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On locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local adaptations of o-minimality. These structures were treated in the past, e.g. in [1], [2]. Meanwhile o-minimal structures have been studied widely, in particular, there is geometric characterization of them by independence relation. We try to consider independence relation in locally o-minimal structures.

1. Introduction

Locally o-minimal structures are some local versions of o-minimal structures. We recall some definitions at first.

Definition 1 A linearly ordered structure $M = (M, <, \cdots)$ is $o-minimal$ if every definable subset of M^1 is a finite union of points and open intervals.

A linearly ordered structure $M = (M, <, \cdots)$ is $weakly\ o-minimal$ if every definable subset of M^1 is a finite union of convex sets.

Definition 2 Let $M = (M, <, \cdots)$ be a densely linearly ordered structure.

M is *locally* $o-minimal$ if for any $a \in M$ and any definable set $A \subset M^1$, there is an open interval $I \ni a$ such that $I \cap A$ is a finite union of points and intervals.

M is $strongly\ locally\ o-minimal$ if for any $a \in M$, there is an open interval $I \ni a$ such that whenever A is a definable subset of M^1, then $I \cap A$ is a finite union of points and intervals.

M is $uniformly\ locally\ o-minimal$ if for any $\varphi(x, \bar{y}) \in L$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$.

Example 3 The following examples are shown in [1] and [2].

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ are locally o-minimal structures.

Let $L = \{<\} \cup \{P_i : i \in \omega\}$ where P_i is a unary predicate. Let $M = (\mathbb{Q}, <^M, P_0^M, P_1^M, \ldots)$ be the structure defined by $P_i^M = \{a \in M : a < 2^{-i}\sqrt{2}\}$. Then M is uniformly locally o-minimal, but it is not strongly locally o-minimal.

Theorem 4 [1] *Weakly o-minimal structures are locally o-minimal.*

Theorem 5 [1] *A structure $\mathcal{M} = (M, <, \dots)$ expanding a dense linear order $(M, <)$ without endpoints is locally o-minimal if and only if for any $a \in M$ and any definable $X \subset M$, there are c, $d \in M$ such that $c < a < d$ and either $X \cap (c, d)$ or $(c, d) \setminus X$ is equal to one of the following :* (1) $\{a\}$, (2) $(c, a]$, (3) $[a, d)$, *or* (4) *the whole interval* (c, d).

Corollary 6 [1] *Local o-minimality is preserved under elementary equivalence. But, strong local o-minimality is not preserved under elementary equivalence.*

It is proved that (weakly) o-minimal structures have no independence property. And there are geometric characterizations of o-minimal structures by independence relation. We try to characterize locally o-minimal structures by independence relation.

2. þ-forking in locally o-minimal structures

At first we argue about some kind of forking, thorn-forking. It is known that this forking notion is available to o-minimal structures, or structures whose theories are NIP unstable.

Definition 7 Let $\mathcal{M}$ be a sufficiently large saturated model.
A formula $\phi(\bar{x}, \bar{a})$ *strongly divides* over A if $tp(\bar{a}/A)$ is nonalgebraic and $\{\phi(x, \bar{a}') ; a' \in \mathcal{M}\}$ with $tp(\bar{a}/A) = tp(\bar{a}'/A)$ is k-inconsistent for some $k < \omega$.
A formula $\phi(\bar{x}, \bar{a})$ þ$-$*divides* (*thorn divides*) over A if for some tuple $\bar{c}$, $\phi(\bar{x}, \bar{a})$ strongly divides over $A\bar{c}$.
A formula $\phi(\bar{x}, \bar{a})$ þ$-$*forks* over A if it implies a finite disjunction of formulas which þ-divides over A.

As the ordinary forking, in [10], they define some local þ-rank for formulas, and theories having finite þ-rank are called *rosy*.

Theorem 8 [10]
þ-independence defines an independence relation in any rosy theory. That is, þ-forking satisfies such axioms : Existence, Extension, Reflexivity, Monotonicity, Finite character, Symmetry, Transitivity.

Here we recall the next $U^{þ}$-rank only.

Definition 9 We define $U^{þ}$-rank (U-thorn rank) inductively as follows.

Let $p(\bar{x})$ be a type over A. Then

(1) $U^{þ}(p(\bar{x})) \geq 0$ if $p(\bar{x})$ is consistent.

(2) For any ordinal α, $U^{þ}(p(\bar{x})) \geq \alpha + 1$ if there is some tuple $\bar{a}$ and some type $q(\bar{x}, \bar{a})$ over $A\bar{a}$ such that $q(\bar{x}, \bar{a}) \supset p(\bar{x})$, $U^{þ}(q(\bar{x}, \bar{a})) \geq \alpha$, and $q(\bar{x}, \bar{a})$ þ-forks over A.

(3) For any λ limit ordinal, $U^{þ}(p(\bar{x})) \geq \lambda$ if $U^{þ}(p(\bar{x})) \geq \beta$ for all $\beta < \lambda$.

Definition 10 A theory T is *superrosy* if $U^{þ}(p(\bar{x})) < \infty$ for any type $p(\bar{x})$.

I introduce a result for o-minimal structures by þ-independence.

Theorem 11 [10]

Let M be an o-minimal structure.

For any definable $A \subset M^n$, $U^{þ}(A) = dim(A)$ in the sense of o-minimal structure.

There are results about o-minimal structures, or expansions of o-minimal structures in relation to rosyness, e.g. in [11].

We can prove the last theorem under the locally o-minimal setting. First we recall a characterization of strongly local o-minimality from [2].

Theorem 12 [2]

The following two conditions are equivalent ;

1. M is strongly locally o-minimal.

2. For any finite subset $\{a_1, \cdots, a_n\}$ of M, there are left-open and right-closed intervals I_i with $a_i \in (I_i)^\circ$ such that, by putting $I = \bigcup_{1 \leq i \leq n} I_i$, I_{def} is o-minimal (I° is the interior of I, and I_{def} is the induced structure on I by definable subsets of M).

Thus we can prove the next proposition.

Proposition 13 *Let M be a strongly locally o-minimal structure and let $a \in M^k$.*

Then there is an open box $B \ni a$ such that for any definable set $A \subset M^k$, $dim(A \cap B) = U^{þ}(A \cap B)$ (where dim means the dimension of some o-minimal structure I_{def}).

3. Forking in locally o-minimal structures

There are many geometric characterizations of o-minimal structures, especially, those of definable groups in o-minimal structures in stability theoretic context.

We recall some definitions.

Definition 14 A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x},\bar{a})$ *forks* over A if $\phi(\bar{x},\bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x},\bar{b}_i)$ and each $\psi_i(\bar{x},\bar{b}_i)$ divides over A.

There is a fundamental result about forking relation in o-minimal structures, first it is proved in [8], after that, it is modified in [9]. The argument is carried out in sufficiently large saturated models.

Theorem 15 [9]
Let $\mathcal{M}$ be a sufficiently large saturated o-minimal structure and $M_0 \prec \mathcal{M}$. Assume that $\{X(a) : a \in S\}$ is an M_0-definable family of closed and bounded subsets of $\mathcal{M}^n$. Let $p(x) \in S_m(M_0)$ be a type of some $a \in S$, and let $P = p(\mathcal{M})$.
Then $\{X(a) : a \in P\}$ has the finite intersection property if and only if there is $c \in M_0$ such that $c \in X(a)$ for every $a \in P$.

We can consider the theorem above under locally o-minimal setting.

Theorem 16 *Let $\mathcal{M}$ be a sufficiently large saturated strongly locally o-minimal structure and $a \in \mathcal{M}^k$.*
Then there is an open box $B \ni a$ satisfying that ;
For any $M_0 \prec \mathcal{M}$ such that M_0 contains the endpoints c of B, and for $p(x) \in S_k(M_0)$ the type of a over M_0 and $P = p(B)$,
if $\{X(ac) : a \in P\}$ is an M_0-definable family of closed and bounded subsets of B,
then $\{X(ac) : a \in P\}$ has the finite intersection property if and only if there is $d \in M_0$ such that $d \in X(ac)$ for every $a \in P$.

4. Small closure in locally o-minimal structures

It is well known that algebraic closure satisfies the exchange property in o-minimal structures. Here we consider another kind of closure operator in locally o-minimal structures.
We recall some definitions.

Definition 17 Let M be a structure.
We call a function cl from $\mathcal{P}(M)$ to $\mathcal{P}(M)$ a *closure operator* if for any A, $B \subset M$, the following hold ; (where $\mathcal{P}(M)$ is the power set of M)
(1) $A \subset cl(A)$,
(2) $A \subset B$ implies $cl(A) \subset cl(B)$,
(3) $cl(cl(A)) = cl(A)$.
A closure operator cl satisfies the *exchange property* if for any a, $b \in M$ and $C \subset M$, if $a \in cl(bC)$ and $a \notin cl(C)$, then $b \in cl(aC)$.

Definition 18 Let M be a structure and $C \subset M$.

The *algebraic closure* of C, $acl(C) = \{a : M \models \phi(a,c) \wedge \exists_{\leq n}\phi(x,c)$ for some $\phi(x,c)$ a formula over $C\}$.

It is easily checked that acl is a closure operator. The next fact is well known.

Theorem 19 [5]

Let M be an o-minimal structure. Then acl satisfies the exchange property in M.

acl also has the exchange property in some locally o-minimal structures.

Definition 20 [1] Let M be a locally o-minimal structure.

We call M has $\emptyset - definable\ strong\ local\ o - minimality$, we denote M has $DSLOM$ if for any $a \in M$, there is b, $c \in acl(\emptyset)$ such that $b < a < c$ and the interval $(b,\ c)$ intersects every definable subset X of M in finitely many isolated points and intervals.

Proposition 21 [1]

Let M be a locally o-minimal structure satisfing DSLOM. Then acl satisfies the exchange proprty in M.

There are such locally o-minimal structures, e.g. $(\mathbb{R}, <, +, \sin)$. However, as strongly local o-minimality is not preserved under elementary equivalence, the next fact is proved.

Theorem 22 [4]

Let M be an expansion of a densely linearly ordered structure and let $Th(M)$ be the theory of M. Suppose that an infinite discrete unary ordered set is definable in M. Then $Th(M)$ can not satisfy the exchange property with respect to acl (or dcl).

Sometimes for a locally o-minimal structure M, we recognize that there is a definable infinite discrete unary set in M to witness non (weakly) o-minimality of M. As we assume that locally o-minimal structures are densely ordered, definable infinite discrete sets are small in some sense.

Definition 23 [11] Let $M = (M, <, \dots)$ be an ordered structure.

A definable set $D \subset M^k$ is *large* if there is some m, an interval $I \subset M$ and a (onto) function $f : D^m \twoheadrightarrow I$.

A definable set D is *small* if it is not large.

The complement of small set is large in group structures.

Theorem 24 [11]

Let $(M, <, +, \dots)$ be an expansion of ordered group. And let $I = (a, b) \subset M$ be a nonempty *interval and $S \subset M$ be a small set.*

Then $I \setminus S$ is large.

Proof ;

Let $f : M^2 \longrightarrow M$ be defined by $(m_1, m_2) \longrightarrow m_1 + m_2$. And let $J = (a + b,\, 2b)$. We show that $f((I \setminus S)^2) \supset J$.

Suppose that $m_0 \in J \setminus f((I \setminus S)^2)$. Thus $m_0 \in \bigcap_{m \notin S \cup I^c} \big(S \cup I^c + m \big)$ where I^c means the complement of I. So $-(S \cup I^c) + m_0 \supset I \setminus S$. As $-I^c + m_0 = (-\infty, -b + m_0) \cup (-a + m_0, \infty)$, we see that $-S + m_0 \supset (-b + m_0, b)$ contradicting the smallness of S. ∎

There are characterizations of some structures in which small sets hold the axioms of closure operator in [11]. This small closure operator, *scl* has the relation to þ-independence there. But although *scl* works in some structure M, *scl* depends on the choice of M unlike the algebraic closure in general.

There are some locally o-minimal structures M in which $acl(\emptyset) = scl(\emptyset)$, or $acl(A) = scl(A)$ for any $A \subset M$. And also some locally o-minimal structures have a definable infinite discrete set which is not contained in algebraic closures of finite sets.

Problem 25 Can we characterize locall o-minimal structures by small sets, or small closure operator ?

5. Further problems

We can consider the application of independence notions mentioned above to concrete locally o-minimal sturctures, e.g. simple products defined in [2].

And we can try analogous argument following up the advance of o-minimal structures, e.g. definably compactness or fsg property of definable groups, and the argument of generic types, and so on.

Problem 26 Can we characterize definably compact groups definable in locally o-minimal structures ?

In addition, we consider whether the argument of measure and that of measure forking are available for locally o-minimal structures.

Problem 27 Can we characterize definably amenable groups definable in locally o-minimal structures ?

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