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Arithmetic and Combinatorics in Galois fundamental groups

Dedicated to Professor Yasutaka Ihara on the occasion of his 80th birthday

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ABSTRACT. In his Annals paper in 1986, Y.Ihara introduced the universal power series for Jacobi sums and showed deep arithmetic phenomena arising in Galois actions on profinite fundamental groups. In particular, the explicit formula established by Anderson, Coleman, Ihara-Kaneko-Yukinari opened remarkable connection to theory of cyclotomic fields (Iwasawa theory) and shed new lights on circle of ideas surrounding Grothendieck's philosophy on anabelian geometry as well as various geometric approaches in inverse Galois theory. In this article, I will illustrate some of these aspects from a viewpoint of Grothendieck-Teichmüller theory.

CONTENTS

I was a graduate student of Ihara in 1987-1989 just when a year had passed since the publication of his influential Annals paper [18]. The paper was motivating many colleagues toward subsequent progresses not only in number theory but also other areas. I was very fortunate to start my research in those illuminating days: for these 30 years the theme has been continuously attracting my interest with deep problems and questions as well as enlightening my humble perception of the mathematical nature.

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0. **Adelic beta function on $\widehat{GT}$**

After the title "Profinite braid groups, Galois representations and complex multiplications" of [18], our generation of students of Ihara in the Univ. of Tokyo called his weekly advanced seminar *the PGC-seminar*, to which I was attending with a feeling of awe and a piece of pride. (The initials of the paper title had also hinted to name this workshop). Impacts of [18] have spread over various related subjects from number theory to topology as well as other new-wave areas of mathematics including what is called the Grothendieck-Teichmüller theory or anabelian geometry (cf. [1]-[37] and references therein).

In this section, I try to give a short overview on some key aspects of the theme. The main stage is the algebraic fundamental group

$$\pi := \pi_1^{\text{ét}}(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}, \overrightarrow{01}) = \langle x, y, z \mid xyz = 1 \rangle \ (\cong \hat{F}_2)$$

isomorphic to the profinite free group $\hat{F}_2$ of rank 2, where x, y, z represent standard loops around the punctures $0, 1, \infty$ respectively on $\mathbf{P}^1(\mathbb{C})$ based at the tangent vector $\overrightarrow{01}$, with outer actions by the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through the *fundamental exact sequence*

$$1 \to \pi \to \pi_1^{\text{ét}}(\mathbf{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \overrightarrow{01}) \to G_{\mathbb{Q}} \to 1.$$

In fact, this sequence splits (in many ways) and the $\mathbb{Q}$-rational tangential base point $\overrightarrow{01}$ determines a homomorphic section $s_{\overrightarrow{01}} : G_{\mathbb{Q}} \to \pi_1^{\text{ét}}(\mathbf{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \overrightarrow{01})$ which induces the standard splitting

$$\pi_1^{\text{ét}}(\mathbf{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \overrightarrow{01}) \cong G_{\mathbb{Q}} \ltimes \pi$$

as well as the Belyi action $\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi)$ lifting the aforementioned outer action.

0.1. **Fermat tower.** Let $\pi \supset \pi' \supset \pi'' \supset \cdots$ be the derived series of the geometric fundamental group π (in the profinite sense). First of all, the abelianization is identified as:

$$\pi^{\text{ab}} = \pi/\pi' = \hat{\mathbb{Z}}\bar{\mathbf{x}} \oplus \hat{\mathbb{Z}}\bar{\mathbf{y}} \cong \hat{\mathbb{Z}}^2$$

with $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \pi^{\text{ab}}$ the images of $x, y \in \pi$, on which $G_{\mathbb{Q}}$ acts simply by multiplication via the cyclotomic character

$$\chi_{\text{cyc}} : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times \quad (\sigma(\zeta_n) = \zeta_n^{\chi_{\text{cyc}}(\sigma)},\ n \geq 1,\ \sigma \in G_{\mathbb{Q}}).$$

Looking at the Galois action on the meta-abelian quotient π/π'' turns out to amount to the $G_{\mathbb{Q}}$-actions on the torsion points of Fermat Jacobians $J_n :=$

$Jac(F_n)$ which has symmetry induced from the covering group $(\mathbb{Z}/n\mathbb{Z})^2$:

$$\begin{array}{ccc} F_n := \{X^n + Y^n = Z^n\} & \hookleftarrow & \{cusps\} \\ \downarrow (\mathbb{Z}/n\mathbb{Z})^2 & & \downarrow \\ F_1 := \{X + Y = Z\} = \mathbf{P}^1_t & \hookleftarrow & \{0, 1, \infty\}. \end{array}$$

Accordingly the Tate module $\hat{T}(J_n) = \varprojlim_k (J_n[k])$ is operated by $G_{\mathbb{Q}}$ and by $\hat{\mathbb{Z}}[(\mathbb{Z}/n\mathbb{Z})^2]$. Climbing up the Fermat tower with identification

$$\varprojlim_n \hat{\mathbb{Z}}[[(\mathbb{Z}/n\mathbb{Z})^2]] \cong \hat{\mathbb{Z}}[[\pi^{\mathrm{ab}}]], \quad \varprojlim_n \hat{T}(J_n) \cong \pi'/\pi'',$$

we eventually find that the $G_{\mathbb{Q}}$-action on the second derived quotient π'/π'' is represented by the adelic beta function

$$\begin{array}{cccc} \mathbb{B}: & G_{\mathbb{Q}} & \longrightarrow & \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]^{\times} \\ & \Psi & & \Psi \\ & \sigma & \longrightarrow & \mathbb{B}_\sigma(\bar{\mathbf{x}}, \bar{\mathbf{y}}). \end{array}$$

Originally, its ℓ-adic version was introduced in [18] as the universal power series for Jacobi sums. Fix a prime ℓ. For each $\sigma \in G_{\mathbb{Q}}$, write $\mathtt{B}_\sigma^{(\ell)}$ for the image of $\mathbb{B}_\sigma$ under the natural ℓ-adic projection

$$\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \twoheadrightarrow \mathbb{Z}_\ell[[u, v]] \hookrightarrow \mathbb{Q}_\ell[[U, V]],$$

where $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \mapsto (1 + u, 1 + v) = (e^U, e^V)$. Here is a list of primary features:

(1) The mapping $\mathtt{B}^{(\ell)} : G_{\mathbb{Q}} \to \mathbb{Z}_\ell[[u, v]]^{\times}$ is unramified outside $\{\ell\}$. The special values at ℓ-power roots of unity $\mathtt{B}_\sigma^{(\ell)}(\zeta_{\ell^n}^a - 1, \zeta_{\ell^n}^b - 1)$ $(n \geq 1; a, b \in \mathbb{Z}/\ell^n\mathbb{Z})$ interpolate the family of Jacobi sum Hecke characters on $G_{\mathbb{Q}(\mu_{\ell^n})}$. In other words, it has values of Jacobi sums at Frobenius elements $\sigma = \sigma_{\mathcal{P}}$ over primes $\mathcal{P} \nmid \ell$ in $G_{\mathbb{Q}(\mu_{\ell^n})}$.

(2) The ℓ-adic Taylor expansion has Soulé character coefficients as follows:

Explicit formula (Anderson/Coleman/Ihara-Kaneko-Yukinari)

$$\mathtt{B}_\sigma^{(\ell)} = \exp\left(\sum_{m \geq 3, odd} \frac{\chi_m^{\text{Soulé}}(\sigma)}{1 - \ell^{m-1}} (U^m + V^m + W^m) \right)$$

with $U + V + W = 0$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$. Here the m-th (ℓ-adic)

Soulé character $\chi_m^{\text{Soulé}} : G_{\mathbb{Q}(\mu_{\ell^\infty})} \to \mathbb{Z}_\ell(m)$ is, by definition, characterized by the (accelerated) Kummer properties:

The m-th (ℓ-adic) Soulé character

$$\Big(\prod_{\substack{1\le a<\ell^n\\ \ell\nmid a}} (1-\zeta_{\ell^n}^a)^{a^{m-1}}\Big)^{\frac{1}{\ell^n}(\sigma-1)} = \zeta_{\ell^n}^{\chi_m^{\text{Soulé}}(\sigma)} \quad (n \ge 1).$$

(3) The local behavior at ℓ is represented by the inertia restriction formula (Coleman-Ihara formula, cf. [18] Theorem C, p.105) in the form:

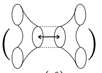
Coleman-Ihara formula

$$\frac{\chi_m^{\text{Soulé}}(\mathrm{rec}(\epsilon))}{\ell^{m-1}-1} = L_\ell(m,\omega^{1-m})\phi_m^{CW}(\epsilon) \quad (\epsilon \in \mathcal{U}_\infty)$$

for $m \ge 3$: odd. Notations: For each $n \ge 1$, we denote by $\mathcal{U}_n$ the group of principal units of $\mathbb{Q}_\ell(\mu_{\ell^n})$ and by $\mathcal{U}_\infty = \varprojlim_n \mathcal{U}_n$ their norm limit. Let Ω_ℓ be the maximal abelian pro-ℓ extension of $\mathbb{Q}(\mu_{\ell^\infty})$ unramified outside ℓ. Then, Ihara's power series $\sigma \mapsto \mathtt{B}_\sigma^{(\ell)}(u,v)$ factors through $\mathrm{Gal}(\Omega_\ell^-/\mathbb{Q})$. Now, fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ and a coherent system of ℓ-power roots of unity $\{\zeta_{\ell^n}\}_{n\ge1}$ to identify $\mathbb{Z}_\ell$ with $\mathbb{Z}_\ell(m)$. This embedding and the local class field theory induce the canonical homomorphism rec : $\mathcal{U}_\infty \to \mathrm{Gal}(\Omega_\ell/\mathbb{Q}(\mu_{\ell^\infty}))$ called the reciprocity map. On the other side, the system $\{\zeta_{\ell^n}\}_n$ determines, for $m \ge 1$, the Coates-Wiles homomorphism $\phi_m^{CW} : \mathcal{U}_\infty \to \mathbb{Z}_\ell$. The coefficient $L_\ell(m,\omega^{1-m})$ is the Kubota-Leopoldt ℓ-adic L-value at m with respect to the power of the Teichmüller character ω.

In regard to the classical decomposition of the beta function into triple gamma functions $B(x,y) = \Gamma(x)\Gamma(y)\Gamma(x+y)^{-1}$, at first sight of the above explicit formula, one may be inclined to consider $\mathit{\Gamma}^\flat := \exp(\sum_{m\ge3,odd} \frac{\chi_m^{\text{Soulé}}(\sigma)}{1-\ell^{m-1}} T^m)$ as a counterpart to the Γ-function. But this turns out a bad idea for arithmetic applications, immediately because, as a power series in $t = e^T - 1$, the coefficients of $\mathit{\Gamma}^\flat$ cannot stay within "integers" (by acquiring big denominators). One useful way to remedy this denominator problem is to consider a "twisted log" of a factor of $\mathtt{B}^{(\ell)}$ defined by

$$\mathsf{g}(t) := \sum_{m\ge1,odd} \chi_m^{\text{Soulé}}(\sigma)\frac{T^m}{m!} \in \mathbb{Z}_\ell[[t]], \quad 1+t = \exp(T)$$

for $\sigma \in G_{\mathbb{Q}(\mu_{\ell^\infty})}$, which differs from $\log(\mathit{\Gamma}^\flat)$ in accompanying $\chi_1^{\text{Soulé}}(\sigma)T$ at $m=1$ while missing divisions $(1-\ell^{m-1})^{-1}$ $(m \geq 3)$. Deep connections to Iwasawa theory of cyclotomic fields emerge in the behavior of g_σ ranged in the "minus part of the Coleman space" $\mathcal{V}^- \subset \mathbb{Z}_\ell[[t]]$. Surprisingly, it is shown in Ichimura-Kaneko [16], $\mathcal{V}^-$ is isomorphic to the expected combinatorial model $\mathfrak{F} \subset \mathbb{Z}_\ell[[u,v]]^\times$ for the collection $\{\mathtt{B}_\sigma^{(\ell)} \mid \sigma \in G_{\mathbb{Q}(\mu_{\ell^\infty})}\}$ defined by certain symmetric relations including one that encodes S_4-symmetry of an amalgamated product of $\pi_1($$)$ (cf. Deligne's idea sketched in [21] p.68). The quotient $\mathcal{V}^-/\{\mathsf{g}_\sigma\}_\sigma \cong \mathcal{F}/\{\mathtt{B}_\sigma^{(\ell)}\}_\sigma$ is called the Vandiver gap, for it vanishes if and only if $\ell \nmid h^+(\mathbb{Q}(\mu_\ell))$ ([7], [21], [16], [15]).

It is G.Anderson's essential idea to extend the coefficients of power series from $\mathbb{Z}_\ell$ to $\mathbb{W}_\ell = W(\bar{\mathbb{F}}_\ell) = \hat{\mathbb{Z}}_\ell^{ur}$, the ring of Witt vectors over $\bar{\mathbb{F}}_\ell$ and to introduce the power series

$$\Gamma_\sigma^{(\ell)}(t) = \exp\Big(\sum_{m\geq 3, odd} \frac{\chi_m^{\text{Soulé}}(\sigma)}{1-\ell^{m-1}} T^m\Big) \cdot (1+t)^{\gamma_\sigma} \in \mathbb{W}_\ell[[t]]^\times$$

which is close to $\mathit{\Gamma}^\flat$ but with complementary factor by a (branch of) Euler-Masheroni cocycle γ_σ such that $\gamma_\sigma - \phi(\gamma_\sigma) = \chi_1^{\text{Soulé}}(\sigma)$ where ϕ is the frobenius automorphism of $\mathbb{W}_\ell$ ([21] (36), [1] (11.3.4), [7] §V-VI). Note also that it recovers the above $\mathsf{g}(t)$ as $\log \Gamma_\sigma^{(\ell)}(t) - \frac{1}{\ell}\log\phi(\Gamma_\sigma^{(\ell)})((1+t)^\ell - 1)$ and satisfies the decomposition $\mathtt{B}_\sigma(u,v) = \Gamma_\sigma^{(\ell)}(u)\Gamma_\sigma^{(\ell)}(v)\Gamma_\sigma^{(\ell)}(w)$ with $(1+u)(1+v)(1+w) = 1$. Taking the product $\mathbb{W} = \prod_\ell \mathbb{W}_\ell$ over all primes ℓ, G.Anderson [1] developed the theory of *hyper-adelic gamma function* $\mathbb{\Gamma}_\sigma \in \mathbb{W}[[\hat{\mathbb{Z}}]]^\times$ and established many properties such as the fact that its special values interpolate Gauss sums, that a hyperadelic analog of the Gauss multiplication formula $\prod_{i=0}^{N-1}\Gamma(\frac{s+i}{N}) = N^{1-s}\prod_{i=1}^{N1}\Gamma(\frac{i}{N})$ holds.

These studies on the Galois image in the meta-abelian reduction $G_\mathbb{Q} \to \mathrm{Aut}(\pi/\pi'')$ bring us first clues to understanding the whole Galois image of the Galois representation $\varphi_{\vec{01}} : G_\mathbb{Q} \to \mathrm{Aut}(\pi)$. In particular, in the pro-ℓ case, the sequence of Soulé characters $\{\chi_m^{\text{Soulé}}\}_{m\geq 3, odd}$ yields a system of virtual generators for the core part $\varphi_{\vec{01}}^{(\ell)}(G_{\mathbb{Q}(\mu_{\ell^\infty})}) \subset \mathrm{Aut}(\hat{F}_2^{\text{pro-}\ell})$, whose (iterated) commutator products diving into deeper anabelian (viz. "entirely non-abelian") sea of $\varphi_{\vec{01}}^{(\ell)}(G_{\mathbb{Q}(\mu_{\ell^\infty})})$ have formed another important subject to research. However, in this article, we content ourselves with just recalling several fundamental references: Ihara [24], Sharifi [35], Hain-Matsumoto [14], Brown [5]...

0.2. **Combinatorial construction.** It is also important to understand the adelic beta function $\mathbb{B}_\sigma$ $(\sigma \in G_\mathbb{Q})$ in terms of profinite combinatorial group theory. Recall once again the derived series of the geometric fundamental group π of $\mathbf{P}^1 - \{0,1,\infty\}$: $\pi \supset \pi' \supset \pi'' \supset \cdots$ with the abelianization $\pi^{\mathrm{ab}} = \pi/\pi' = \hat{\mathbb{Z}}\bar{\mathbf{x}} \oplus \hat{\mathbb{Z}}\bar{\mathbf{y}} \cong \hat{\mathbb{Z}}^2$. It is known that the second derived quotient π'/π'' (equipped with $(G_{\mathbb{Q}}, \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]])$-action) is a free cyclic $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$-module generated by $\overline{[x,y]} \in \pi'/\pi''$, the image of $[x,y] = xyx^{-1}y^{-1} \in \pi'$. It follows from this remark that, for $\sigma \in G_{\mathbb{Q}}$, one can introduce $\mathbb{B}'_\sigma \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]^\times$ by the equation $\sigma(\overline{[x,y]}) = \mathbb{B}'_\sigma \cdot \overline{[x,y]}$ in π'/π'': Then, it turns out that

$$\mathbb{B}'_\sigma = \left(\frac{\bar{\mathbf{x}}^\lambda - 1}{\bar{\mathbf{x}} - 1} \cdot \frac{\bar{\mathbf{y}}^\lambda - 1}{\bar{\mathbf{y}} - 1} \right) \mathbb{B}_\sigma \qquad (\sigma \in G_{\mathbb{Q}})$$

where $\lambda = \chi_{\mathrm{cyc}}(\sigma)$ (cf. [23] §1.4 (2)).

It is not difficult to see that the $G_{\mathbb{Q}}$-action on π'/π'' as the projective limit of the torsions of Fermat Jacobians $\varprojlim_n (\hat{T} J_n)$ comes from the Belyi (faithful) action of $G_{\mathbb{Q}}$ on $\pi = \pi_1(\mathbf{P}^1_{\bar{\mathbb{Q}}} - \{0,1,\infty\}, \overrightarrow{01})$ which is uniquely characterized as a homomorphism $\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}} \hookrightarrow \widehat{GT} \subset \mathrm{Aut}(\pi)$ into the Grothendieck-Teichmüller group $\widehat{GT}$:

$$\widehat{GT} := \left\{ \alpha \in \mathrm{Aut}(\pi) \;\middle|\; \begin{array}{l} \alpha(x) = x^\lambda \quad (\exists \lambda \in \hat{\mathbb{Z}}^\times), \\ \alpha(y) = f^{-1} y^\lambda f \;\; (\exists f \in \pi'), \\ \alpha(z) \sim z^\lambda \quad (\pi\text{-conjugate}), \\ \qquad s.t. \\ (\lambda, f) \text{ satisfies conditions "(I),(II),(III)" where} \\ \text{(I),(II)} \Leftrightarrow S_3\text{-symmetry of } \mathbf{P}^1 - \{0,1,\infty\} \\ \text{(III)} \Leftrightarrow \text{"pentagon" on the moduli space } M_{0,5} \end{array} \right\}.$$

Each element $\alpha \in \widehat{GT}$ can be characterized by the two parameters $\lambda \in \hat{\mathbb{Z}}^\times$ and $f \in \hat{F}'_2$ appearing in the defining condition. We often write $\widehat{GT} \ni \sigma \mapsto (\lambda_\sigma, f_\sigma) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2$ for the set-theoretical embedding. The former parameter λ_σ just extends the cyclotomic character: $\lambda_\sigma = \chi_{\mathrm{cyc}}(\sigma)$ $(\sigma \in G_{\mathbb{Q}})$; thus, to control the latter parameter $f_\sigma \in \hat{F}_2$ (ranged in the space of pro-words in non-commutative two variables x, y) should be one of the ultimate goals for the Grothendieck-Teichmüller theory. The above Grothendieck-Teichmüller group was introduced as a combinatorial model of the Galois image $\varphi_{\overrightarrow{01}}(G_{\mathbb{Q}})$ in $\mathrm{Aut}(\hat{F}_2)$ by Drinfeld and Ihara (see [9], [20]). In [23], after extending $\mathbb{B}$ and $\mathbb{\Gamma}$ to functions on $\widehat{GT}$ and deriving "pentagon $\Rightarrow$ Γ-factorization", Ihara introduced intermediate subgroups of $\widehat{GT}$:

$$\begin{array}{ccccc} & & GTK & & \\ & \nearrow & & \searrow & \\ G_{\mathbb{Q}} & & & & GTA' \subset \widehat{GT} \subset \mathrm{Aut}(\pi), \\ & \searrow & & \nearrow & \\ & & GTA & & \end{array}$$

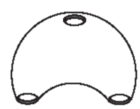
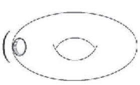
where, GTA is designed to hold the Gauss-multiplication formula for the exntended $\mathbb{\Gamma}$, and GTK is designed to respect compatibility condition in the following 'Kummer covering vs. open immersion'-diagram:

$$\begin{array}{ccc} \mathbf{G}_m - \mu_N & & \\ \downarrow \phi(\mathbb{Z}/N\mathbb{Z}) & \searrow & \\ \mathbf{P}^1 - \{0,1,\infty\} & \hookrightarrow & \mathbf{G}_m - \mu_N. \end{array}$$

After years later, B.Enriquez [10] remarkably proved $GTK = \widehat{GT}$.

1. **An elliptic analog on $\widehat{GT}_{ell}$**

Already in [19], Ihara introduces Fox calculus in profinite context and aims to study not only the Fermat tower but also other important towers including the elliptic modular tower over $\mathbf{P}^1 - \{0,1,\infty\}$ (λ-line). This influential paper motivated M.Ohta [34] to invent a new theory of "ordinary p-adic Eichler-Shimura cohomology" in his subsequent series of works. On the other hand, Ihara delivered lectures in the Spring Term of 1984 at Chicago (cf. Acknowledgements of [18]) which hinted S.Bloch to consider Galois representations in fundamental groups of once-punctured elliptic curves. Note that topological fundamental groups of both once punctured torus and 3 point punctured sphere are isomorphic to a free group with two generators:

$\pi_1($  $) \cong F_2 \cong \pi_1($  $)$

A mimeographed copy of Bloch's letter to Deligne [4] was brought to the author before Ihara left from Tokyo to Kyoto in 1990. H.Tsunogai and I began to study [4] by summer of 1992: For an elliptic curve E over a number field k, Bloch's letter [4] looked at the action of Galois group G_{k_∞}, where k_∞ is the field obtained by adjoining all coordinates of ℓ-power torsion points of E to k, on the meta-abelian quotient π/π'' for $\pi = \pi_1^{\text{pro-}\ell}(E_{\bar{k}} - \{O\})$ and constructed a certain map from G_{k_∞} into $\mathbb{Z}_\ell[[T_\ell(E)(1)]]$ modulo constant terms, whose non-triviality follows from properties of modular units exhibited in the book of Kubert-Lang. In [37], Tsunogai figured out the missing constant term, and in [27] I expressed the other coefficients in terms of (accelerated) Kummer properties of special values of elliptic modular units. Unfortunately, extension of the above power series from G_{k_∞} to the whole G_k had been obstructed by technical reason due to the different inertia structures between $\pi_1(E \setminus \{O\})$ and $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$. This obstruction problem together with Ibukiyama's hint concerning similarity between Jacobi sums and Dedekind sums (this was also brought to me in 1993) had been kept mysterious in my mind for long years till a solution was obtained in 2009 (cf. [29] Note and acknowledgements).

1.1. **Universal elliptic curves.** Let us quickly summarize the solution to the above last problem obtained in [29] and subsequent works. The main setup is the monodoromy representation arising in the universal family of Weierstrass elliptic curves $E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\}$ over the parameter space $\mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\}$ ([29, §5]). We consider both $E \setminus \{O\}$ and $\mathfrak{M}$ as affine varieties over $\mathbb{Q}$.

$$\begin{array}{ccc} E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\} & \dashleftarrow & \mathrm{Tate}(q) \\ \Downarrow\Uparrow \tilde{w} & & \Downarrow\Uparrow \\ \mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\} & \dashleftarrow & \mathrm{Spec}\,\mathbb{Q}((q)) \end{array}$$

The natural projection $E\setminus\{O\} \to \mathfrak{M}$ is the Weierstrass family of once punctured elliptic curves. We have a tangential section $\tilde{w} : \mathfrak{M} \dashrightarrow E \setminus \{O\}$ (normalized with $t := -2x/y$) and a tangential fiber $\mathrm{Tate}(q) \hookrightarrow E \setminus \{O\}$ whose Weierstrass coefficients $g_2(q), g_3(q) \in \mathbb{Q}[[q]]$ are well known power series in q of Eisenstein type. The images of $\mathrm{Spec}\,\mathbb{Q}((q))$ on the individual spaces in the above diagram will be most useful as base points of those étale fundamental groups. From the van-Kampen construction of the degeneration of Tate curve, one can introduce standard loops $\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}$ of $\hat{\pi}_{1,1} := \pi_1^{\text{ét}}(\mathrm{Tate}(q) \otimes \overline{\mathbb{Q}})$ based at $\mathrm{Im}(\tilde{w}) \cap \mathrm{Tate}(q)$ with $[\mathbf{x}_1, \mathbf{x}_2]\mathbf{z} = 1$ ($[\mathbf{x}_1, \mathbf{x}_2] := \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}$). Note that $\hat{\pi}_{1,1}$ is isomorphic to a free profinite group $\hat{F}_2$ freely generated by $\mathbf{x}_1, \mathbf{x}_2$. From these setup, we obtain the splitting of arithmetic fundamental groups

$$\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}, \quad \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3,$$

and the monodromy representation into the *elliptic Grothendieck-Teichmüller group* $\widehat{GT}_{ell}$ introduced by B.Enriquez [11]:

$$\begin{array}{ccc} \varphi_{1,1} : \pi_1(\mathfrak{M}_{1,1}) & \hookrightarrow & \mathrm{Aut}^*(\pi) = \{\alpha \in \mathrm{Aut}(\pi) \mid \alpha(\mathbf{z}) = \mathbf{z}^\lambda\} \\ \| & & \uparrow \\ \hat{B}_3 \rtimes G_{\mathbb{Q}} & \hookrightarrow & \widehat{GT}_{ell} = \hat{B}_3 \rtimes \widehat{GT}. \end{array}$$

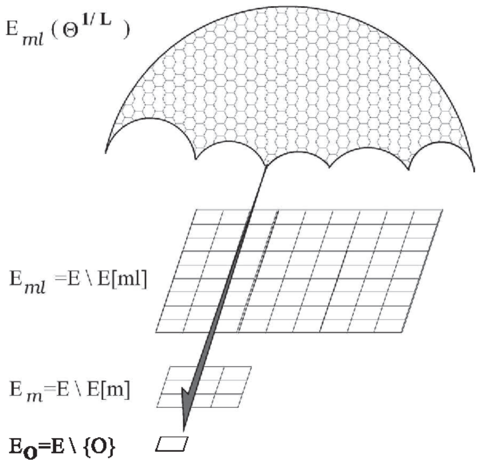
1.2. **Adelic Eisenstein function.**

Theorem 1.1. *Let $\mathbb{Q}_f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$. The $\pi_1(\mathfrak{M})$-action on π/π'' is represented by a single function (called the adelic Eisenstein function)*

$$\mathbb{E} : \widehat{GT}_{ell} \times \mathbb{Q}_f^2 \longrightarrow \hat{\mathbb{Z}}.$$

On $B_3 \times \mathbb{Q}^2$, $\mathbb{E}_\sigma(\frac{u}{m}, \frac{v}{m})$ is described in terms of generalized Rademacher functions (Dedekind sums). On $\widehat{GT} \times \mathbb{Q}_f^2$, it is described in terms of $\mathbb{B} : \widehat{GT} \to \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$.

To illustrate a core idea behind the above theorem, we focus on how to capture monodromy effect on the meta-abelian covers over a once-punctured elliptic curve $E \setminus \{O\}$ (cf. [27], [29]). Let $\{E^N\}_{N \in \mathbb{N}}$ be the isogeny tower on $E^1 = E$, where all E^N are the same E but $E^N \to E^1$ is given as the isogeny of multiplcation by $N \in \mathbb{N}$. Then, each $E_N := E \setminus E[N]$ is geometrically the étale $(\mathbb{Z}/N\mathbb{Z})^2$-cover of $E_O = E_1 = E \setminus \{O\}$. Let $H_N \subset \pi$ be the corresponding open normal subgroup of $\pi := \hat{\pi}_{1,1}$.



Consider a sequence of étale covers $E_{ml}(\Theta^{1/L}) \to E_{ml} \to E_m \to E_O$ for $m \in \mathbb{N}$, l: a prime and $L = l^k$ $(k > 0)$, where $E_{ml}(\Theta^{1/L})$ is the L-th Kummer covering by the unit function $\Theta : E_{ml} \to \mathbf{G}_m$ whose divisor is supported on $div(\Theta) = E^{ml}[l] - l^2[O]$ (i.e., $E_{ml}(\Theta^{1/L})$ is the fiber product of Θ and degree L isogeny of $\mathbb{G}_m$). Let $H_{ml,L}$ be the subgroup of $H_{ml} (\subset \pi)$ corresponding to $E_{ml}(\Theta^{1/L}) \to E_{ml}$. The monodromy permutation on the $H_{ml,L}$-conjugacy classes of inertia subgroups over the missing points in $E^{ml}[ml] \setminus E^{ml}[l]$ of E_{ml} is encoded in the Kummer monodromy properties of values $\{\Theta(P)^{1/L} \mid P \in E^{ml}[ml] \setminus E^{ml}[l]\}$. But since $\Theta(P)$ is (a quotient of) Siegel modular units which are essentially of the form $exp(\int Eis^{wt=2}_{level=ml} d\tau)$, the focused Kummer monodromy corresponds to period integrals of Eisenstein forms of weight 2 and level ml. The geometric monodromy moving moduli of elliptic curves amounts to the period function classically known as the generalized Rademacher function computing *generalized Dedekind sums.* The arithmetic action of $G_{\mathbb{Q}}$ at "Tate-section" can also be computed by the action on the first coefficients of involved q-series, which turns out to be reduced to looking at adelic beta function. It turns out that the process of letting $m, k \to \infty$ exhausts monodromy effects on the meta-abelian quotient π/π'' so as to determine $\mathbb{E}_\sigma(\frac{u}{m}, \frac{v}{m})$.

* * *

At the end of my talk in the workshop, I remarked that the above illustration looks like a parasol covering stages of isogeny-tower of ellitptic curves. In Japan, 80th anniversary is called 傘寿, where the first Kanji-character indicates a parasol under which many people (人) live. I would like to celebrate Ihara's 80th happy birthday and once again express my gratitude for his being the great mentor to us to discover wonderful and deep world of mathematics.

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