A note on strictly stable generic structures

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Abstract

We show that there is a generic structure in a finite language such that the theory is strictly stable and not ω-categorical, and has finite closures.

1 The class K

It is assumed that the reader is familiar with the basics of generic structures. For details, see Baldwin-Shi [1] and Wagner [3].

Let R, S be binary relations with irreflexivity, symmetricity and $R \cap S = \emptyset$. Let $L = \{R, S\}$.

Definition 1.1 Let $\mathbf{K}_0$ be the class of finite L-structures A with the following properties:

1. $A \models R(a, b)$ implies that a, b are not S-connected;
2. If $A \models R(a, b) \wedge R(b, c)$, then a, c are not S-connected;
3. If $A \models R(a, b) \wedge R(b', c)$ and b, b' are S-connected, then a, c are not S-connected;
4. A has no S-cycles.

Definition 1.2 Let $A \in \mathbf{K}_0$.

- For $a, b \in A$, aEb means that a and b are S-connected.
- For $a \in A$, let $a_E = a/E$, and let $A_E = \{a_E : a \in A\}$.

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- A binary relation R_E on A_E is defined as follows: for any $a, b \in A$, $A_E \models R_E(a_E, b_E)$ iff there are some $a', b' \in A$ with $a'Ea, b'Eb$ and $A \models R(a', b')$. By Definition 1.1, the structure $A_E = (A_E, R_E)$ can be considered as an R-structure (or an R-graph) with irreflexivity and symmetricity.

Notation 1.3 Let $A \in \mathbf{K}_0$.

- Let $s(A)$ denote the number of the S-edges in A.
- Let $x(A) = |A| - s(A)$.
- Let $r(A)$ denote the number of the R-edges in A.
- For α with $0 < \alpha \leq 1$, let $\delta(A) = x(A) - \alpha \cdot r(A)$.

Definition 1.4 Let $A, B, C \in \mathbf{K}_0$.

- Let $\delta(B/A)$ denote $\delta(BA) - \delta(A)$.
- For $A \subset B$, A is said to be closed in B, denoted by $A \leq B$, if $\delta(X/A) \geq 0$ for any $X \subset B - A$.
- For $A = B \cap C$, B and C are said to be free over A, denoted by $B \bot_A C$, if $R^{B \cup C} = R^B \cup R^C$ and $S^{B \cup C} = S^B \cup S^C$.
- When $B \bot_A C$, we write $B \oplus_A C$ for an L-structure $B \cup C$.

Lemma 1.5 $(\mathbf{K}_0, \leq)$ has the free amalgamation property, i.e., whenever $A \leq B \in \mathbf{K}_0$, $A \leq C \in \mathbf{K}_0$ and $B \bot_A C$ then $B \oplus_A C \in \mathbf{K}_0$.

Proof. Let $D = B \oplus_A C$. We have to check that D satisfies conditions 1-4 in Definition 1.1. Here, for simplicity, we see condition 2 in Definition??. Take any $a, b, c \in D$ with $R(a, b) \wedge R(b, c)$. If abc is contained in either B or C, then it is clear that a and c are not S-connected. So we can assume that $a \in B - A, b \in A$ and $c \in C - A$. Suppose for a contradiction that a and c are S-connected. Then there is some $d \in A$ with $R(d, c)$. So $\delta(c/A) \leq 1 - (\alpha + 1) < 0$, and hence $A \not\leq C$, a contradiction. Hence a and c are not S-connected.

Remark 1.6 In [2], Hrushovski proved that there were an $\alpha \in (0, 1)$ and a function $f : \mathbb{N} \to \mathbb{R}$ such that

1. $f(0) = 0, f(1) = 1$;
2. f is unbounded and convex;

3. $f'(n) \leq \min\{r : r = \dfrac{p - q\alpha}{m} > 0, m \leq n \text{ and } m, p, q \in \omega\}$ for each $n \in \omega$.

Definition 1.7 For a function f in Remark 1.6, let $\mathbf{K} = \{A \in \mathbf{K}_0 : \delta(A') \geq f(x(A'))$ for any $A' \subset A\}$.

Lemma 1.8 $(\mathbf{K}, \leq)$ has the free amalgamation property.

Proof. Let $A, B, C \in \mathbf{K}$ be such that $A \leq B, A \leq C$ and $B \perp_A C$. Let $D = B \oplus_A C$. We want to show that $D \in \mathbf{K}$. By Lemma 1.5, we have $D \in \mathbf{K}_0$. So it is enough to see that $f(|D|) \leq \delta(D)$. Without loss of generality, we can assume that $\delta(C/A) \geq \delta(B/A)$. By Remark??, we have $\dfrac{\delta(B) - \delta(A)}{|B| - |A|} \geq f'(|B|)$. On the other hand, since $B \in \mathbf{K}$, we have $\delta(B) \geq f(|B|)$. Hence we have $\delta(D) \geq f(|D|)$.

Definition 1.9

- Let $\overline{\mathbf{K}}$ denote the class of L-structure A satisfying $A_0 \in \mathbf{K}$ for every finite $A_0 \subset A$.
- For $A \subset B \in \overline{\mathbf{K}}$, $A \leq B$ is defined by $A \cap B_0 \leq B_0$ for any finite $B_0 \subset B$.
- For $A \subset B \in \overline{\mathbf{K}}$, we write $\mathrm{cl}_B(A) = \bigcap\{C : A \subset C \leq B\}$.
- It can be checked that there exists a countable L-structure M satisfying
  1. if $M \in \overline{\mathbf{K}}$;
  2. if $A \leq B \in \mathbf{K}$ and $A \leq M$, then there exists a copy B' of B over A with $B' \leq M$;
  3. if $A \subset_{\mathrm{fin}} M$, then $\mathrm{cl}_M(A)$ is finite.

  This M is called a $(\mathbf{K}, \leq)$-generic structure.

2 Theorem

In what follows, let M be the $(\mathbf{K}, \leq)$-generic structure, $T = \mathrm{Th}(M)$ and $\mathcal{M}$ a big model of T.

Lemma 2.1 T has finite closures, i.e., for any finite $A \subset \mathcal{M}$, $\mathrm{cl}_{\mathcal{M}}(A)$ is finite.

Proof. For each $t \in \mathbf{R}$, let $H_t = \{(x, y) : x, r \in \omega, y = x-\alpha r, f(x) \leq y \leq t\}$. Since f is unbounded, each H_t is finite. Hence any $A \subset_{\text{fin}} \mathcal{M}$ has finite closures.

Lemma 2.2 T is not ω-categorical.

Proof. Let $a_0, a_1, ...$ be vertices with the relations $S(a_0, a_1)$, $S(a_1, a_2)$,.... Since $a_0a_1... \in \overline{\mathbf{K}}$, we can assume that $a_0a_1... \subset \mathcal{M}$. It can be checked that $\text{tp}(a_0a_n) \neq \text{tp}(a_0a_m)$ for each distinct $m, n \in \omega$. Then $S_2(T)$ is infinite. Hence T is not ω-categorical.

For $A \subset_{\text{fin}} \mathcal{M}$ and $n \in \omega$, A is said to be n-closed, if $\delta(X/A) \geq 0$ for any $X \subset \mathcal{M} - A$ with $|X| \leq n$.

Notation 2.3 Let $A \leq_{\text{fin}} \mathcal{M}$ and $n \in \omega$.

- $\text{cltp}_n(A) = \{X \cong A\} \cup \{X \text{ is } n\text{-closed}\}$
- $\text{cltp}(A) = \bigcup_{i\in\omega} \text{cltp}_i(A)$
- $E(A) = \{B \in \mathbf{K} : A \leq B\}$
- $E^+(A) = \{B \in E(A) : \text{there is a copy of } B \text{ over } A \text{ in } \mathcal{M}\}$
- $E^-(A) = E(A) - E^+(A)$
- $\text{ptp}(A) = \{\exists Y(XY \cong AB) : B \in E^+(A)\}$
- $\text{ntp}(A) = \{\neg\exists Y(XY \cong AB) : B \in E^-(A)\}$
- $\text{gtp}(A) = \text{cltp}(A) \cup \text{ptp}(A) \cup \text{ntp}(A)$
- $\text{gtp}_n(A) = \text{cltp}_n(A) \cup \text{ptp}(A) \cup \text{ntp}(A)$

Definition 2.4 Let $A \subset B \in \mathbf{K}_0$. Then B_A is an $L \cup \{R_E, S_E\}$-structure with the following properties:

1. the universe is $\{b_E : b \in B - A\} \cup A$;
2. the restriction of B on A is the L-structure A;
3. for $a \in A$ and $b \in B-A$, $B_A \models R_E(a, b_E)$ iff there is a $b' \in B-A$ with $b'Eb$ and $B \models R(a, b')$, and $B_A \models R_E(b_E, a)$ iff there is a $b' \in B - A$ with $b'Eb$ and $B \models R(b', a)$;
4. for $a \in A$ and $b \in B-A$, $B_A \models S_E(a, b_E)$ iff there is a $b' \in B-A$ with $b'Eb$ and $B \models S(a, b')$, and $B_A \models S_E(b_E, a)$ iff there is a $b' \in B - A$ with $b'Eb$ and $B \models S(b', a)$;

5. for $b, c \in B - A$, $B_A \models R(b_E, c_E)$ iff there are $b', c' \in B - A$ with $b'Eb, c'Ec$ and $B \models R(b', c')$.

Note 2.5 By the similar argument as in Definition 1.2, the structure B_A is canonically considered as an L-structure.

Lemma 2.6 Let $A \leq_{\text{fin}} \mathcal{M}$ and $n \in \omega$. Then $\text{gtp}_n(A)$ is finitely generated.

Proof. Take a sequence $(S_i)_{i \in \omega}$ of finite subsets of $\text{gtp}_n(A)$ with $S_0 \subset S_1 \subset \cdots$ and $\bigcup S_i = \text{gtp}_n(A)$. For $i \in \omega$, let $\sigma_i(X) = \bigwedge S_i$. We can assume that $\models \sigma_i(A')$ implies $A' \cong A$. Since f is unbounded, $\mathcal{C}_i = \{C'_{A'} : M \models \sigma_i(A'), C' = \text{cl}_M(A')\}$ is finite. So there is some $i_0 \in \omega$ such that $\mathcal{C}_j = \mathcal{C}_{i_0}$ for every $j > i_0$. Hence S_{i_0} generates $\text{gtp}_n(A)$.

Lemma 2.7 If $\text{gtp}(A) = \text{gtp}(B)$ and $A \leq C \leq_{\text{fin}} \mathcal{M}$, then there is a D with $\text{gtp}(AC) = \text{gtp}(BD)$.

Proof. Let $\Sigma(XY) = \text{gtp}(AC)$ and let $\Sigma_n(XY) = \text{gtp}_n(AC)$ for $n \in \omega$. We want to show that $\Sigma(BY)$ is consistent. To show this, it is enough to see that $\Sigma_n(BY)$ is consistent for each n. On the other hand, by Lemma 2.6, $\Sigma_n(XY)$ can be considered as some formula $\sigma(XY)$. So we want to show that $\sigma(BY)$ has a realization. For this, we prove that $\sigma(XY) \wedge \phi(X)$ has a realization for each $\phi(X) \in \text{tp}(B)$. Let $\tau(X) = \sigma(XY)|_X$. Note that $\tau(X) \wedge \phi(X) \in \text{tp}(B)$ and $\tau(X) \vdash \text{gtp}_n(A) = \text{gtp}_n(B)$. Take $B' \models \tau \wedge \phi$ in M. Take $A'C' \models \sigma$ in M with $A' \cup \text{cl}(A') \cong B' \cup \text{cl}(B')$. Let DE be such that $DE \cup \text{cl}(B') \cong C' cl(C') \cup \text{cl}(A')$. By genericity, we can assume that $E \leq M$. Then we have $\models \sigma(B'D)$, and hence $\sigma(XY) \wedge \phi(X)$ has a realization.

Corollary 2.8 Let $A \leq_{\text{fin}} \mathcal{M}$. Then $\text{gtp}(A) \vdash \text{tp}(A)$.

Definition 2.9 Let $A, B, C \subset \mathcal{M}$ with $A = B \cap C$. Then the notation $B \downarrow^*_A C$ is defined as follows: for each $n \in \omega$ and $A^* B^* C^* \models \text{gtp}_n(ABC)$ in M,

1. $\text{cl}(B^*) \cap \text{cl}(C^*) = \text{cl}(A^*)$;
2. $\text{cl}(B^*) \perp_{\text{cl}(A^*)} \text{cl}(C^*)$.

Lemma 2.10 Let $A \leq B \leq \mathcal{M}, A \leq E \leq \mathcal{M}$ and $E \downarrow^*_A B$. Then $\text{gtp}(E/A) \vdash \text{gtp}(E/B)$.

Proof. For simplicity, we assume that A, B and E are finite. Take any $E_1 \models \mathrm{gtp}(E/A)$ with $E_1 \downarrow^*_A B$ in M. Fix any n. Then there are realizations $E^*A^*, E_1^*A^* \models \mathrm{gtp}_n(EA)$ in M with $\mathrm{cl}(E^*) \cong_{\mathrm{cl}(A^*)} \mathrm{cl}(E_1^*)$. Since $E \downarrow^*_A B$ and $E_1 \downarrow^*_A B$, there is $B^*A^* \models \mathrm{gtp}_n(BA)$ with $\mathrm{cl}(E^*) \cong_{\mathrm{cl}(B^*)} \mathrm{cl}(E_1^*)$. Hence $E_1 \models \mathrm{gtp}(E/B)$.

Lemma 2.11 T is strictly stable.

Proof. Let $N \prec \mathcal{M}$ with $|N| = \lambda$. Take any $e \in \mathcal{M} - N$. Then there is a countable $A \leq N$ with $e \downarrow^*_A N$. Let $E = \mathrm{cl}(eA)$. We can assume that $E \cap N = A$. We want to show that $\mathrm{gtp}(E/A) \vdash \mathrm{gtp}(E/N)$. Take any $E_1, E_2 \models \mathrm{gtp}(E/A)$ with $E_i \downarrow^*_A N$. Take any countable $N_0 \leq N$. Take $E_i^*A^* \subset M$ such that $E_1^*A^*, E_2^*A^* \models \mathrm{gtp}_n(EA)$ and $\mathrm{cl}(E_1^*A^*) \cong \mathrm{cl}(E_2^*A^*)$. Hence $\mathrm{gtp}(E_1/N) = \mathrm{gtp}(E_2/N)$. It follows that $|S(N)| \leq 2^\omega \cdot \lambda^\omega = \lambda^\omega$. Hence T is stable.

Theorem 2.12 There is a generic structure M with the following properties:

1. the language is finite;
2. $\mathrm{Th}(M)$ is not ω-categorical;
3. $\mathrm{Th}(M)$ has finite closures;
4. $\mathrm{Th}(M)$ is strictly stable.

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