数理解析研究所講究録 1357 巻 2004 年 119-123

SOME DEGENERATE UNIPOTENT BLOCKS

NAGOYA UNIVERSITY HYOHE MIYACHI (名古屋大学 宮地 兵衛)

1. PRELIMINARY

Definition 1. *Let G be a finite group of Lie type. Let A be a unipotent block ideal of G with Φ_e-defect torus T and canonical character λ in $M = Z_G(T)$. Let $W(M, \lambda)$ be the inertial group of A. We say that A is a Rouquier block if there exists a Levi subgroup L of G such that*

(i) *there exists parabolic subgroup P of G with Levi decomposition $P = LU_P$,*
(ii) *L contains M,*
(iii) *$H = L \cdot W(M, \lambda)$ is a proper subgroup of G,*
(iv) *There exists a block B of H with canonical character λ such that A is Morita equivalent to B.*

Remark 2. *By L. Puig* [Pui90], *if ℓ dose not divide the order of Weyl group W of G and dose $q-1$, then the principal block of G is Morita (Puig) equivalent to the principal block of $T.W$. In the case of type A, there is a generalization of this theorem which was conjectured by R.Rouquier* [Rou98] *in the context of symmetric groups. This generalization is intensively studied by J.Chuang, R. Kessar, K.Tan, W. Turner, A. Hida and the author* [CK02], [CT02a], [Tur02],[HM00],[Miy01] . *In this case there is also an interesting interpretation on these Rouquier blocks in terms of Fock space over quantum affine algebra of type A* [CT02b],[LM02].

One aim of this note is to report that there are Rouquier blocks in type E_6 and E_8 (see Theorem 4 and Remark 5 below) which are not included in [Pui90].

In through this note, we assume that a prime number ℓ and a prime power q satisfy the following condition:

(1) $$\text{(i) } \ell \neq 2, 3, \quad \text{(ii) } \ell \text{ divides } q^2+1, \text{ and } \quad \text{(iii) } q \text{ is odd.}$$

Remark 3. (i) *and* (ii) *are essential in this note.* (iii) *is needed to use Kawanaka's generalized Gelfand-Graev character* [GP92]. *So, this might be removed once one gets the lower unitriangularity of decomposition matrix.*

The main strategy (which I learned from T. Okuyama) is analogous to that in [KM00]. Namely, we construct a stable equivalence between two blocks by Broué's theorem [Bro92, 6.3. Theorem] and checking the assumption, and then we chase the images of simple modules. The most powerful tool in this approach is Linckelmann's theorem [Lin96]. In this note we choose a numbering of simple roots of type E_8 as follows:

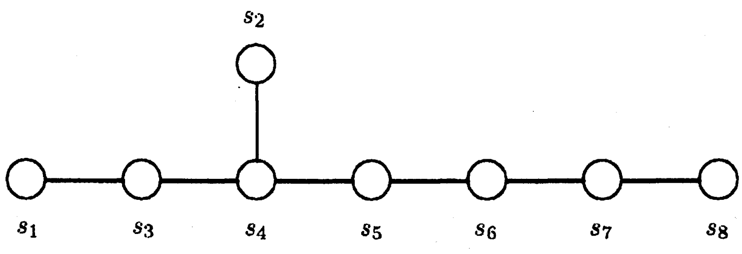

TABLE E_8

Date: 2003.

2. The statment of Main Theorem

Let $\mathbf{G}$ be the Chevalley group of type E_6 with its defining field $\overline{\mathbb{F}_q}$. Let F be its standard Frobenius map. Let $\mathbf{L}$ be the Levi subgroup corresponding to $\{s_2, s_3, s_4, s_5\}$. It is of type D_4.

Let A (resp. B) be the principal block ideal of $\mathbb{k}\mathbf{G}^F$ (resp. $\mathbb{k}\mathbf{L}^F.\mathfrak{S}_3$). Moreover, $\mathbf{L}^F.\mathfrak{S}_3$ contains $\mathcal{N}_{\mathbf{G}}^F(D)$. Hence,

(2) there is a Green correspondence $\mathbf{G}^F \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightleftarrows}} \mathbf{L}^F.\mathfrak{S}_3$

(see [Alp86].) Let $\Delta(D)$ be $\{(x,x) \mid x \in D\}$. Let $\mathbf{X}$ be the Green correspondent of an indecomposable (A,A)-bimodule A in $G \times H$ with vertex $\Delta(D)$. (In other words, $\mathbf{X}$ is the Scott (A,B)-bimodule $\mathbf{S}_{\mathbf{G}^F \times (\mathbf{L}^F.\mathfrak{S}_3)}(\Delta(D))$ with vertex $\Delta(D)$.)

Now, we can state the main result of this note as follows:

Theorem 4. *The functor* $\mathbf{X} \otimes_B -$ *induces a Morita equivalence between* A *and* B.

Remark 5. *By* [Miy03] *we know that the principal block* $B_0(\mathbb{k}\mathbf{G}^F)$ *is Morita equivalent to the unipotent block ideal of* $E_8(q)$ *with canonical character* $\phi_{23,01}$ *in the notation* [BMM93]. *Moreover, thanks to Broué's abelian defect conjecture and our main chart* [BMM93], *our main result is expected to be useful to settle Broué's abelian defect conjecture for the following unipotent* Φ_4*-blocks:*

(i) *Group* $E_7(q)$: *canonical characters* ϕ_2^3, ϕ_{11}^3,

(ii) *Group* $E_8(q)$: *canonical characters* $\phi_{3.1}, \phi_{123,013}, \phi_{12,03}$.

These will be discussed in elsewhere.

3. The decomposition matrix

Now, we recall what is known for the decomposition matrix of A without Theorem 4. Using [Gec93],[GH97], [GP92],etc, we can approximate the decomposition matrix as follows:

Lemma 6. *The decomposition matrix of the unipotent characters lying in the principal block of* $\mathbf{G}^F$ *has the following shape:*

	a	*name*	ps	ps	ps	ps	D_4	ps	ps	ps	D_4	ps	A_3			D_4	D_4	D_4
E_6	0	$\phi_{1,0}$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_1)$	1	$\phi_{6,1}$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_3)$	3	$\phi_{15,5}$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$\phi_{15,4}$	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$D_4, 1$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_4	6	$\phi_{81,6}$	.	1	1	.	$*$	1	.	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{80,7}$	.	1	.	1	$*$	1	1	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{90,8}$	.	.	1	.	$*$	1	.	1	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	D_4, r	.	.	.	.	$*$	.	.	.	1	.	.	.	.	.	.	.
$2A_2 + A_1$	7	$\phi_{10,9}$	1	.	.	1	$*$	.	.	.	$*$	1	.	.	.	.	.	.
A_3	10	$\phi_{81,10}$	.	.	.	.	$*$	1	1	1	$*$	.	1	.	.	.	.	.
A_2	15	$\phi_{15,17}$	.	.	.	.	$*$	.	.	1	$*$	.	$*$	1	.	.	.	.
$3A_1$	15	$\phi_{15,16}$	.	.	.	1	$*$	.	1	.	$*$	1	$*$	$*$	1	.	.	.
$3A_1$	15	D_4, ϵ	.	.	.	.	$*$	.	.	.	$*$	.	$*$	$*$	$*$	1	.	.
A_1	25	$\phi_{6,25}$	.	.	.	.	$*$	.	1	.	$*$	.	$*$	$*$	$*$	$*$	1	.
1	36	$\phi_{1,36}$	.	.	.	.	$*$	.	.	.	$*$	1	$*$	$*$	$*$	$*$	$*$	1

Here, $*$ *means an unknown non-negative integer.*

I would like to remove these unknown parameters $*$'s in Lemma 6 as possible. Using Theorem 4, Lemma 6 and the precise correspondence on the simple modules over A and B, we get the following new result:

Theorem 7. *The decomposition matrix of the unipotent characters lying in the principal block of* $\mathbf{G}^F$ *is given as follows:*

	a	*name*	ps	ps	ps	ps	D_4	ps	ps	ps	D_4	ps	A_3	D_4	A_3	D_4	D_4	D_4
E_6	0	$\phi_{1,0}$	1	.	.	.	.	.	.		.	.	.	.	.	.	.	.
$E_6(a_1)$	1	$\phi_{6,1}$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_3)$	3	$\phi_{15,5}$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$\phi_{15,4}$	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$D_4, 1$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_4	6	$\phi_{81,6}$	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{80,7}$	.	1	.	1	.	1	1	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{90,8}$	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	D_4, r	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$2A_2 + A_1$	7	$\phi_{10,9}$	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.
A_3	10	$\phi_{81,10}$	.	.	.	.	.	1	1	1	.	.	1	.	.	.	.	.
A_2	15	$\phi_{15,17}$	.	.	.	.	*	.	.	1	.	.	1	1	.	.	.	.
$3A_1$	15	$\phi_{15,16}$	.	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.
$3A_1$	15	D_4, ϵ	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
A_1	25	$\phi_{6,25}$	.	.	.	.	.	.	1	.	*	.	1	.	1	.	1	.
1	36	$\phi_{1,36}$	.	.	.	.	.	.	.	.	.	1	.	.	1	*	.	1

Here, $*$ *is equal to the decomposition number* $d_{(.1111),\varphi}$ *in type* D_4.

4. Remarks on Hecke algebras

By Theorem 4, we can know that

Theorem 8. *Let* F *be a field with an invertible element* q. *We assume that*

(i) *The characteristic of* F *is not* 2, 3.
(ii) $q^4 = 1, q^2 \neq 1$.
(iii) F *contains* $q^{\frac{1}{2}}$.
(iv) *If the characteristic of* F *is positive, then,* q *lie in the prime field of* F.

Then, $B_0(\mathcal{H}_{F,q}(E_6))$ *and* $B_0(\mathcal{H}_{F,q}(D_4)).\mathfrak{S}_3$ *are Morita equivalent.*

We can construct $B_0(\mathcal{H}_{F,q}(D_4)).\mathfrak{S}_3$ as a block ideal of a well-known Iwahori-Hecke algebra in the following way. $\mathcal{H}_{F,q}(D_4)$ is a q-deformation of the group algebra of Weyl group $W(D_4)$ of type D_4. And, in our situation, $W(D_4).\mathfrak{S}_3$ is nothing but the Weyl group $W(F_4)$ of type F_4. Moreover, $W(D_4)$ is realized as a reflection subgroup of $W(F_4)$ generated by all the reflections of $W(F_4)$ whose roots are long. So, let us recall the definition of Iwahori-Hecke algebra of type F_4.

Definition 9. *Let* R *be an integral domain with invertible elements* $u^{\frac{1}{2}}, v^{\frac{1}{2}}$. *The Iwahori-Hecke algebra* $\mathcal{H}_{R,u,v}(F_4)$ *over* R *with parameter* u, v *is an associative algebra with generators* T_1, T_2, T_3, T_4 *and relations*

$$(T_i - u)(T_i + 1) = 0 \text{ for } i = 1, 2, (T_j - v)(T_j + 1) = 0 \text{ for } j = 3, 4,$$
$$T_i T_i T_i = T_{i+1} T_i T_{i+1} \text{ for } i = 1, 3,$$
$$T_2 T_3 T_2 T_3 = T_3 T_2 T_3 T_2,$$
$$T_i T_j = T_j T_i \text{ for } 1 \leq i < j - 1 \leq 3.$$

From now on, we consider the Iwahori-Hecke algebra $\mathcal{H}_{\Bbbk,q,1}(F_4)$ of type F_4 with parameter q and 1. Put a_1, a_2, a_3, a_4 respectively to be

$$a_1 = T_2, a_2 = T_1, a_3 = T_3 T_2 T_3, a_4 = T_4 T_3 T_2 T_3 T_4.$$

$\mathcal{H}_{\Bbbk,q}(D_4)$ is isomorphic to the subalgebra $\mathcal{H}'$ of $\mathcal{H}_{\Bbbk,q,1}(F_4)$ generated by a_1, a_2, a_3, a_4. One can easily check that a_i's satisfy the quodratic relations and braid relations. Moreover, clearly, $\Bbbk\langle T_3, T_4 \rangle$ is isomorphic to the group algebra $\Bbbk\mathfrak{S}_3$ since $T_3^2 = 1 = T_4^2$. By definition, the action of T_3 and T_4 on $\mathcal{H}'$ is also clear. Since the principal block idempotent of $\mathcal{H}_{\Bbbk,q}(D_4)$ is normalized by $\Bbbk\langle T_3, T_4 \rangle$, it is lifted to the idempotent of the whole algebra $\mathcal{H}_{k,q,1}(F_4)$. The decomposition matrix of $\mathcal{H}_{k,q,1}(F_4)$ is first calculated by Bremke [Bre94, p.342]. So, it is worth saying the correspondences among characters, simple modules, PIM's, etc over $B_0(\mathcal{H}_{k,q}(E_6))$ and $B_0(\mathcal{H}_{k,q,1}(F_4))$. The correspondence is given as follows:

E_6										F_4								
$\phi_{1,0}$	1	.	.	.	.	.	.	.		$\phi_{1,0}$	1	.	.	.	.	.	.	.
$\phi_{6,1}$	.	1	.	.	.	.	.	.		$\phi''_{2,4}$	.	1	.	.	.	.	.	.
$\phi_{15,5}$	.	.	1	.	.	.	.	.		$\phi''_{1,12}$	.	.	1	.	.	.	.	.
$\phi_{15,4}$	1	1	.	1	.	.	.	.		$\phi_{9,2}$	1	1	.	1	.	.	.	.
$\phi_{81,6}$	.	1	1	.	1	.	.	.		$\phi''_{9,6}$	.	1	1	.	1	.	.	.
$\phi_{90,8}$	.	.	1	.	1	1	.	.		$\phi''_{8,9}$	.	.	1	.	1	1	.	.
$\phi_{80,7}$	.	1	.	1	1	.	1	.	$\leftrightarrow$	$\phi_{16,5}$	.	1	.	1	1	.	1	.
$\phi_{10,9}$	1	.	.	1	.	.	.	1		$\phi'_{8,3}$	1	.	.	1	.	.	.	1
$\phi_{81,10}$	.	.	.	.	1	1	1	.		$\phi_{9,10}$	.	.	.	.	1	1	1	.
$\phi_{15,16}$	.	.	.	1	.	.	1	1		$\phi'_{9,6}$	.	.	.	1	.	.	1	1
$\phi_{15,17}$	.	.	.	.	.	1	.	.		$\phi_{1,24}$	.	.	.	.	.	1	.	.
$\phi_{6,25}$	.	.	.	.	.	.	1	.		$\phi'_{2,16}$	.	.	.	.	.	.	1	.
$\phi_{1,36}$	.	.	.	.	.	.	.	1		$\phi'_{1,12}$	.	.	.	.	.	.	.	1

References

[Alp76] J. L. Alperin. Isomorphic blocks. *J. Algebra*, 43:694–698, 1976.

[Alp86] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[BM89] M. Broué and J. Michel. Blocs et séris de Lusztig dans un groupe réductif fini. *J. Reine Angew. Math.*, 395:56–67, 1989.

[BMM93] Michel Broué, Gunter Malle, and Jean Michel. Generic blocks of finite reductive groups. *Astérisque*, *212:7–92, 1993.

[Bre94] K. Bremke. The decomposition numbers of Hecke algebras of type F_4 with unequal parameters. *Manuscripta Math.*, 83(3-4):331–346, 1994.

[Bro90] M. Broué. Isométries caractères et équivalences de Morita ou dérivées. *Inst. Haustes Études Sci. Publ. Math.*, 71:45–63, 1990.

[Bro92] M. Broué. Equivalences of blocks of group algebras. In V.Dlab and L.L.Scott, editors, *In Proceedings of the International Conference on Representations of Algebras*, Finite Dimensional Algebra and Related Topics, pages 1–26, Ottawa, Aug 1992. Kluwer Academic Publishers.

[Car72] R. W. Carter. *Simple Groups of Lie Type*. John Wiley, London, 1972.

[Car85] R. W. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*. John Wiley, New York, 1985.

[CK02] J. Chuang and R. Kessar. Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture. *Bull. London Math. Soc.*, 34:174–184, 2002.

[CT02a] J. Chuang and K.M. Tan. Filtrations in Rouquier blocks of symmetric groups and Schur algebras. *preprint*, 2002.

[CT02b] J. Chuang and K.M. Tan. Some canonical basis vectors in the basic $U_q(\widehat{\mathfrak{sl}}_n)$-modules. *J. Algebra*, 248(2):765–779, 2002.

[Dad77] E. C. Dade. Remarks on isomorphic blocks. *J. Algebra*, 45:254–258, 1977.

[DF91] D. I. Deriziotis and A.P. Fakiolas. The maximal tori in the finite Chevalley groups of type E_6, E_7 and E_8. *Comm. Algebra*, 19(3):889–903, 1991.

[Dip90] R. Dipper. On quotients of Hom-functors and representations of finite general linear groups I. *J. Algebra*, 130:235–259, 1990.

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite groups. *Ann. of Math.*, 103:103–161, 1976.

[Erd95] K. Erdmann. On Auslander-Reiten components for group algebras. *J. Pure Appl. Algebra*, 104:149–160, 1995.

[FJ93] P. Fleischmann and I. Janiszczak. The semisimple conjugacy classes of finite groups of Lie type E_6 and E_7. *Comm. Algebra*, 21(1):93–161, 1993.

[FS80] P. Fong and B. Srinivasan. Blocks with cyclic defect groups in $GL(n,q)$. *Bull. Amer. Math. Soc.*, 3:1041–1044, 1980.

[FS82] P. Fong and B. Srinivasan. The blocks of finite general linear and unitary groups. *Invent. math.*, 69:109–153, 1982.

[FS90] P. Fong and B. Srinivasan. Brauer trees in classical groups. *J. Algebra*, 131:179–225, 1990.

[Gec92] M. Geck. On the classification of ℓ-blocks of finite groups of Lie type. *J. Algebra*, 151:180–191, 1992.

[Gec93] M. Geck. The decomposition numbers of the Hecke algebra of type E_6^*. *Math. Comp.*, 61:889–899, 1993.

[Gec98] M. Geck. Kazhdan-Lusztig cells and decomposition numbers. *Represent. theory*, 2:264–277, 1998.

[GH97] M. Geck and G. Hiss. Modular representations of finite groups of Lie type in non-defining characteristic. In M. Cabanes, editor, *Finite reductive groups: Related structures and representations*, volume 141 of *Progress in Mathematics*, pages 195–249. Birkhäuser,Basel, 1997.

[GHM94] M. Geck, G. Hiss, and G. Malle. Cuspidal unipotent Brauer characters. *J. Algebra*, 168:182–220, 1994.

[GP92] M. Geck and G. Pfeiffer. Unipotent characters of the Chevalley groups $D_4(q)$, q odd. *Manuscripta Math.*, 76:281–304, 1992.

[His93] G. Hiss. Harish-Chandra series of Brauer characters in a finite groups with split BN-pair. *J. London Math. Soc.*, 48:219–228, 1993.

[HM00] Akihiko Hida and Hyohe Miyachi. Module correspondences in some blocks of finite general linear groups. *preprint*, 2000.

[Hum86] J. F. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[Kaw97] S. Kawata. On the Auslander-Reiten components and simple modules for finite group algebras. *Osaka J. Math.*, 34:681–688, 1997.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.

[Kle87] P. B. Kleidman. The maximal subgroups of the 8-dimentional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups. *J. Algebra*, 110:173–242, 1987.

[Kle88] P. B. Kleidman. The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups. *J. Algebra*, 115:182–199, 1988.

[KM00] S. Koshitani and H. Miyachi. The principal 3-blocks of four- and five-dimensional projective special linear groups in non-defining characteristic. *J. Algebra*, 226:788–806, 2000.

[Kun00] N. Kunugi. Morita equivalent 3-blocks of the 3-dimensional projective special linear groups. *Proc. London Math. Soc.*, 80(3), 2000.

[Lin96] M. Linckelmann. Stable equivalences of Morita type for self-injective algebras and p-groups. *Math.Z.*, 223:87–100., 1996.

[LM02] B. Leclerc and H. Miyachi. Some closed formulas for canonical bases of Fock space. *Represent. Theory (electronic)*, 6:290–312, 2002.

[Lus84] G. Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Math. Studies*. Princeton University Press, 1984.

[Mar96] A. Marcus. On equivalences between blocks of group algebras: reduction to the simple components. *J. Algebra*, 184:372–389, 1996.

[Miy01] H. Miyachi. *Unipotent Blocks of Finite General Linear Groups in Non-defining Characteristic*. PhD thesis, Chiba univ., March 2001.

[Miy03] H. Miyachi. A Scopes equivalence in type E. *preprint*, 2003.

[Miz77] K. Mizuno. The conjugate classes of Chevalley groups of type E_6. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24:525–563, 1977.

[NT90] H. Nagao and Y. Tsushima. *Representations of Finite Groups*. Academic Press, New York, 1990.

[Oku87] T. Okuyama. On the Auslander-Reiten quiver of a finite group. *J. Algebra*, 110:425–430, 1987.

[Pui90] L. Puig. Algèbres de source de certains blocs des groupes de Chevalley. *Astérisque*, 181-182:221–236, 1990.

[Ric96] J. Rickard. Splendid equivalences: derived categories and permutaion modules. *Proc. London Math. Soc*, 72(3):331–358, 1996.

[Rou98] R. Rouquier. Représentations et catégories dérivées. *Rapport d'habilitation*, 1998.

[Sch85] Klaus-Dieter Schewe. *Blöcke exzeptioneller Chevalley-Gruppen. (German)*. PhD thesis, Rheinische Friedrich-Wilhelm-Universität, Bonn, Bonner Mathematische Schriften, Universität Bonn,Mathematisches Institut, 1985.

[Sco73] L. Scott. Modular permutation representations. *Trans. Amer. Math. Soc.*, 175:101–121, 1973.

[SS70] T.A. Springer and R. Steinberg. *Conjugacy classes*, volume 131 of *Lecture Notes in Math.*, chapter E, pages 167–247. Springer-Verlag,Berlin, 1970.

[Tur02] W. Turner. Equivalent blocks of finite general linear groups in non-describing characteristic. *J. Algebra*, 247(1):244–267, 2002.

[Wak93] K. Waki. The Loewy structure of the projective indecomposable modules of the Mathieu groups in characteristic 3. *Comm. Algebra*, 21(5):1457–1485, 1993.

[Web82] P. Webb. The Auslander-Reiten quiver of a finte group. *Math. Z*, 179:97–121, 1982.