0,\infty)$. More precisely, the process $T(m;t)$ can be represented as follows (See [6, Corollary2.6]):

$$T(m;t)=\lim_{\epsilon\to+0}\int_\epsilon^\infty \ell(\ell^{-1}(t,0),x)dm(x)+m(\epsilon)t\ (t\geq 0). \tag{8.1}$$

Here ℓ denotes the local time of a standard Brownian motion. We note that when 0 is a regular boundary, the process $\int_0^\infty \ell(\ell^{-1}(t,0),x)dm(x)$ is the inverse local time at 0 of the diffusion with the speed measure dm. When $m(0+)=-\infty$, it holds that $\int_0^\infty \ell(\ell^{-1}(t,0),x)dm(x)=\infty$ for every $t>0$. This is the reason we call $T(m;t)$ a renormalized inverse local time at 0. Under assumptions on the tail behavior of m, they showed the scaling limit of the process $T(m;t)$ exists. They applied these results to the studies of the occupation times of one-dimensional diffusions.

Kotani[7] has revealed that the class of Lévy processes without negative jumps $T(m;t)$ have a one-to-one correspondence to a class of functions which we call spectrally characteristic functions and also showed that the convergence of strings in a certain sense is equivalent to the pointwise convergence of their spectrally characteristic functions.

8.2 On jumping-in diffusions

Feller[2] and Itô[4] have shown that jumping-in diffusions are characterized by the speed measures and the jumping-in measures and gave an explicit representations of their excursion measures.

Yano[10] has studied the scaling limit of jumping-in diffusions. He showed that the scaling limit of a jumping-in diffusion $X_{m,j}$ exists under assumptions on the tail behavior of m and j. We note that our results do not overlap with Yano[10] since we mainly treat the case when the scaling limit of $X_{m,j}$ does not exist.

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