数理解析研究所講究録 第2116巻 2019年 85-94

A formulation of quasi-regular non-local Dirichlet forms on Féchet spaces with application to a stochastic quantization of Φ_3^4 field

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February 20, 2019

1 Introduction

We consider a space S that is a real Banach space l^p, $1 \leq p \leq \infty$ with suitable weights. Let μ be a Borel probability measure on S. On the *real* $L^2(S;\mu)$ space, for each $0 < \alpha \leq 1$, we give an explicit formulation of α-stable type (cf., e.g., section 5 of [Fukushima,Uemura 2012] for corresponding formula on $L^2(\mathbb{R}^d)$, $d < \infty$) *non-local* strictly quasi-regular (cf. section IV-3 of [M,R 92]) Dirichlet forms $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ (with a domain $\mathcal{D}(\mathcal{E}_{(\alpha)})$), and show the existence of S-valued Hunt processes properly associated to $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. These general theorems are applied to a stochastic quantization of (α-stable type) Euclidean Φ_3^4 field on $\mathbb{R}^3$.

The objective of the present paper is to announce the above developments that are part of *general (e.g. for* $0 < \alpha < 2$*) and detailed results* given in [A,Kagawa,Yahagi,Y 2018] (cf. also [A,Y 2018]), where the state spaces S are assumed to be either the above l^p, $1 \leq p \leq \infty$, spaces or the direct product $\mathbb{R}^{\mathbb{N}}$ (with $\mathbb{R}$ and resp. $\mathbb{N}$ the spaces of real numbers and resp. natural numbers), both understood as Fréchet spaces, and for each $0 < \alpha < 2$, an explicit formulation of α-stable type non-local quasi-regular (cf. section IV-3 of [M,R 92]) Dirichlet forms is considered.

2 Markovian symmetric forms individually adapted to each measure space

The state space S, on which we define the Markovian symmetric forms, is a weighted l^p space, denoted by $l^p_{(\beta_i)}$, such that, for some $p \in [1,\infty)$ and a weight $(\beta_i)_{i\in\mathbb{N}}$ with $\beta_i \geq 0, i \in \mathbb{N}$,

$$S = l^p_{(\beta_i)} \equiv \big\{ \mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \, : \, \|\mathbf{x}\|_{l^p_{(\beta_i)}} \equiv (\sum_{i=1}^{\infty} \beta_i |x_i|^p)^{\frac{1}{p}} < \infty \big\}. \qquad (2.1)$$

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We denote by $\mathcal{B}(S)$ the Borel σ-field of S. Suppose that we are given a Borel probability measure μ on $(S, \mathcal{B}(S))$. For each $i \in \mathbb{N}$, let σ_{i^c} be the sub σ-field of $\mathcal{B}(S)$ that is generated by the Borel sets

$$B = \left\{ \mathbf{x} \in S \;\middle|\; x_{j_1} \in B_1, \ldots x_{j_n} \in B_n \right\}, \quad j_k \neq i,\ B_k \in \mathcal{B}^1,\ k = 1, \ldots, n,\ n \in \mathbb{N}, \tag{2.2}$$

where $\mathcal{B}^1$ denotes the Borel σ-field of $\mathbb{R}^1$, i.e., σ_{i^c} is the smallest σ-field that includes every B given by (2.2). *Namely, σ_{i^c} is the sub σ-field of $\mathcal{B}(S)$ generated by the variables $\mathbf{x} \setminus x_i$, i.e., all variables except for the i-th variable x_i.* For each $i \in \mathbb{N}$, let $\mu(\cdot \,|\, \sigma_{i^c})$ be the conditional probability, a one-dimensional probability distribution-valued σ_{i^c} measurable function, (μ-every where defined) that is characterized by (cf. (2.4) of [A,R91])

$$\mu\big(\{\mathbf{x} \,:\, x_i \in A\} \cap B\big) = \int_B \mu(A \,|\, \sigma_{i^c})\, \mu(d\mathbf{x}), \quad \forall A \in \mathcal{B}^1,\ \forall B \in \sigma_{i^c}. \tag{2.3}$$

Define

$$L^2(S; \mu) \equiv \left\{ f \;\middle|\; f : S \to \mathbb{R}, \text{ measurable and } \|f\|_{L^2} = \Big(\int_S |f(\mathbf{x})|^2 \mu(d\mathbf{x}) \Big)^{\frac{1}{2}} < \infty \right\}, \tag{2.4}$$

and

$$\mathcal{F}C_0^\infty \equiv \text{the } \mu \text{ equivalence class of } \left\{ f \;\middle|\; \exists n \in \mathbb{N},\ f \in C_0^\infty(\mathbb{R}^n \to \mathbb{R}) \right\} \subset L^2(S; \mu), \tag{2.5}$$

where $C_0^\infty(\mathbb{R}^n \to \mathbb{R})$ denotes the space of *real valued* infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

On $L^2(S; \mu)$, for any $0 < \alpha \leq 1$ (*for the case of general $0 < \alpha < 2$, cf. [A,Kagawa,Yahagi,Y 2018]*), we are going to define the Markovian symmetric forms $\mathcal{E}_{(\alpha)}$ called *individually adapted Markovian symmetric forms of index α relative to the measure μ*. They have a natural analogy of the one for α-stable type (*non local* Dirichlet forms on $\mathbb{R}^d$, $d < \infty$ (cf. Remark 1 given below and (5.3), (1.4) of [Fukushima,Uemura 2012]), and can be seen as non local analogy of local classical Dirichlet forms on infinite dimensional topological vector spaces (cf. [A,R 89, 90, 91]). The latter are defined by making use of directional derivatives. The definition of our forms is as follows: Firstly, for each $0 < \alpha \leq 1$ and $i \in \mathbb{N}$, and for the variables y_i, $y_i' \in \mathbb{R}^1$, $\mathbf{x} = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \in S$ and $\mathbf{x} \setminus x_i \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)$, we consider the bilinear expression

$$\begin{aligned}
&\Phi_\alpha(u, v; y_i, y_i', \mathbf{x} \setminus x_i) \\
&\quad \equiv \frac{1}{|y_i - y_i'|^{\alpha+1}} \times \big\{ u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots) - u(x_1, \ldots, x_{i-1}, y_i', x_{i+1}, \ldots) \big\} \\
&\quad \times \big\{ v(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots) - v(x_1, \ldots, x_{i-1}, y_i', x_{i+1}, \ldots) \big\},
\end{aligned} \tag{2.6}$$

and set

$$\mathcal{E}_{(\alpha)}^{(i)}(u, v) \equiv \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i)\, \Phi_\alpha(u, v; y_i, x_i, \mathbf{x} \setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathbf{x}), \tag{2.7}$$

$$\mathcal{E}_{(\alpha)}(u, v) \equiv \sum_{i \in \mathbb{N}} \mathcal{E}_{(\alpha)}^{(i)}(u, v). \tag{2.8}$$

where $I_{\{\cdot\}}$ denotes the indicator function. For $y_i \neq y'_i$, (2.6) is well defined for any real valued $\mathcal{B}(S)$-measurable functions u and v. For *the Lipschiz continuous* functions $\tilde{u} \in C_0^\infty(\mathbb{R}^n \to \mathbb{R}) \subset \mathcal{F}C_0^\infty$ resp. $\tilde{v} \in C_0^\infty(\mathbb{R}^m \to \mathbb{R}) \subset \mathcal{F}C_0^\infty$, $n, m \in \mathbb{N}$ which are representations of $u \in \mathcal{F}C_0^\infty$ resp. $v \in \mathcal{F}C_0^\infty$, $n, m \in \mathbb{N}$, (2.7) and (2.8) are well defined (the right hand side of (2.8) has only a finite number of sums). In Theorem 1 given below we see that (2.7) and (2.8) are well defined for $\mathcal{F}C_0^\infty$, the space of μ-equivalent class.

Remark **1** *We can also derive the following equivalent expressions for* $\mathcal{E}^i_{(\alpha)}(u, v)$.

$$
\begin{aligned}
\mathcal{E}^{(i)}_{(\alpha)}(u, v) &= \int_S \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}\, \Phi_\alpha(u, v; y_i, x_i, \mathbf{x} \setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu\big(dx_i \,\big|\, \sigma_{i^c}\big)\, \mu(d\mathbf{x}) \\
&= \int_S \Big\{ \int_{\mathbb{R}^2} I_{\{y_i \neq y'_i\}}\, \Phi_\alpha(u, v; y_i, y'_i, \mathbf{x} \setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\, \mu\big(dy'_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathbf{x}) \qquad (2.9) \\
&= \int_{S \setminus x_i} \Big\{ \int_{\mathbb{R}^2} I_{\{y_i \neq y'_i\}}\, \Phi_\alpha(u, v; y_i, y'_i, \mathbf{x} \setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\, \mu\big(dy'_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d(\mathbf{x} \setminus x_i)),
\end{aligned}
$$

where $\mu(d(\mathbf{x} \setminus x_i))$ *is the marginal probability distribution of the variable* $\mathbf{x} \setminus x_i$, *i.e., for any* $A \in \sigma_{i^c}$, $\int_A \mu(d(\mathbf{x} \setminus x_i)) = \int_S I_{\mathbb{R}}(x_i)\, I_A(\mathbf{x} \setminus x_i)\, \mu(d\mathbf{x})$. *The third and fourth formulas give more symmetric definitions for* $\mathcal{E}^{(i)}_{(\alpha)}(u, v)$ *with respect to the variables* y_i *and* x_i *(analogous to (1.2.1) of [Fukushima 80]). These will be used in section 4*

The following is the main theorem on the closability part of this paper.

Theorem 1 *The symmetric non-local forms* $\mathcal{E}_{(\alpha)}$, $0 < \alpha \leq 1$ *given by (2.8) are*
i) well-defined on $\mathcal{F}C_0^\infty$;
ii) Markovian;
iii) closable in $L^2(S; \mu)$.
For each $0 < \alpha \leq 1$, *the closed extension of* $\mathcal{E}_{(\alpha)}$ *is denoted by* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *with the domain* $\mathcal{D}(\mathcal{E}_{(\alpha)})$, *which is a non-local Dirichlet form on* $L^2(S; \mu)$.
Moreover it holds that $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$.

3 Proof of Theorem 1.

Suppose that $0 < \alpha \leq 1$
For the statement i), we have to show that
i-1) for any real valued $\mathcal{B}(S)$-measurable function u on S, such that $u = 0$, $\mu - a.e.$, it holds that $\mathcal{E}_{(\alpha)}(u, u) = 0$ (cf. (3.8) given below), and
i-2) for any $u, v \in \mathcal{F}C_0^\infty$, there corresponds only one value $\mathcal{E}_{(\alpha)}(u, v) \in \mathbb{R}$,
For the statement ii), we have to show that (cf. [Fukushima 80]) for any $\epsilon > 0$ there exists a real function $\varphi_\epsilon(t)$, $-\infty < t < \infty$, such that $\varphi_\epsilon(t) = t$, $\forall t \in [0, 1]$, $-\epsilon \leq \varphi_\epsilon(t) \leq 1 + \epsilon$, $\forall t \in (-\infty, \infty)$, and $0 \leq \varphi_\epsilon(t') - \varphi_\epsilon(t) \leq t' - t$ for $t < t'$, such that for any $u \in \mathcal{F}C_0^\infty$ it holds that $\varphi_\epsilon(u) \in \mathcal{F}C_0^\infty$ and

$$\mathcal{E}_{(\alpha)}(\varphi_\epsilon(u), \varphi_\epsilon(u)) \leq \mathcal{E}_{(\alpha)}(u, u). \qquad (3.1)$$

For the statement iii), we have to show the following: For a sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in \mathcal{F}C_0^\infty$, $n \in \mathbb{N}$, if

$$\lim_{n\to\infty} \|u_n\|_{L^2(S;\mu)} = 0, \tag{3.2}$$

and

$$\lim_{n,m\to\infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) = 0, \tag{3.3}$$

then

$$\lim_{n\to\infty} \mathcal{E}_{(\alpha)}(u_n, u_n) = 0. \tag{3.4}$$

i-1) can be seen as follows:
For each $i \in \mathbb{N}$ and any real valued $\mathcal{B}(S)$-measurable function u, note that for each $\epsilon > 0$,

$$I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)$$

defines a $\mathcal{B}(S\times\mathbb{R})$-measurable function. Here we use an extension of the function $\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)$, for $v=u$, $x=x_i$, defined by (2.6) to a general $\mathcal{B}(S)$-measurable function u (instead of a function in $\mathcal{F}C_0^\infty$). $\mathcal{B}(S\times\mathbb{R})$ is the Borel σ-field of $S\times\mathbb{R}$. $\mathbf{x} = (x_i, i\in\mathbb{N}) \in S$ and $y_i \in \mathbb{R}$. Then, for any compact subset K of $\mathbb{R}$, $0 \le I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)$ converges monotonically to $I_{\{y_i\neq x_i\}}(y_i)\,\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)$ as $K\uparrow\mathbb{R}$ and $\epsilon\downarrow 0$, for every $y_i\in\mathbb{R}$, $\mathbf{x}\in S$, and by the Fatou's Lemma, we have

$$\begin{aligned}
&\int_S \Big\{\int_{\mathbb{R}} I_{\{y_i\neq x_i\}}(y_i)\,\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x}) \qquad (3.5)\\
&\quad= \int_S \liminf_{K\uparrow\mathbb{R}}\liminf_{\epsilon\downarrow 0}\Big\{\int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x})\\
&\quad\le \liminf_{K\uparrow\mathbb{R}}\liminf_{\epsilon\downarrow 0}\int_S \Big\{\int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\,\Phi_\alpha(u,u;y_i,x_i,\mathbf{x}\setminus x_i)\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x}),
\end{aligned}$$

I_K denotes the indicator function of K. Through the definition of the conditional probability distributions and conditional expectations, we see that, for any $\epsilon > 0$,

$$\begin{aligned}
&\int_S \Big\{\int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\,\frac{1}{|y_i-x_i|^{\alpha+1}}\big(u(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots)\big)^2\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x})\\
&\quad\le \frac{1}{\epsilon^{\alpha+1}}\int_S \Big\{\int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\,\big(u(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots)\big)^2\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x})\\
&\quad\le \frac{1}{\epsilon^{\alpha+1}}\int_S \Big\{\int_{\mathbb{R}} \big(u(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots)\big)^2\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x})\\
&\quad= \frac{1}{\epsilon^{\alpha+1}}\int_S \big(u(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots)\big)^2\mu(d\mathbf{x}), \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_S \big(u(x_1,\dots)\big)^2\Big\{\int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\,\frac{1}{|y_i-x_i|^{\alpha+1}}\,\mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\,\mu(d\mathbf{x})\\
&\qquad\qquad\le \frac{1}{\epsilon^{\alpha+1}}\int_S \big(u(x_1,\dots)\big)^2\mu(d\mathbf{x}). \qquad (3.7)
\end{aligned}$$

From (3.6), by making use of the Cauchy Schwaz's inequality we have

$$\Big| \int_S u(x_1,\dots,x_n)\Big\{ \int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i)\, \frac{1}{|y_i-x_i|^{\alpha+1}} \times u(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big)\Big\}\mu(d\mathbf{x})\Big| \le \frac{1}{\epsilon^{\alpha+1}} \int_S \big(u(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)\big)^2 \mu(d\mathbf{x}).$$

By this and (3.6), (3.7), from (3.5) we have proven i-1):

$$\mathcal{E}^{(i)}_{(\alpha)}(u,u)=0, \quad \forall i\in\mathbb{N}, \qquad \mathcal{E}_{(\alpha)}(u,u)=0,$$

for any real valued $\mathcal{B}(S)$-measurable function u such that $u=0$, μ-a.e.. (3.8)

In order to show i-2), for $0<\alpha\le 1$, take any *representation* $\tilde u\in C_0^\infty(\mathbb{R}^n)$ of $u\in\mathcal{F}C_0^\infty$, $n\in\mathbb{N}$. Using $0<\alpha+1\le 2$, it is easy to see from the definition (2.6) that there exists an $M<\infty$ depending on $\tilde u$ such that

$$0\le \Phi_\alpha(\tilde u,\tilde u;y_i,y_i',\mathbf{x}\setminus x_i)\le M, \quad \forall \mathbf{x}\in S, \text{ and } \forall y_i,\ y_i'\in\mathbb{R}. \tag{3.9}$$

Since, $u=\tilde u+\overline{0}$ for some real valued $\mathcal{B}(S)$-measurable function $\overline{0}$ such that $\overline{0}=0$, μ-a.e., by (3.9) together with i-1) (cf. (3.8)) and the the Cauchy Schwarz's inequality, for $u\in\mathcal{F}C_0^\infty$, $\mathcal{E}_{(\alpha)}(u,u)\in\mathbb{R}$, $0<\alpha\le 1$, is identical with $\mathcal{E}_{(\alpha)}(\tilde u,\tilde u)$ and well-defined (in fact, for only a finite number of $i\in\mathbb{N}$. we have $\mathcal{E}^{(i)}_{(\alpha)}(u,u)\ne 0$, cf. also (2.8)). Then by the Cauchy Schwarz's inequality i-2) follows.

The proof of ii) is very similar to the one given in section 1 of [Fukushima 80], and it is omitted.

iii) can be proved as follows (cf. section 1 of [Fukushima 80]): Suppose that a sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies (3.2) and (3.3). Then, by (3.2) there exists a measurable set $\mathcal{N}\in\mathcal{B}(S)$ and a sub sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\mu(\mathcal{N})=0$, $\lim_{n_k\to\infty} u_{n_k}(\mathbf{x})=0$, $\forall \mathbf{x}\in S\setminus\mathcal{N}$. Define

$$\tilde u_{n_k}(\mathbf{x})=u_{n_k}(\mathbf{x}) \quad \text{for } \mathbf{x}\in S\setminus\mathcal{N}, \qquad \text{and} \qquad \tilde u_{n_k}(\mathbf{x})=0 \quad \text{for } \mathbf{x}\in\mathcal{N}.$$

Then,

$$\tilde u_{n_k}(\mathbf{x})=u_{n_k}(\mathbf{x}),\ \mu-a.e., \qquad \lim_{n_k\to\infty}\tilde u_{n_k}(\mathbf{x})=0, \quad \forall\mathbf{x}\in S. \tag{3.10}$$

By the fact i-1), precisely by (3.8), shown above and (3.10), for each i, we see that

$$\begin{aligned}
&\int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i\ne x_i\}}(y_i)\, \Phi_\alpha(u_n,u_n;y_i,x_i,\mathbf{x}\setminus x_i)\, \mu\big(dy_i\,\big|\,\sigma_{i^c}\big)\Big\}\mu(d\mathbf{x})\\
&= \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i\ne x_i\}}(y_i) \lim_{n_k\to\infty} \Phi_\alpha(u_n-\tilde u_{n_k},u_n-\tilde u_{n_k};y_i,x_i,\mathbf{x}\setminus x_i)\, \mu\big(dy_i\,\big|\,\sigma_{i^c}\big)\Big\}\mu(d\mathbf{x})\\
&\le \liminf_{n_k\to\infty} \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i\ne x_i\}}\, \Phi_\alpha(u_n-\tilde u_{n_k},u_n-\tilde u_{n_k};y_i,x_i,\mathbf{x}\setminus x_i)\, \mu\big(dy_i\,\big|\,\sigma_{i^c}\big)\Big\}\mu(d\mathbf{x})\\
&= \liminf_{n_k\to\infty} \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i\ne x_i\}}\, \Phi_\alpha(u_n-u_{n_k},u_n-u_{n_k};y_i,x_i,\mathbf{x}\setminus x_i)\, \mu\big(dy_i\,\big|\,\sigma_{i^c}\big)\Big\}\mu(d\mathbf{x})\\
&\equiv \liminf_{n_k\to\infty} \mathcal{E}^{(i)}_{(\alpha)}(u_n-u_{n_k},u_n-u_{n_k}).
\end{aligned} \tag{3.11}$$

Now, by using the assumption (3.3) on the right hand side of (3.11), we get

$$\lim_{n\to\infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n, u_n) = 0, \qquad \forall i \in \mathbb{N}. \tag{3.12}$$

(3.12) together with i) show that for each $i \in \mathbb{N}$, $\mathcal{E}_{(\alpha)}^{(i)}$ with the domain $\mathcal{F}C_0^\infty$ is closable in $L^2(S;\mu)$. Since, $\mathcal{E}_{(\alpha)} \equiv \sum_{i\in\mathbb{N}} \mathcal{E}_{(\alpha)}^{(i)}$, by using Fatou's Lemma, from (3.12) and the assumption (3.3) we see that

$$\mathcal{E}_{(\alpha)}(u_n, u_n) = \sum_{i\in\mathbb{N}} \lim_{m\to\infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n - u_m, u_n - u_m) \le \liminf_{m\to\infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) \to 0 \,\text{as}\, n \to \infty.$$

This proves (3.4) (cf. Proposition I-3.7 of [M,R 92] for a general argument of this type). This completes the proof of iii). Thus, by the closed extension the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is defined.

In order to see that $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$, we take $\eta \in C_0^\infty(\mathbb{R} \to \mathbb{R})$ such that $\eta(x) \ge 0$, $|\frac{d}{dx}\eta(x)| \le 1$ for $x \in \mathbb{R}$, and $\eta(x) = 1$ for $|x| < 1$; $\eta(x) = 0$ for $|x| > 3$, and define $u_M(x_1, x_2, \dots) \equiv \eta(x_1 \cdot M^{-1}) \prod_{i\ge 2} I_{\mathbb{R}}(x_i) \in \mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E}_{(\alpha)})$ for each $M \in \mathbb{N}$. Then it is possible to show that (cf. (2.6) and (2.7)) $\sup_{M\in\mathbb{N}} \mathcal{E}_{(\alpha)}(u_M, u_M) < \infty$. Since, $\lim_{M\to\infty} u_M(\mathbf{x}) = 1 = \prod_{i\ge1} I_{\mathbb{R}}(x_i)$ point wise, and hence $\mu - a.e.$, from Lemma I-2.12 of [M,R 92] we have $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$. This complete the proof of Theorem 1. ■

4 Quasi-regularity

For each $i \in \mathbb{N}$, we denote by X_i the random variable (i.e., measurable function) on $(S, \mathcal{B}(S), \mu)$, that represents the coordinate x_i of $\mathbf{x} = (x_1, x_2, \dots)$, precisely,

$$X_i \,: S \ni \mathbf{x} \longmapsto x_i \in \mathbb{R}. \tag{4.1}$$

By making use of the random variable X_i, we have the following probabilistic expression:

$$\int_S 1_B(x_i)\, \mu(d\mathbf{x}) = \mu(X_i \in B), \qquad \text{for} \quad B \in \mathcal{B}(S). \tag{4.2}$$

Theorem 2 *Let* $0 < \alpha \le 1$, *and let* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *be the closed Markovian symmetric form defined through Theorem 1 on the state space* S. *For* $S = l^p_{(\beta_i)}$, $1 \le p < \infty$, *if there exists a positive* l^p *sequence* $\{\gamma_i^{-\frac{1}{p}}\}_{i\in\mathbb{N}}$, *and an* $0 < M < \infty$ *such that*

$$\sum_{i=1}^\infty \beta_i^{\frac{2}{p}} \gamma_i^{\frac{2}{p}} \cdot \mu\Big(\beta_i^{\frac{1}{p}} |X_i| > M \cdot \gamma_i^{-\frac{1}{p}}\Big) < \infty, \tag{4.3}$$

holds, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a (strictly) quasi-regular Dirichlet form.*

Proof of Theorem 2. It is possible to verify that the Dirichlet forms $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ satisfy the definition of the quasi-regularlity given by Definition 3.1 in section IV-3 of [M,R 92]. Namely, by using the same notions adopted in [M,R 92], we have to certify that the following i), ii) and iii) are satisfied by $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$:

i) There exists an $\mathcal{E}_{(\alpha)}$-nest $(D_M)_{M\in\mathbb{N}}$ consisting of compact sets.
ii) There exists a subset of $\mathcal{D}(\mathcal{E}_{(\alpha)})$, that is dense with respect to the norm $\|\cdot\|_{L^2(S;\mu)}+\sqrt{\mathcal{E}_{(\alpha)}}$. And the elements of this subset have $\mathcal{E}_{(\alpha)}$-quasi continuous versions.
iii) There exists $u_n\in\mathcal{D}(\mathcal{E}_{(\alpha)})$, $n\in\mathbb{N}$, having $\mathcal{E}_{(\alpha)}$-quasi continuous μ-versions $\tilde{u}_n$, $n\in\mathbb{N}$, and an $\mathcal{E}_{(\alpha)}$-exceptional set $\mathcal{N}\subset S$ such that $\{\tilde{u}_n : n\in\mathbb{N}\}$ separates the points of $S\setminus\mathcal{N}$.
The fact that the quasi-regular Dirichlet form $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$ is looked upon a *strictly* quasi-regular Dirichlet form can be guaranteed by showing (cf. Proposition V-2.15 of [M,R 92])
iv) $1\in\mathcal{D}(\mathcal{E}_{(\alpha)})$

In fact, by Theorem 1 in section 2, the above ii) and iii) hold for $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$: since $\mathcal{F}C_0^\infty\subset C(S\to\mathbb{R})$, and $\mathcal{D}(\mathcal{E}_{(\alpha)})$ is the closure of $\mathcal{F}C_0^\infty$ by Theorem 1, we can take $\mathcal{F}C_0^\infty$ as the subset of $\mathcal{D}(\mathcal{E}_{(\alpha)})$ mentioned in the above ii). Moreover, since $\mathcal{F}C_0^\infty$ separates the points S, we see that the above iii) holds. Also, iv) is the last statement of Theorem 1.

Hence, we have only to show that the above i) holds for $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$. Equivalently (cf. Definition 2.1. in section III-2 of [M,R 92]), we have to show that there exists an increasing sequence $(D_M)_{M\in\mathbb{N}}$ of compact subsets of S such that $\cup_{m\geq 1}\mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}$ is dense in $\mathcal{D}(\mathcal{E}_{(\alpha)})$ (with respect to the norm $\|\cdot\|_{L^2(S;\mu)}+\sqrt{\mathcal{E}_{(\alpha)}}$), where $\mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}$ is the subspace of $\mathcal{D}(\mathcal{E}_{(\alpha)})$ the elements of which are functions with supports belonging to D_M. For this, by Theorem 1, since $\mathcal{D}(\mathcal{E}_{(\alpha)})$ is the closure of $\mathcal{F}C_0^\infty$, it suffices to show the following: there exists a sequence of compact sets

$$D_M\subset S,\quad M\in\mathbb{N} \tag{4.4}$$

and a subset $\tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)})\subset L^2(S;\mu)$ that satisfies

$$\tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)})\subset\bigcup_{M\geq 1}\mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}; \tag{4.5}$$

for any $u\in\mathcal{F}C_0^\infty$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n\in\tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)})$, $n\in\mathbb{N}$, such that

$$\lim_{n\to\infty}u_n=u,\quad \text{in } \mathcal{D}(\mathcal{E}_{(\alpha)}) \quad \text{with respect to the norm}\quad \|\cdot\|_{L^2(S;\mu)}+\sqrt{\mathcal{E}_{(\alpha)}}. \tag{4.6}$$

■

5 Associated Markov processes and a standard procedure of application of stochastic quantizations on $\mathcal{S}'$

Let $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$, $0<\alpha\leq 1$, be the family of strictly quasi-regular Dirichlet forms on $L^2(S;\mu)$ with a state space S defined by Theorems 2. By Theorem IV-3.5 and Proposition V-2.15 of [M,R 92] we conclude that to $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$, there exists a properly associated S-valued Hunt process

$$\mathbb{M}\equiv\Big(\Omega,\mathcal{F},(X_t)_{t\geq 0},(P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\Big). \tag{5.1}$$

$\triangle$ is a point adjoined to S as an isolated point of $S_\triangle\equiv S\cup\{\triangle\}$. Let $(T_t)_{t\geq 0}$ be the strongly continuous contraction semigroup associated with $(\mathcal{E}_{(\alpha)},\mathcal{D}(\mathcal{E}_{(\alpha)}))$, and $(p_t)_{t\geq 0}$ be the corresponding transition semigroup of kernels of the Hunt process $(X_t)_{t\geq 0}$. Then for any $u \in \mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E}_{(\alpha)})$ the following holds:

$$\frac{d}{dt}\int_S \big(p_t u\big)(\mathbf{x})\,\mu(d\mathbf{x}) = \frac{d}{dt}\big(T_t u, 1\big)_{L^2(S;\mu)} = \mathcal{E}_{(\alpha)}(T_t u, 1) = 0. \tag{5.2}$$

By this, we see that

$$\int_S \big(p_t u\big)(\mathbf{x})\,\mu(d\mathbf{x}) = \int_S u(\mathbf{x})\,\mu(d\mathbf{x}), \quad \forall t \geq 0, \quad \forall u \in \mathcal{F}C_0^\infty, \tag{5.3}$$

and hence,

$$\int_S P_{\mathbf{x}}(X_t \in B)\,\mu(d\mathbf{x}) = \mu(B), \qquad \forall B \in \mathcal{B}(S). \tag{5.4}$$

Thus, we have proven the following Theorem 3.

Theorem 3 *Let $0 < \alpha \leq 1$, and let $\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ be a strictly quasi-regular Dirichlet form on $L^2(S;\mu)$ that is defined through Theorem 2. Then for $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, there exists a properly associated S-valued Hunt process (cf. Definitions IV-1.5, 1.8 and 1.13 of [M,R 92] for its precise definition) $\mathbb{M}$ defined by (5.1), the invariant measure of which is μ (cf. (5.4)).*

■

We shall now present some examples.

Consider

$$H^{-1} \equiv (|x|^2+1)^{-\frac{d+1}{2}}(-\Delta+1)^{-\frac{d+1}{2}}(|x|^2+1)^{-\frac{d+1}{2}}, \tag{5.5}$$

as a pseudo differential operators on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R}) \equiv \mathcal{S}'(\mathbb{R}^d)$, where Δ is the d-dimensional Laplace operator Δ. Let

$$\mathcal{H}_{-n}\text{be the completion of } \mathcal{S}'(\mathbb{R}^d) \text{ with respect to the norm } \|f\|_{-n}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \tag{5.6}$$

where $\|f\|_{-n}^2 = (f,f)_{-n}$ with

$$(f,g)_{-n} = ((H^{-1})^n f, (H^{-1})^n g)_{\mathcal{H}_0}, \qquad f,\, g \in \mathcal{S}(\mathbb{R}^d). \tag{5.7}$$

Now, the restriction of H^{-1} to Borel functions in $\mathcal{H}_0 = L^2(\mathbb{R}^d \to \mathbb{R})$ is a strictly positive self-adjoint operator in $L^2(\mathbb{R}^d \to \mathbb{R})$, which is a Hilbert-Schmidt operator and thus a compact operator. By *Hilbert-Schmidt theorem* (cf., e.g., Theorem VI 16, Theorem VI 22 of [Reed,Simon 80]) we have an orthonormal base (O.N.B.) of $\mathcal{H}_0$. The spectrum of H^{-1} consists of eigenvalues $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots > 0$, and we have

$$\sum_{i\in\mathbb{N}}(\lambda_i)^2 < \infty, \quad \text{i.e.,} \quad \{\lambda_i\}_{i\in\mathbb{N}} \in l^2. \tag{5.8}$$

Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be the system of normalized eigen functions corresponding to the eigenvalues λ_i, $i \in \mathbb{N}$ (adequately indexed corresponding to the finite multiplicity of each λ_i), which forms an O.N.B. of $\mathcal{H}_0$.

By the definition (5.6) and (5.7), for each $n \in \mathbb{N} \cup \{0\}$, we have that

$$\{(\lambda_i)^{-n}\varphi_i\}_{i\in\mathbb{N}} \quad \text{is an O.N.B. of } \mathcal{H}_{-n} \tag{5.9}$$

Thus, by denoting $\mathbb{Z}$ the set of integers, by the Fourier series expansion of functions in $\mathcal{H}_m$, $m \in \mathbb{Z}$ (cf. (5.6), (5.7)), such that for $f \in \mathcal{H}_m$,

$$f = \sum_{i \in \mathbb{N}} a_i (\lambda_i^m \varphi_i), \quad \text{with} \quad a_i \equiv \big(f, (\lambda_i^m \varphi_i)\big)_m = \lambda_i^{-m} (f, \varphi_i)_{L^2}, \; i \in \mathbb{N}, \tag{5.10}$$

we have an *isometric isomorphism* τ_m from $\mathcal{H}_m$ to $l^2_{(\lambda_i^{-2m})}$ defined by, for each $m \in \mathbb{Z}$

$$\tau_m \, : \, \mathcal{H}_m \ni f \longmapsto (\lambda_1^m a_1, \lambda_2^m a_2, \dots) \in l^2_{(\lambda_i^{-2m})}, \tag{5.11}$$

where $l^2_{(\lambda_i^{-2m})}$ is the weighted l^2 space defined by (2.1) with $p = 2$, and $\beta_i = \lambda_i^{-2m}$.

By making use of the results given by [Brydges,Föhlich,Sokal 83] and applying the Bochner-Minlos's Theorem the Φ_3^4 Euclidean field measure can be realized as a Borel probability measure discussed in [Brydges,Föhlich,Sokal 83] ν on $\mathcal{H}_{-3}$. We can then define a probability measure μ on $l^2_{(\lambda_i^6)}$ such that

$$\mu(B) \equiv \nu \circ \tau_{-3}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^6)}). \tag{5.12}$$

We set $S = l^2_{(\lambda_i^6)}$ in Theorems 1, 2 and 3, with the weight $\beta_i = \lambda_i^6$. We can take $\gamma_i{}^{-\frac{1}{2}} = \lambda_i^2$ in Theorem 2 with $p = 2$, then, from (5.9) we have

$$\sum_{i=1}^{\infty} \beta_i \gamma_i \cdot \mu\Big(\beta_i^{\frac{1}{2}} |X_i| > M \cdot \gamma_i^{-\frac{1}{2}}\Big) \le \sum_{i=1}^{\infty} \beta_i \gamma_i = \sum_{i=1}^{\infty} (\lambda_i)^2 < \infty \tag{5.13}$$

(5.15) shows that the condition (4.3) holds.

Thus, by Theorem 2 and Theorem 4, for each $0 < \alpha \le 1$, there exists an $l^2_{(\lambda_i^6)}$-valued Hunt process

$$\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t \ge 0}, (P_{\mathbf{x}})_{\mathbf{x} \in S_\triangle}\big), \tag{5.14}$$

associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. We can then define an $\mathcal{H}_{-3}$-valued process $(Y_t)_{t \ge 0}$ such that $(Y_t)_{t \ge 0} \equiv \big(\tau_{-2}^{-1}(X_t)\big)_{t \ge 0}$.
Equivalently, by (5.13) for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^6)}$, $P_{\mathbf{x}} - a.e.$, by setting $A_i(t)$ such that $A_i(t) = \lambda_i^3 X_i(t)$ (cf. (5.11) and (5.12)), then Y_t is given by

$$Y_t = \sum_{i \in \mathbb{N}} A_i(t) (\lambda_i^{-3} \varphi_i) = \sum_{i \in \mathbb{N}} X_i(t) \varphi_i \in \mathcal{H}_{-3}, \qquad \forall t \ge 0, \; P_{\mathbf{x}} - a.e.. \tag{5.15}$$

By (5.4) and (5.13), it is an $\mathcal{H}_{-3}$-valued Hunt process that can be looked upon a *stochastic quantization* with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\mathcal{H}_{-3}, \nu)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-3}. See [A,Kagawa,Yahagi,Y 2018] for more details.

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