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Mekler's Construction on NTP_1 Theories

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Abstract

For any model M of T, one can find a graph model which is bi-interpretable with it. Extending this folklore, Mekler developed a construction method to get a group preserving many model-theoretic properties of the original one, such as stability, which Mekler himself proved. Recently, Chernikov and Hempel found another result for the case of NTP_2 by using array lemmas on mutually indiscernible sequences. Following their results, we will show that Mekler's constrction preserves NTP_1 by using tree lemmas on strongly indiscernible trees.

1 Introduction

Mekler's construction is the way to construct a pure nil-2 group $G(C)$ of exponent p from any given nice graph C. It was proved that the construction preserves λ-stability for any $\lambda > \omega$, that is, for any infinite nice graph C, $\mathrm{Th}(C)$ is λ-stable if and only if $\mathrm{Th}(G(C))$ is λ-stable [6]. Hence, by the construction, we can find a example of pure group which is stable but not superstable from given nice graph with the same property.

It was later found that more model-theoretic properties are preserved by Mekler's construction; simplicity, NIP, and NTP_2 [1, 3]. Thus we naturally expect that the other generalized stability like NTP_1 could be preserved by the construction, which is our goal in this paper. To prove it, We will follow the argument in [3].

In section 2, we introduce the notions about strong indiscerniblility on trees. Then, we find equivalent conditions of NTP_1. In section 3, we describe Mekler's construction following by [3] and [4]. In section 4, we sketch the proof that the Mekler's construction preserves NTP_1.

2 Tree property of the first kind

Consider a tree ${}^{<\lambda}\kappa$ of height λ which has κ many branches. We denote $\langle\rangle$ as an empty string, 0^α as a string of α many zeros, and α as a string $\langle\alpha\rangle$ of length one.

Definition 2.1. Let $\eta, \nu, \xi \in {}^{<\lambda}\kappa$.

(1) (Ordering) $\eta \lhd \nu$ if $\nu\lceil\alpha = \eta$ for some ordinal $\alpha \in \mathrm{dom}(\nu)$.

(2) (Meet) $\xi = \eta\wedge\nu$ if ξ is the meet of η and ν, i.e., $\xi = \eta\lceil\beta$, when $\beta = \bigcup\{\alpha \leq \mathrm{dom}(\eta)\cap\mathrm{dom}(\nu) \mid \eta\lceil\alpha = \nu\lceil\alpha\}$. For $\bar{\eta} \in {}^{<\lambda}\kappa$, $\bar{\nu}$ is the meet closure of $\bar{\eta}$ if $\bar{\nu} = \{\eta_1 \wedge \eta_2 | \eta_1, \eta_2 \in \bar{\eta}\}$

(3) (Incomparability) $\eta\perp\nu$ if they are $\unlhd$-incomparable, i.e., $\neg(\eta \unlhd \nu)$ and $\neg(\nu \unlhd \eta)$.

(4) (Lexicographic order) $\eta <_{lex} \nu$ if

(a) $\eta \lhd \nu$, or

(b) $\eta \perp \nu$ and for ordinal $\alpha = \text{dom}(\eta \wedge \nu)$, $\eta(\alpha) < \nu(\alpha)$.

Tree language is a collection of symbols in 2.1 or more with appropriate interpretations, which is introduced and studied on [5, 7, 8]. We use one of the tree language called strong language from [8].

Definition 2.2. A *strong language* L_0 is defined by the collection $\{\lhd, \wedge, <_{lex}\}$.
We may view the tree ${}^{<\lambda}\kappa$ as an L_0-structure.

Fix a complete first order theory T (with language L). Let $\mathfrak{C} \models T$ be a monster model. From now on, we will work in this $\mathfrak{C}$.

Definition 2.3. Let L_0-structure ${}^{<\lambda}\kappa$ be an index structure. For a tree $(b_\eta | \eta \in {}^{<\lambda}\kappa)$ in $\mathfrak{C}$, we say it is *strongly indiscernible* if for any finite tuple $\bar{\eta}$ and $\bar{\nu}$ in ${}^{<\lambda}\kappa$,

$$\text{qftp}_{L_0}(\bar{\eta}) = \text{qftp}_{L_0}(\bar{\nu}) \Rightarrow (b_\eta)_{\eta \in \bar{\eta}} \equiv_{\mathfrak{C}} (b_\nu)_{\nu \in \bar{\nu}}.$$

The strong indiscernibility is a generalized form of indiscernibility. It satisfies many properties proved on indiscernibility. For instance, we can produce an strongly indiscernible tree based on any given tree. This is so called the modeling property [5, 8].

Definition 2.4. Let $\phi(x, y)$ be a formula in T. Fix an integer $k > 1$. We say $\phi(x, y)$ has *the tree property of the first kind(TP$_1$)* if there is a tree $(a_\eta)_{\eta \in {}^{<\omega}\omega}$ such that

(1) For all $\eta \in \omega^\omega$, $\{\phi(x, a_{\eta\lceil\alpha}) \mid \alpha < \omega\}$ is consistent, and

(2) For all $\eta \perp \nu \in \omega^{<\omega}$, $\{\phi(x, a_\eta), \phi(x, a_\nu)\}$ is inconsistent.

We say T has TP$_1$ if it has a TP$_1$ formula. We say T is NTP$_1$ if it does not have TP$_1$.

Example 2.5. Th$((\mathbb{Q}, <))$ has TP$_1$. Choose $\phi(x, y_1y_2) := y_1 < x < y_2$. Since it is a theory of dense linear ordering, ϕ has TP$_1$.

We apply the notion of strong indiscernibility to the NTP$_1$ theories.

Proposition 2.6. *Let κ be a sufficiently large regular cardinal. For a given complete theory T, TFAE.*

(1) T is NTP$_1$

(2) For any strongly indiscernible tree $(a_\eta \mid \eta \in {}^{<\kappa}\kappa)$ and for any finite tuple b, there is some b' and a sequence $(a'_i \mid i < \omega)$ such that

(a) There is β, $0 < \beta < \kappa$, such that $a'_i = a_{0^\beta \frown i}$ for each $i < \omega$,

(b) $tp(b/a'_0) = tp(b'/a'_0)$,

(c) $(a'_i \mid i < \omega)$ is indiscernible over b'.

We can prove 2.6 following the same argument in [2], substituting the role of mutually indiscernible sequences to strongly indiscernible trees.

3 Mekler's construction

We follow the definitions and facts from [3] and [4].

For a graph A and its vertices a and b, $R(a, b)$ means that a and b are connected by a single edge in A.

Definition 3.1. A graph A which has at least two vertices is called *nice* if

(a) For any two distinct vertices a and b, there is some vertex c different from a and b such that $R(a, c) \wedge \neg R(b, c)$

(b) There are no triangles nor squares.

Note that for any structure, there is a nice graph which is bi-interpretable with it.

Definition 3.2. Fix an odd prime p. For a nice graph A, let $F(A)$ be the free nilpotent group of class 2 and exponent p that is generated freely by the vertices of A. Then the *Mekler group* of A, denoted by $G(A)$, is defined as follows;

$$G(A) = F(A)/\langle\{[a, b] \,|\, a, b \in A,\ A \models R(a, b)\}\rangle$$

Fact 3.3. Let A be a nice graph. Then there is an interpretation Γ such that for any model G of $\mathrm{Th}(G(A))$, $\Gamma(G) \models \mathrm{Th}(A)$

Fact 3.4. Let C be an infinite nice graph and G be a model of $\mathrm{Th}(G(C))$. Then there exists some subsets of G, say transversals $X^\nu, X^p, X^\iota, X = X^\nu \cup X^p \cup X^\iota$ such that G is isomorphic to $\langle X\rangle \times H$ for some $H \subseteq Z(G)$.

Moreover,

(i) the elements of X^ν corresponds to vertices of $\Gamma(G)$ and the commutativity between two elements in X^ν corresponds to the existence of edge relation between two vertices in $\Gamma(G)$

(ii) H is an elementary abelian p-group, which is a vector space.

From the fact, we may say G is of the form $\langle X\rangle \times \langle H\rangle$.

4 The main result

It is known that Mekler's construction preserves many model-theoretic properties.

Fact 4.1. [1, 3, 6] Let C be an infinite nice graph.

(1) (Mekler, 1981) $Th(C)$ is λ-stable if and only if $Th(G(C))$ is λ-stable.

(2) (Baudisch, 2002) $Th(C)$ is simple if and only if $Th(G(C))$ is simple.

(3) (Chernikov, Hempel, 2017) $Th(C)$ is NIP if and only if $Th(G(C))$ is NIP.

(4) (-) $Th(C)$ is NTP_2 if and only if $Th(G(C))$ is NTP_2.

We extend these results to NTP_1.

Theorem 4.2. *$Th(C)$ is NTP_1 if and only if $Th(G(C))$ is NTP_1.*

Proof. (Sketch)

$\Leftarrow$. Use the interpretability.

$\Rightarrow$. Suppose $\mathrm{Th}(C)$ is NTP_1 but $\mathrm{Th}(G(C))$ is TP_1. Let $G = \langle X \rangle \times \langle H \rangle$ be a monster model of $\mathrm{Th}(G(C))$. WMA there is a formula $\phi(x, y)$ and a strongly indiscernible tree (c_η) in G witnessing TP_1 where c_η is of the form $x^\nu_\eta x^p_\eta x^\iota_\eta h_\eta$ in X and H.

Let b be a realization of $\bigwedge \phi(x, c_{0^i})$. b is $t(x^\nu, x^p, x^\iota, h)$ for some term t and tuples x^ν, x^p, x^ι in X and h in H. Putting the term t inside the ϕ, we may assume b is the tuple $x^\nu x^p x^\iota h$.

Note that the tree (x^ν_η) and x^ν lie inside X^ν, which can be considered as a graph of $\mathrm{Th}(C)$. Apply the Proposition to (x^ν_η) and x^ν, and then extend the result to $(x^\nu_\eta x^p_\eta x^\iota_\eta h_\eta)$ and $x^\nu x^p x^\iota h$. This will make a contradiction that $\phi(x, c_{\eta_1}) \wedge \phi(x, c_{\eta_2})$ is inconsistent for any incomparable η_1 and η_2. □

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