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Some results on the Möbius gyrovector space

メビウスジャイロベクトル空間に関するいくつかの結果

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アブストラクト. Hilbert 空間の間の線形作用素に対応する Möbius ジャイロベクトル空間の間の写像について考察する. また, Möbius の演算に関連する Cauchy 型の不等式, Möbius の和に関する級数の収束条件, さらに Möbius の和に関連する実または複素内積空間における Cauchy-Bunyakovsky-Schwarz 型の不等式についても述べる.

1 Möbius gyrovector space

Definition.(Ungar [U]) Let $\mathbb{V} = (\mathbb{V}, \cdot)$ be a real inner product space with a positive definite inner product $\cdot$ and let $\mathbb{V}_s$ be the open ball

$$\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V} : \ ||\boldsymbol{a}|| < s\}$$

for any fixed $s > 0$. Then the Möbius addition $\oplus_{\mathrm{M}}$ and the Möbius scalar multiplication $\otimes_{\mathrm{M}}$ are given by the equations

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes_{\mathrm{M}} \boldsymbol{a} = s\,\tanh\left(r\tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \quad r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$$

for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$. The operations $\oplus_{\mathrm{M}}$, $\otimes_{\mathrm{M}}$ in the interval $(-s, s)$ are defined by

$$a \oplus_{\mathrm{M}} b = \frac{a + b}{1 + \frac{1}{s^2}ab}$$

$$r \otimes_{\mathrm{M}} a = s\tanh\left(r\tanh^{-1}\frac{a}{s}\right)$$

for $a, b \in (-s, s)$, $r \in \mathbb{R}$.

Then, $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ satisfies the axioms of gyrovector space.

We simply denote $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ as $\oplus, \otimes$, respectively.

We use $\oplus_s, \otimes_s$ also, if we want to indicate the parameter s.

If several kinds of operations appear in a formula simultaneously, we always give priority by the following order (i) ordinary scalar multiplication (ii) gyroscalar multiplication $\otimes$ (iii) gyroaddition $\oplus$, that is,

$$r_1 \otimes w_1 \boldsymbol{a}_1 \oplus r_2 \otimes w_2 \boldsymbol{a}_2 = \{r_1 \otimes (w_1 \boldsymbol{a}_1)\} \oplus \{r_2 \otimes (w_2 \boldsymbol{a}_2)\},$$

and parentheses are omitted in such cases.

Gyro-operations are generally not commutative, associative or distributive:

$$\boldsymbol{a} \oplus \boldsymbol{b} \neq \boldsymbol{b} \oplus \boldsymbol{a}$$

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) \neq (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c}$$

$$r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) \neq r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}$$

$$t(\boldsymbol{a} \oplus \boldsymbol{b}) \neq t\boldsymbol{a} \oplus t\boldsymbol{b}.$$

Proposition.(Ungar)

$$\boldsymbol{a} \oplus_s \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b} \quad (s \to \infty)$$

$$r \otimes_s \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty).$$

2 On mappings between Möbius gyrovector spaces corresponding to Hilbert space operators

T. Abe raised the following problem in an oral presentation [A].

Problem. What are mappings between gyrolinear spaces corresponding to linear mappings between linear spaces ?

The author thinks that the following two results (due to the author) will not provide any satisfactory answer to the problem above.

Theorem. Let $\mathbb{V}$ be a real Hilbert space with $\dim \mathbb{V} \geq 2$. If $f : \mathbb{V}_1 \to (-1, 1)$ satisfies that

$$f(\boldsymbol{x} \oplus \boldsymbol{y}) = f(\boldsymbol{x}) \oplus f(\boldsymbol{y})$$
$$f(r \otimes \boldsymbol{x}) = r \otimes f(\boldsymbol{x})$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}_1$, $r \in \mathbb{R}$, then we have $f \equiv 0$.

Theorem. Let $\mathbb{V}$ be a real Hilbert space. If $f : \mathbb{V} \to \mathbb{R}$ is a continuous map and satisfies that

$$f(\boldsymbol{x} \oplus_s \boldsymbol{y}) - \{f(\boldsymbol{x}) \oplus_s f(\boldsymbol{y})\} \to 0 \ \ (s \to \infty)$$
$$f(r \otimes_s \boldsymbol{x}) - r \otimes_s f(\boldsymbol{x}) \to 0 \ \ (s \to \infty)$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$, $r \in \mathbb{R}$, then there exists a unique $\boldsymbol{c} \in \mathbb{V}$ such that $f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{c}$ $(\boldsymbol{x} \in \mathbb{V})$. The converse is also true.

Theorem.(Molnár and Virosztek, 2015) Let $\beta : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ be a continuous map. We have β is an algebraic endomorphism with respect to the operation $\oplus_{\mathrm{E}}$, i.e., β satisfies

$$\beta(\boldsymbol{u} \oplus_{\mathrm{E}} \boldsymbol{v}) = \beta(\boldsymbol{u}) \oplus_{\mathrm{E}} \beta(\boldsymbol{v}) \quad (\boldsymbol{u},\, \boldsymbol{v} \in \mathbb{R}^3_1)$$

if and only if either (i) or (ii) of the following holds:
(i) there is an orthogonal matrix $O \in \mathrm{M}_3(\mathbb{R})$ such that

$$\beta(\boldsymbol{v}) = O\boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^3_1$$

(ii)
$$\beta(\boldsymbol{v}) = \boldsymbol{0}, \quad \boldsymbol{v} \in \mathbb{R}^3_1.$$

Theorem.(Frenkel, 2016) For $n \geq 2$, continuous endomorphisms of the Einstein gyrogroup $(\mathbb{R}^n_1, \oplus_{\mathrm{E}})$ are precisely the restrictions to $\mathbb{R}^n_1$ of the orthogonal transformations of $\mathbb{R}^n$ and 0-map.

• An attempt to formulate a class of mappings between Möbius gyrovector spaces that is corresponding to Hilbert space operators.

Definition. We denote by $\mathrm{M}_{n,m}(\mathbb{R})$ the set of all $n \times m$ matrices whose entries are real numbers.

The ordinary operation of matrices on vectors:

$$A\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

$$A : x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \mapsto (a_{11}x_1 + a_{12}x_2)\boldsymbol{f}_1 + (a_{21}x_1 + a_{22}x_2)\boldsymbol{f}_2.$$

For simplicity, let us assume that $\mathbb{U}$ and $\mathbb{V}$ are finite dimensional real inner product spaces. Suppose that $\{\boldsymbol{e}_\imath\}_{i=1}^m$ (resp. $\{\boldsymbol{f}_\jmath\}_{j=1}^n$) be an orthonormal basis in $\mathbb{U}$ (resp. $\mathbb{V}$). For an arbitrary element $\boldsymbol{x} \in \mathbb{U}_1$, we can apply the orthogonal gyroexpansion (see [W2]) to get a unique m-tuple $(r_1, \cdots, r_m)$ of real numbers such that

$$\boldsymbol{x} = r_1 \otimes_1 \frac{\boldsymbol{e}_1}{2} \oplus_1 \cdots \oplus_1 r_m \otimes_1 \frac{\boldsymbol{e}_m}{2}.$$

Then we can define a map $f : \mathbb{U}_1 \to \mathbb{V}_1$ by the equation

$$f(\boldsymbol{x}) = (a_{11}r_1 + \cdots + a_{1m}r_m) \otimes_1 \frac{\boldsymbol{f}_1}{2} \oplus_1$$

$$\cdots \oplus_1 (a_{n1}r_1 + \cdots + a_{nm}r_m) \otimes_1 \frac{\boldsymbol{f}_n}{2}.$$

We say that f is the induced map from the matrix A.

For any map $f : \mathbb{U}_1 \to \mathbb{V}_1$ and any $s > 0$, we can define a map $f_s : \mathbb{U}_s \to \mathbb{V}_s$ by the equation

$$f_s(\boldsymbol{x}) = sf\left(\frac{\boldsymbol{x}}{s}\right) \quad (\boldsymbol{x} \in \mathbb{U}_s). \tag{1}$$

Theorem. Let $\mathbb{U}$ and $\mathbb{V}$ be two real Hilbert spaces and $A \in \mathrm{M}_{n,m}(\mathbb{R})$. If $f : (\mathbb{U}_1, \oplus_1, \otimes_1) \to (\mathbb{V}_1, \oplus_1, \otimes_1)$ is the induced map from the matrix A and f_s is defined by (1), then, for arbitrary $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{U}$ and $r \in \mathbb{R}$, we have

$$f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to A(\boldsymbol{x} + \boldsymbol{y})$$

$$f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) \to A\boldsymbol{x} + A\boldsymbol{y}$$

$$f_s(r \otimes_s \boldsymbol{x}) \to Ar\boldsymbol{x}$$

$$r \otimes_s f_s(\boldsymbol{x}) \to rA\boldsymbol{x}$$

as $s \to \infty$.

Theorem. Let $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$ be real Hilbert spaces. Let $A = (a_{\imath\jmath}) \in \mathrm{M}_{n,m}(\mathbb{R})$, $B = (b_{\imath\jmath}) \in \mathrm{M}_{p,n}(\mathbb{R})$. Suppose that $\{\boldsymbol{e}_\imath\}_{i=1}^m$, $\{\boldsymbol{f}_\jmath\}_{j=1}^n$, $\{\boldsymbol{g}_k\}_{k=1}^p$ be an orthonormal basis in $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$, respectively. Let f (resp. g) be the induced map from matrix A (resp. B). Then the composed map $g \circ f$ is also an induced map from the matrix BA.

Theorem. Let f be the induced map from a matrix A and f^* be the induced map from the adjoint matrix A^*. Then

$$-f_s^*(\boldsymbol{x}) \boldsymbol{\cdot} \boldsymbol{y} \oplus_s \boldsymbol{x} \boldsymbol{\cdot} f_s(\boldsymbol{y}) \to 0 \quad (s \to \infty).$$

Definition. Let $\mathbb{U}$, $\mathbb{V}$ be two real Hilbert spaces. A mapping $f : \mathbb{U}_1 \to \mathbb{V}_1$ is said to be *quasi gyrolinear operator* if there exists an linear operator $T : \mathbb{U} \to \mathbb{V}$ such that, for any $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{U}$ and $r \in \mathbb{R}$,

$$f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to T(\boldsymbol{x} + \boldsymbol{y})$$

$$f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) \to T\boldsymbol{x} + T\boldsymbol{y}$$

$$f_s(r \otimes_s \boldsymbol{x}) \to Tr\boldsymbol{x}$$

$$r \otimes_s f_s(\boldsymbol{x}) \to rT\boldsymbol{x}$$

as $s \to \infty$. It might be necessary to impose some conditions on order of convergence, such as $o(s^\alpha), O(s^\alpha)$, in some contexts.

3 Some inequalities

Properties of operations $\oplus$ and $\otimes$ on the open interval $(-1, 1)$. For $a, b, c \in (-1, 1)$, $r, r_1, r_2 \in \mathbb{R}$,

(1) $a \oplus b = b \oplus a$

(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(3) $0 \oplus a = a \oplus 0 = a, \qquad (-a) \oplus a = a \oplus (-a) = 0$

(4) $1 \otimes a = a$

(5) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$

(6) $r \otimes (a \oplus b) = r \otimes a \oplus r \otimes b$

(7) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a.$

It seems not appropriate to say "$((-1, 1), \oplus, \otimes)$ is a one-dimensional real linear space".

Theorem.(Cauchy, 1821) If x_j, y_j are real numbers, then

$$x_1y_1 + \cdots + x_ny_n \le \left(x_1{}^2 + \cdots + x_n{}^2\right)^{\frac{1}{2}} \left(y_1{}^2 + \cdots + y_n{}^2\right)^{\frac{1}{2}}.$$

Theorem. If $r_1, \cdots, r_n \ge 0$ and $0 \le x_1, \cdots, x_n < 1$, then we have

$$r_1 \otimes x_1 \oplus \cdots \oplus r_n \otimes x_n \le \left(r_1{}^2 + \cdots + r_n{}^2\right)^{\frac{1}{2}} \otimes \left(x_1{}^2 \oplus \cdots \oplus x_n{}^2\right)^{\frac{1}{2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $r_j = 0 \ (j = 1, \cdots, n)$

(ii) $x_j = 0 \ (j = 1, \cdots, n)$

(iii) $r_j = x_j = 0$ except for precisely one j.

Theorem. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence satisfying $0 \le x_n < 1$ $(n = 1, 2, \cdots)$. Then, the series $x_1 \oplus x_2 \oplus \cdots \oplus x_n \oplus \cdots$ converges in the open interval $(-1, 1)$ if and onlly if the series $\sum_{n=1}^{\infty} x_n < \infty$ in the ordinary sense.

Example. $x_n = \dfrac{1}{2n}$. Then $x_1 \oplus \cdots \oplus x_n = 1 - \dfrac{1}{n+1}$.

Theorem.(Cauchy, Bunyakovsky 1859, Schwarz 1885) Let $\mathbb{V}$ be an inner product space. Then

$$|\langle u, v\rangle| \le ||u|| ||v|| \quad (u, v \in \mathbb{V}).$$

The following theorem is a CBS type inequality related to the Möbius addition in complex inner product spaces. One can get the classical CBS inequality by putting $v = 0$.

Theorem. Let $\mathbb{V}$ be a complex inner product space and let $w \in \mathbb{V}$ be a fixed element with $||w|| \le 1$. For any $u, v \in \mathbb{V}$ and for any $s > \max\{||u||, ||v||\}$, the following inequality holds

$$\left| \frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \frac{1}{s^2}\overline{\langle u, w\rangle}\langle v, w\rangle} \right| \le \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $u = v$

(ii) $||w|| = 1$ and $u = \lambda w, v = \mu w$ for some complex numbers λ, μ.

We state the following theorem for the real inner product spaces, showing relation between the Möbius addition (subtraction) and inner product.

Theorem. Let $\mathbb{V}$ be a real inner product space and let $w \in \mathbb{V}$ be a fixed element with $||w|| \leq 1$. For any u, $v \in \mathbb{V}$ and for any $s > \max\{||u||, ||v||\}$, the following inequality holds

$$|\langle u, w\rangle \ominus_s \langle v, w\rangle| \leq ||u \ominus_s v||.$$

That is,

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \frac{1}{s^2}\langle u, w\rangle\langle v, w\rangle}\right| \leq \sqrt{\frac{||u||^2 - 2\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $u = v$

(ii) $u = \lambda w$ and $v = \mu w$ for some real numbers λ, μ.

Remark. If $||w|| \leq 1$, then the CBS inequality trivially shows that

$$|\langle u, w\rangle - \langle v, w\rangle| = |\langle u - v, w\rangle| \leq ||u - v||. \qquad (2)$$

On the contrary, note that generally

$$\langle u, w\rangle \ominus_s \langle v, w\rangle \neq \langle u \ominus_s v, w\rangle.$$

So we cannot prove Theorem as (2).

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