数理解析研究所講究録 1358 巻 2004 年 9-23

On the global structure of solutions to the equation of the minimal curvature energy

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1 Introduction

This is a joint work with Waichiro Matsumoto and Shoji Yotsutani (Ryukoku University).

Let Ω be a planar domain with smooth boundary Γ. M.Kac([1]) showed that if $Spect_D(\Omega) = Spect_D(B_r)$, then $\Omega \equiv B_r$, where $Spect_D(\Omega)$ denotes a Dirichlet spectrum and B_r is disk with radius r. K.Watanabe ([2],[3]) showed the following theorem.

Theorem A.[K.Watanabe] Let $L > 0$ and $M > 0$ be given.

1) If $10L^2/49\pi \leq M < L^2/4\pi$, then there exists a non-disk planar domain Ω with boundary Γ which has the Dirichlet(Neumann) spectrum of Laplacian under the condition that $\frac{1}{2}\int_0^L \kappa(s)^2 ds$ takes the minimum, $Length(\Gamma) = L$ and $Area(\Omega) = M$. Further it has even number of axes of symmetry.

2) Under the above condition, if M is sufficiency close to $L^2/4\pi$,the domain is (essentially) unique, and Γ is oval with four vertices.

By using the numerical computation, Watanabe suggested the existence of non-convex curve for suitable M.

To prove Theorem A, K.Watanabe considered the following variational problem:

$$(VP)\begin{cases} For\ given\ L\ and\ M\ with\ L^2 - 4\pi M > 0,\ find\ \Omega\ so\ that \\ the\ functional\ \dfrac{1}{2}\displaystyle\int_0^L \kappa(s)^2 ds\ subject\ to\ \int_0^L \kappa(s)ds = 2\pi \\ takes\ the\ minimum, \end{cases}$$

where s is an arc-length parameter and $\kappa(s)$ is the curvature of Γ.
He derived the Euler-Lagrange Equation, and investigated properties of solutions of (VP).

Theorem B.[K.Watanabe] Suppose that $\kappa(s)$ is a solution of (VP), then $\kappa(s)$ satisfies following three properties:

1) $\kappa(s)$ is C^∞ function.

2) $\kappa(s)$ satisfies the following Euler-Lagrange equation:

$$(E)\begin{cases} \{\kappa_{ss} + \frac{1}{2}\kappa^3 + \tilde{\mu}\kappa\}_s = 0, \quad s \in [0, L], \\ \kappa(0) = \kappa(L),\ \kappa_s(0) = \kappa_s(L), \\ \int_0^L \kappa(s) ds = 2\pi, \\ \tilde{\mu} := \frac{1}{L^2 - 4\pi M}\left\{M\int_0^L \kappa(s)^3 ds - \frac{L}{2}\int_0^L \kappa(s)^2 ds\right\}. \end{cases}$$

3) There exists a natural number n such that $\kappa(s)$ is periodic function with period $s = L/n$, where n denotes the number of minimum (maximum) points of $\kappa(s)$. (We call this solution "$n - mode\ solution$".)

Our final aim is to investigate the the global structure of existence of solutions of (E) and the shape of solutions.

The first theorem is a non-existence result.

Theorem 1.1. *There does not exist 1-mode solution of* (E).

We prepare notations to show n-mode solution $\kappa(s; M, n)$ of (E). We define the complete elliptic integral of first, second and third kind by

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi,$$

$$\Pi(\nu, k) := \int_0^{\pi/2} \frac{d\varphi}{(1 + \nu \sin^2 \varphi)\sqrt{1 - k^2 \sin^2 \varphi}}.$$

We define the amplitude $\mathrm{am}(u, k)$ by

$$u = \int_0^{\mathrm{am}(u,k)} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

and Jacobi's elliptic function $\mathrm{cn}(u,k)$ by

$$\mathrm{cn}(u,k) := \cos(\mathrm{am}(u,k)).$$

Let us set a positive number m_n $(n = 2,3,4\cdots)$ by

$$m_n := \frac{\sqrt{7n^2-8n+2-2(2n-1)\sqrt{(3n-1)(n-1)}}}{4\pi(2n-1)(2n-1-\sqrt{(3n-1)(n-1)})}L^2.$$

Now we consider the case $n \geq 2$.

Theorem 1.2. *For $M \in [-m_n, L^2/4\pi)$, there exists an n-mode solution $\kappa(s;M,n)$ with property $\kappa(0) = \max_{0\leq s\leq L}\kappa(s)$. It is represented by*

$$\kappa(s;M,n) :=$$

$$2\frac{P\sqrt{Q^2+\delta}+Q\sqrt{P^2+\delta}+(P\sqrt{Q^2+\delta}-Q\sqrt{P^2+\delta})cn(\frac{4n}{L}K(k)s,k)}{\sqrt{P^2+\delta}+\sqrt{Q^2+\delta}-(\sqrt{P^2+\delta}-\sqrt{Q^2+\delta})cn(\frac{4n}{L}K(k)s,k)}$$
$$-\frac{P+Q}{2},$$

where (k,H) is the unique solution of $Z_{1n}(k,H) = 0$ and $M_1(k,H) = nM$ with $0 < k < 1$ and $2k^2-1 < H < 2$. Here

$$Z_{1n}(k,H) := \{H^2+8k^2(1-k^2)(H+2)+H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}\}$$
$$\Bigg\{\bigg((4k^2-3)H^2-8k^2(1-k^2)(H+8k^2-6)$$
$$-H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}\bigg)K(k)$$
$$+4\,(1-k^2)\Big(H^2+16k^2(1-k^2)\Big)\cdot$$
$$\Pi\left(\frac{2k^4(H-2)^2}{(1-2k^2)H^2+8k^4H+8k^2(1-2k^2)+H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}},k\right)\Bigg\}$$
$$-\frac{\sqrt{2}\pi}{n}k\sqrt{1-k^2}\sqrt{H^2+16k^2(1-k^2)}\{2\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}-H^2\}\cdot$$
$$\sqrt{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)+H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}},$$

$$M_1(k,H) := \sqrt{2}\sqrt{H^2+16k^2(1-k^2)}$$
$$\sqrt{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)+H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}}$$
$$\Big[\{k^2H^2-4(1-k^2)(H+1-4k^2)\}H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}K(k)$$
$$-\{H^2-4(1-2k^2)H-4(8k^4-8k^2+1)\}H\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}E(k)$$
$$+(1-k^2)\{(H+2)^2-8k^2H\}\{(1-k^2)H^2+4k^2H+4k^2(3-4k^2)\}K(k)$$
$$-\{(H-2)^2+8k^2H\}\{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)\}E(k)\Big]L^2$$
$$\Big/\Big[16k\sqrt{1-k^2}\{2\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}-H^2\}$$
$$\{(-3k^4+3k^2-1)H^4-8k^2(1-k^2)(1-2k^2)H^3-8k^2(1-k^2)(8k^4-8k^2+5)H^2$$
$$-32k^2(1-k^2)(1-2k^2)H-16k^2(1-k^2)\}K(k)^2\Big]\ ,$$

$$P :=$$
$$8\sqrt{2}k\sqrt{1-k^2}\Big\{8k^2(1-k^2)+\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}\Big\}nK(k)$$
$$\Big/\Big\{\sqrt{H^2+16k^2(1-k^2)}\cdot$$
$$\sqrt{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)+H\sqrt{H^2+16k^2(1-k^2)}}L\Big\},$$

$$Q :=$$
$$8\sqrt{2}k\sqrt{1-k^2}\{-H^2-8k^2(1-k^2)+\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}\}nK(k)$$
$$\Big/\Big\{\sqrt{H^2+16k^2(1-k^2)}\cdot$$
$$\sqrt{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)+H\sqrt{H^2+16k^2(1-k^2)}}L\Big\},$$

and

$$\delta := 8\Big\{(1-2k^2)H^2+8k^2(1-k^2)(H+2-4k^2)$$
$$+\sqrt{(1-2k^2)^2H^2+16k^2(1-k^2)}\Big\}n^2K(k)^2\Big/\Big\{\Big(H^2+16k^2(1-k^2)\Big)L^2\Big\}.$$

Theorem 1.3. *For any $M \in (-L^2/4\pi, -m_n)$, there exists an n-mode solution $\kappa(s; M, n)$ with property $\kappa(0) = \max_{0\le s\le L} \kappa(s)$. It is represented by*

$$\begin{aligned}
&\kappa(s; M, n) \\
&:= \Big\{ -2(1-k^2)(A-B)(A-\eta)(B-\eta)\Big\} \\
&/\Big[\{(A-\eta)k^2 - (A-B)\}\cdot \\
&\quad \Big\{\{(A-\eta)k^2 - (A-B)\}cn(\tfrac{2n}{L}K(k)s, k)^2 + (1-k^2)(A-\tilde{\delta})\Big\}\Big] \\
&+\{2(A-\eta)Bk^2 - 2(A-B)\eta\}/\{(A-\eta)k^2 - (A-B)\} - (A+B)/2,
\end{aligned}$$

where (k, h) be the unique solution of $Z_{2n}(k, h) = 0$, and $M_2(k, h) = nM$ with $0 < k < 1$ and $0 < h < 1$. Here

$$\begin{aligned}
&Z_{2n}(k, h) := \\
&\{(2-k^2)h^2 - 2(2+3k^2)h + 8k^2 - (2-h)\sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}\}K(k) \\
&- 4(1-h)\{2k^2 + hk^2 - 2h - \sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}\}\cdot \\
&\Pi\left(\frac{1}{4}k^2h - \frac{1}{2}k^2 - \frac{1}{2}h - \frac{1}{4}\sqrt{(2-h)^2k^4 + 4(1-k^2)}, k\right) \\
&- \frac{2\pi}{n}\sqrt{h}\sqrt{1-h}\sqrt{(2-k^2)h + \sqrt{(2-h)^2k^4 + 4(1-k^2)}},
\end{aligned}$$

$$\begin{aligned}
&M_2(k, h) := -\sqrt{h}\sqrt{1-h}\sqrt{(2-k^2)h + \sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}} \\
&[\{(2-k^2)h + \sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}\}(2-h)^2K(k) - 16h(1-h)E(k)]\,L^2 \\
&/[\,2(2-h)\{(2-k^2)h^2 + 4k^2(1-h) + h\sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}\} \\
&\{(2-k^2)h^2 - 4k^2(1-h) + h\sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}\}K(k)^2],
\end{aligned}$$

$$A := \frac{\sqrt{(2-k^2)h + \sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}}}{2\sqrt{h}\sqrt{1-h}}\left(\frac{4n}{L}K(k)\right),$$

$$B := \frac{\sqrt{1-h}\sqrt{(2-k^2)h + \sqrt{(2-h)^2k^4 + 4(1-k^2)h^2}}}{2\sqrt{h}}\left(\frac{4n}{L}K(k)\right),$$

and

$$\eta := \frac{k^2\sqrt{1-h}}{\sqrt{(2-k^2)h+\sqrt{(2-h)^2k^4+4(1-k^2)h^2}}}\left(\frac{4n}{L}K(k)\right).$$

As for the shape of curves, we obtain the following theorem:

Theorem 1.4. *There exist an unique positive constant $A_n(L)$ with $0 < A_n(L) < L^2/4\pi$ such that , the curve which is generated by the solution $\kappa(s; M, n)$ appeared in theorem 1.2,1.3, is oval for $M \in [A_n(L), L^2/4\pi)$ and it is not oval for $M \in (-L^2/4\pi, A_n(L))$. Moreover,*

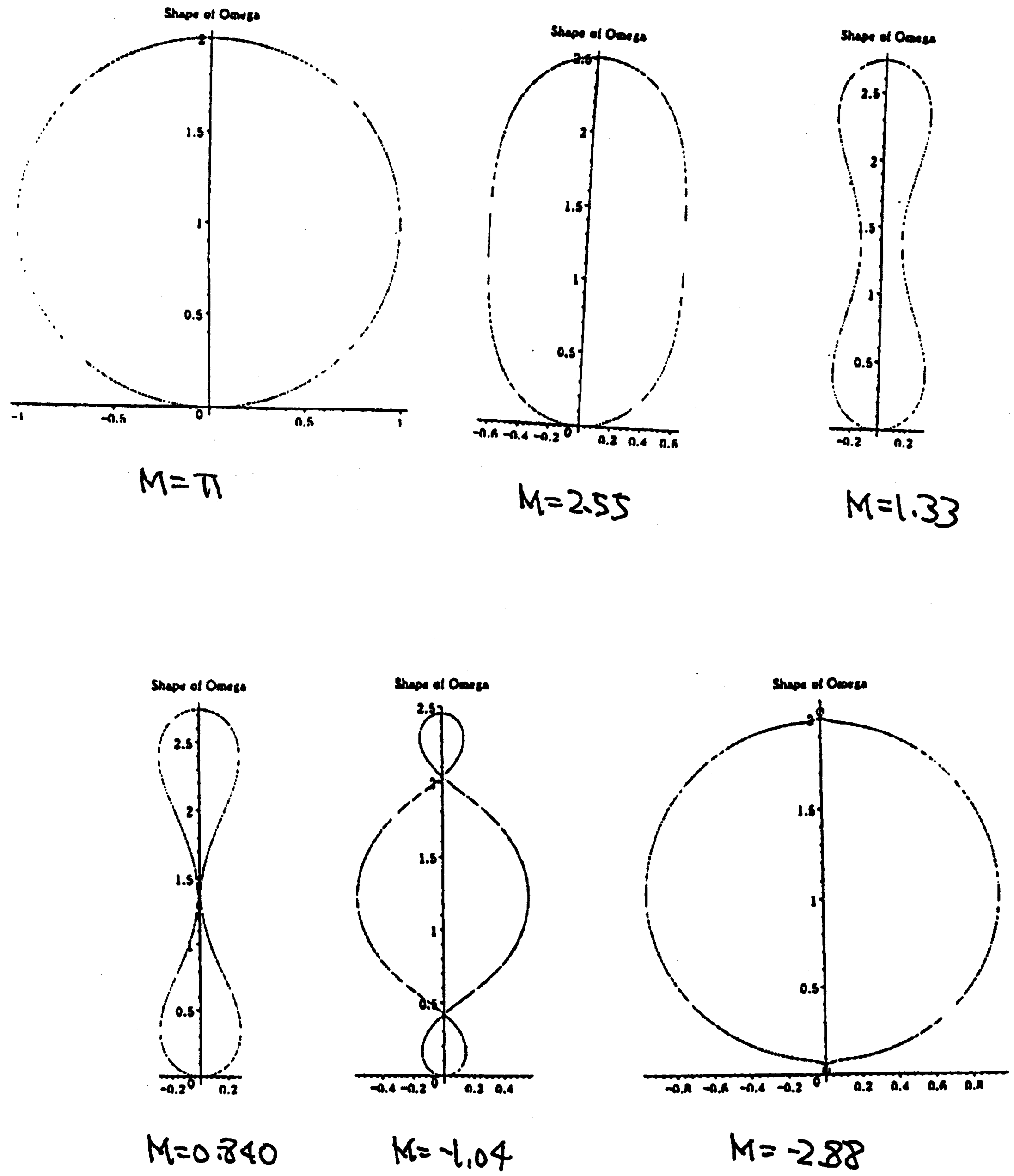
$$A_n(L) := \frac{1}{n}M_1\left(k_n, \frac{2}{3}\sqrt{16k_n^4 - 16k_n^2 + 1}\right),$$

where k_n is an unique solution of

$$Z_{1n}\left(k, \frac{2}{3}\sqrt{16k^4 - 16k^2 + 1}\right) = 0.$$

We show the curves corresponding to 2-mode solution in the case of $L = 2\pi$.

Curve by 2-mode solution in the case of $L = 2\pi$.



2 Outline of proof of Theorem1.2.

We show the idea of proof of Theorem 1.2, since the essential ideas already appear in this case.

To prove theorem 1.2, we rewrite the original problem to the following problem:

$$(E')\begin{cases} \left\{\kappa_{ss}+\frac{1}{2}\kappa^3+\mu\kappa\right\}_s=0, & s\in[0,L], \\ \kappa(0)=\kappa(L),\ \kappa_s(0)=\kappa_s(L), & \end{cases}$$

with conditions

$$\int_0^L \kappa(s)ds-2\pi=0,$$

and

$$\frac{\mu L^2+\frac{L}{2}\int_0^L\kappa(s)^2ds}{4\pi\mu+n\int_0^L\kappa(s)^3ds}=M,$$

where μ is constant. By noting theorem C, we can rewrite the above problem to the following equivalent problem:

$$(E_n)\begin{cases} \left\{\kappa_{ss}+\frac{1}{2}\kappa^3+\mu\kappa\right\}_s=0, & s\in\left[0,\frac{L}{2n}\right] \\ \kappa_s(0)=\kappa_s\left(\frac{L}{2n}\right)=0, & \\ \kappa_s(s)<0 & s\in\left(0,\frac{L}{2n}\right). \end{cases}$$

with conditions

$$\int_0^{L/2n}\kappa(s)ds=\pi/n, \tag{2.1}$$

and

$$\frac{\mu L^2+nL\int_0^{L/2n}\kappa(s)^2ds}{4\pi\mu+2n\int_0^{L/2n}\kappa(s)^3ds}=M. \tag{2.2}$$

For $L>0$ given, we will represent all solution $(\kappa(s),\mu)$ of (E_n) and adjust μ so that (2.1) and (2.2) are satisfied. This kind of method was first proposed by Lou-Ni-Yotsutani[6]. Later, Ikeda-Kondo-Okamoto-Yotsutani [4] and Kosugi-Morita-Yotsutani [5] developed the method. In our problem, the nonlinear term κ^3 is included. Thus the arguments become terribly complicated and new devices are needed.

Integrating (E_n), we get

$$\kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa + C_1 = 0, \qquad s \in \left[0, \frac{L}{2n}\right],$$

where C_1 is constant. Multiplying $2\kappa_s$, we have

$$\frac{d}{ds}\left\{\kappa_s(s)^2 + \frac{1}{4}\kappa(s)^4 + \mu\kappa(s)^2 + 2C_1\kappa(s)\right\} = 0,$$

which implies

$$\kappa_s(s)^2 + \frac{1}{4}\kappa(s)^4 + \mu\kappa(s)^2 + 2C_1\kappa(s) + C_2 = 0. \tag{2.3}$$

where C_2 is constant.

Let us set $p := \kappa(0), q := \kappa(L/2n)$. By the Neumann boundary condition of (E_n), we can rewrite (2.3) as follows:

$$\frac{d\kappa}{ds} = -\frac{1}{2}\sqrt{(p-\kappa)(\kappa-q)\left\{\left(\kappa + \frac{p+q}{2}\right)^2 + 4\delta\right\}}, \tag{2.4}$$

where δ is constant.

We note that $\delta \geq 0$ and $\delta < 0$ corresponding to Theorem 1.2 and 1.3 respectively. Now we assume $\delta \geq 0$.

Let us set $\hat{\kappa} := \frac{1}{2}\left(\kappa + \frac{p+q}{2}\right)$, $P := \frac{3p+q}{4}, Q := \frac{p+3q}{4}$. (2.4) is rewritten as follows:

$$\frac{d\hat{\kappa}}{ds} = -\sqrt{(P-\hat{\kappa})(\hat{\kappa}-Q)(\hat{\kappa}^2+\delta)}.$$

Thus we get

$$s = \int_{\hat{\kappa}}^{P} \frac{d\xi}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}}. \tag{2.5}$$

We introduce change of variables from ξ to ϕ in the right hand side of (2.5):

$$\begin{aligned} \xi &:= Q + \frac{1}{\eta} \\ \eta &:= \frac{1}{P-Q} + \frac{1}{P-Q}\sqrt{\frac{P^2+\delta}{Q^2+\delta}}\tan^2\frac{\phi}{2}. \end{aligned} \tag{2.6}$$

Then we have
(2.7)

$$s=\begin{cases}\dfrac{1}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}}\displaystyle\int_0^{\phi(\hat\kappa)}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, & s\in\left[0,\dfrac{L}{4n}\right]\\ \dfrac{1}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}}\left(2\mathrm{K}(k)-\displaystyle\int_0^{\pi-\phi(\hat\kappa)}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}\right), & s\in\left[\dfrac{L}{4n},\dfrac{L}{2n}\right],\end{cases}$$

where

$$\tan^2\frac{\phi(\hat\kappa)}{2}=\frac{P-\hat\kappa}{\hat\kappa-Q}\sqrt{\frac{Q^2+\delta}{P^2+\delta}},\qquad k^2=\frac{1}{2}\left(1-\frac{PQ+\delta}{\sqrt{(P^2+\delta)(Q^2+\delta)}}\right).$$

At $s=\dfrac{L}{2n}$, we see from (2.7) that

$$\sqrt[4]{(P^2+\delta)(Q^2+\delta)}=\frac{4n}{L}\mathrm{K}(k). \tag{2.8}$$

Now we introduce change of variables from (k,h) to (P,Q) by

$$\begin{cases}k^2=\dfrac{1}{2}\left(1-\dfrac{PQ+\delta}{\sqrt{(P^2+\delta)(Q^2+\delta)}}\right),\\ \sqrt[4]{(P^2+\delta)(Q^2+\delta)}=\dfrac{4n}{L}\mathrm{K}(k),\\ Q=(1-h)P, \qquad (h>0),\end{cases}$$

which implies

$$\begin{cases}P=\dfrac{8n\sqrt{2}\sqrt{k}\sqrt{1-k^2}\mathrm{K}(k)}{L\sqrt{h}\sqrt{(1-2k^2)h+\sqrt{h^2-4k^2(1-k^2)(2-h)^2}}},\\ Q=\dfrac{8n\sqrt{2}(1-h)\sqrt{k}\sqrt{1-k^2}\mathrm{K}(k)}{L\sqrt{h}\sqrt{(1-2k^2)h+\sqrt{h^2-4k^2(1-k^2)(2-h)^2}}},\end{cases} \tag{2.9}$$

$$\text{in}\ \ \{(k,h)\,|\,0<k<1/\sqrt{2},\ D(k)\le h<2\}$$
$$\cup\ \{(k,h)\,|\,1/\sqrt{2}\le k<1,\ 2k^2-1<h<2\}$$

and

$$
(2.10) \quad \begin{cases} P = \dfrac{8n\sqrt{2}\sqrt{k}\sqrt{1-k^2}\mathrm{K}(k)}{L\sqrt{h}\sqrt{(1-2k^2)h - \sqrt{h^2-4k^2(1-k^2)(2-h)^2}}}, \\[2ex] Q = \dfrac{8n\sqrt{2}(1-h)\sqrt{k}\sqrt{1-k^2}\mathrm{K}(k)}{L\sqrt{h}\sqrt{(1-2k^2)h - \sqrt{h^2-4k^2(1-k^2)(2-h)^2}}}, \end{cases}
$$

$$
\text{in} \quad \{(k,h) |\ 0 < k < 1/\sqrt{2},\ D(k) \le h < 1\},
$$

where

$$
D(k) := \frac{4k\sqrt{1-k^2}}{1+2k\sqrt{1-k^2}}.
$$

Further, we introduce change of variables from (k,h) to (k,H) by

$$
H := \begin{cases} \sqrt{h^2-4k^2(1-k^2)(2-h)^2}, & \textit{for the case } (2.9), \\ -\sqrt{h^2-4k^2(1-k^2)(2-h)^2}, & \textit{for the case } (2.10). \end{cases}
$$

Consequently, we obtain

(2.11)

$$
\begin{cases} P = \\ 8\sqrt{2}k\sqrt{1-k^2}\{8k^2(1-k^2) + \sqrt{(1-2k^2)^2H^2 + 16k^2(1-k^2)}\}n\mathrm{K}(k) \\ \Big/\Big[\sqrt{H^2+16k^2(1-k^2)} \\ \{(1-2k^2)H^2 + 8k^2(1-k^2)(H+2-4k^2) + H\sqrt{H^2+16k^2(1-k^2)}\}^{1/2}L\ \Big], \\ Q = \\ 8\sqrt{2}k\sqrt{1-k^2}\{-H^2 - 8k^2(1-k^2) + \sqrt{(1-2k^2)^2H^2 + 16k^2(1-k^2)}\}n\mathrm{K}(k) \\ /[\sqrt{H^2+16k^2(1-k^2)} \\ \{(1-2k^2)H^2 + 8k^2(1-k^2)(H+2-4k^2) + H\sqrt{H^2+16k^2(1-k^2)}\}^{1/2}L\ ], \end{cases}
$$

$$
(k,H) \in \{(k,H) | 0 < k < 1, 2k^2 - 1 < H < 2\}.
$$

We write (2.1),(2.2) by (P,Q,δ):

$$
\int_0^{L/2n} \kappa(s)ds \ = \ \int_Q^P \frac{2\left(\xi - \frac{P+Q}{4}\right)}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}} d\xi
$$

(2.12)

$$= \left[\frac{4(PQ+\delta)\{2\sqrt{(P^2+\delta)(Q^2+\delta)}+P^2+Q^2+2\delta\}}{(P-Q)^2(P+Q)} \right.$$

$$\left. -\frac{\{2\sqrt{(P^2+\delta)(Q^2+\delta)}+P^2+Q^2+2\delta\}^2}{(P-Q)^2(P+Q)} \right] \frac{\mathrm{K}(k)}{\sqrt[4]{(P^2+\delta)(Q^2+\delta))}}$$

$$+2\frac{\{2\sqrt{(P^2+\delta)(Q^2+\delta)}+P^2+Q^2+2\delta\}}{(P+Q)\sqrt[4]{(P^2+\delta)(Q^2+\delta)}}\cdot$$

$$\Pi\left(-\frac{1}{4}\left(2-\frac{P^2+Q^2+2\delta}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}}\right),k\right),$$

$$\begin{aligned}
\int_0^{L/2n} \kappa(s)^2 ds &= \int_Q^P \frac{4\left(\xi-\frac{P+Q}{4}\right)^2}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}} d\xi \\
&= \left(\frac{1}{2}\frac{P^2+Q^2-6PQ-8\delta}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}} - 4\sqrt[4]{(P^2+\delta)(Q^2+\delta)}\right)\mathrm{K}(k) \\
&\quad +8\sqrt[4]{(P^2+\delta)(Q^2+\delta)}\mathrm{E}(k), \\
\int_0^{L/2n} \kappa(s)^3 ds &= \frac{1}{4}(3P^2-2PQ+3Q^2-8\delta)\int_Q^P \frac{2\left(\xi-\frac{P+Q}{4}\right)}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}} d\xi \\
&\quad +\frac{1}{4}(P+Q)\{(P-Q)^2+4\delta\}\int_Q^P \frac{1}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}} d\xi.
\end{aligned}$$

We see from (2.1) and (2.8) that

$$\int_0^{L/2n} \kappa(s)^3 ds = \frac{\pi}{4n}(3P^2-2PQ+3Q^2-8\delta)+\frac{L}{8n}(P+Q)\{(P-Q)^2+4\delta\}.$$

On the other hand, it follows from (2.3), (2.4) that

$$\mu = -\frac{1}{8}(3P^2-2PQ+3Q^2-8\delta).$$

Thus (2.2) can be expressed

$$\frac{\mu L^2 + nL\int_0^{L/(2n)} \kappa(s)^2 ds}{4\pi\mu + 2n\int_0^{L/(2n)} \kappa(s)^3 ds}$$

$$(2.13)\qquad \begin{aligned} &= \Big[\, 4n(P^2 - 6PQ + Q^2 - 8\delta - 8\sqrt{(P^2+\delta)(Q^2+\delta)}) \\ &\quad -(3P^2 - 2PQ + 3Q^2 - 8\delta)\ {}^4\sqrt{(P^2+\delta)(Q^2+\delta)}L \\ &\quad +64n\sqrt{(P^2+\delta)(Q^2+\delta)}\mathrm{E}(k)\Big] \\ &/ \Big[2(P+Q)\{(P-Q)^2+4\delta\}\ {}^4\sqrt{(P^2+\delta)(Q^2+\delta)}\Big]. \end{aligned}$$

Substituting (2.11) to (2.12)and (2.13), we obtain $Z_{1n}(k,H)$ and $M_1(k,H)$.

Next we express $\kappa(s)$ with P, Q and δ. By (2.7), we have

$$\phi(\hat{\kappa}) = \mathrm{am}\left(\frac{4n}{L}\mathrm{K}(k)s, k\right) \quad s \in \left[0, \frac{L}{4n}\right]$$

Hence we get

$$\begin{aligned} \mathrm{cn}\left(\tfrac{4n}{L}\mathrm{K}(k)s,k\right) &= \cos\phi(\hat{\kappa}) \\ &= \frac{1-\tan^2\frac{\phi(\hat{\kappa})}{2}}{1+\tan^2\frac{\phi(\hat{\kappa})}{2}} \\ &= \frac{\sqrt{P^2+\delta}(\hat{\kappa}-Q) - \sqrt{Q^2+\delta}(\hat{\kappa}-P)}{\sqrt{P^2+\delta}(\hat{\kappa}-Q) + \sqrt{Q^2+\delta}(\hat{\kappa}-P)}, \quad s \in \left[0, \frac{L}{4n}\right]. \end{aligned}$$

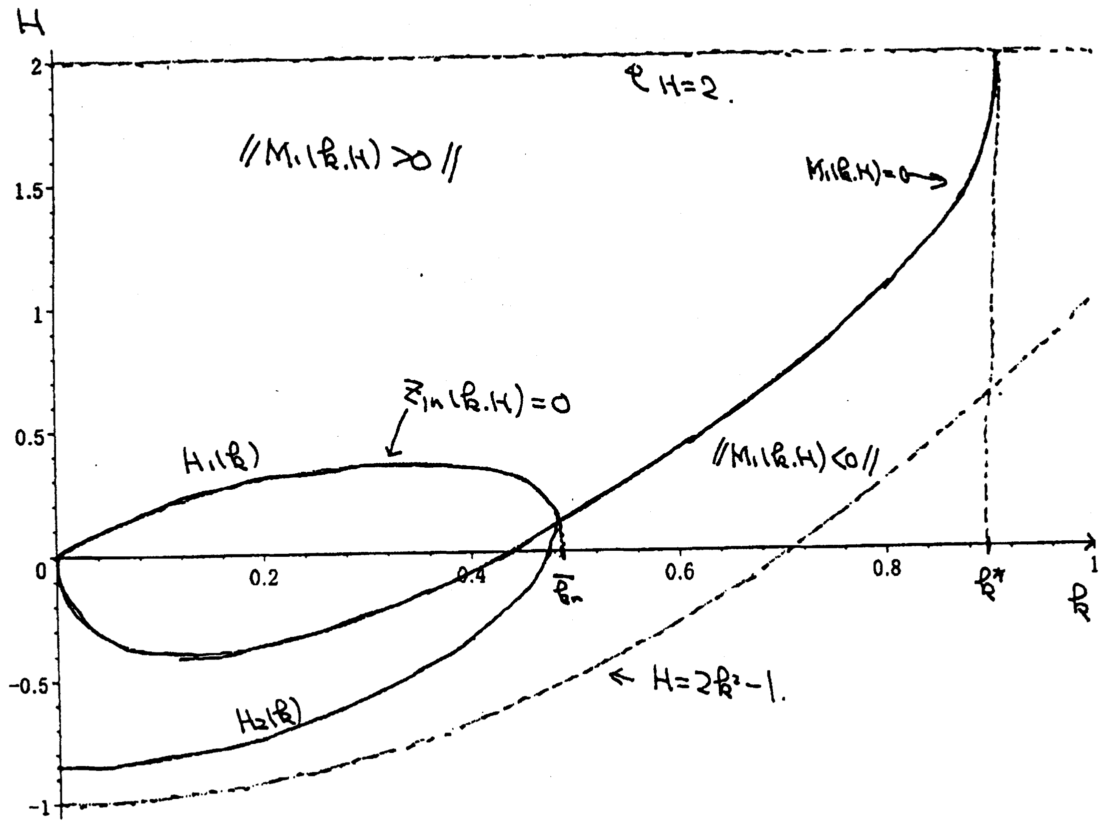
In the same way, we have

$$\mathrm{cn}\left(\frac{4n}{L}\mathrm{K}(k)s,k\right) = \frac{\sqrt{P^2+\delta}(\hat{\kappa}-Q) - \sqrt{Q^2+\delta}(\hat{\kappa}-P)}{\sqrt{P^2+\delta}(\hat{\kappa}-Q) + \sqrt{Q^2+\delta}(\hat{\kappa}-P)}, \quad s \in \left[\frac{L}{4n}, \frac{L}{2n}\right].$$

Therefore we obtain

$$\begin{aligned} \kappa(s) &= -\frac{P+Q}{2} \\ &+2\frac{P\sqrt{Q^2+\delta} + Q\sqrt{P^2+\delta} + (P\sqrt{Q^2+\delta} - Q\sqrt{P^2+\delta})\mathrm{cn}(\frac{4n}{L}\mathrm{K}(k)s,k)}{\sqrt{P^2+\delta} + \sqrt{Q^2+\delta} - (\sqrt{P^2+\delta} - \sqrt{Q^2+\delta})\mathrm{cn}(\frac{4n}{L}\mathrm{K}(k)s,k)}, \\ &\qquad s \in \left[0, \frac{L}{2n}\right]. \end{aligned}$$

We show the level curves of $M_1(k,H)=0$ and $Z_{1n}(k,H)=0$ with $n=2$ in the following figure.



Let us set

$$\Sigma := \{(k,H)\,|0<k<1, 2k^2-1\le H<2\}.$$

Let k^* be the unique solution of $2\mathrm{E}(k)-\mathrm{K}(k)=0$ in $(0,1)$.
The following Propositions hold.

Proposition 2.1. *There exists the unique smooth function* $\tilde{H}(k)$ *such that* $M_1(k,\tilde{H}(k))\equiv 0$ *in* $(0,k^*]$.

We see that there exists unique $\bar{k}_n \in (0, k^*)$ such that $Z_{1n}(k, \tilde{H}(k)) \equiv 0$.

Proposition 2.2. *There exist the unique smooth function* $H_1(k)$ *such that* $Z_{1n}(k, H_1(k)) \equiv 0$ *in* $(0, \bar{k}_n)$.

Proposition 2.3. $M_1(k, H_1(k))$ *is a decreasing function in* $(0, \bar{k}_n)$. *Especially* $\frac{1}{n} M_1(0,0) = \frac{L^2}{4\pi}$ *and* $M_1(\bar{k}_n, H_1(\bar{k}_n)) = 0$.

Proposition 2.4. *There exists the unique smooth function* $H_2(k)$ *such that* $Z_{1n}(k, H_2(k)) \equiv 0$ *in* $(0, \bar{k}_n)$

Proposition 2.5. $M_1(k, H_2(k))$ *is a increasing function in* $(0, \bar{k}_n)$ *and* $H_2(\bar{k}_n) = H_1(\bar{k}_n)$.

By using Proposition 2.1 - 2.5, we complete Theorem1.2.

References

[1] M.Kac, Can one hear the shape of a drum, Amer.Math.Mon.**73**(1966),1-23.

[2] K.Watanabe, Plane domains which are spectrally determined,Ann.Global Anal.Geom.**18**(2000),447-475.

[3] K.Watanabe, Plane domains which are spectrally determined II, J.of Ineq.and Appl.**7** (2002),25-47.

[4] H.Ikeda, K.Kondo, H.Okamoto and S.Yotsutani, On the global branches of the solutions to a nonlocal boundary-value problem arising in Ossen's spiral flows, Commun. Pure. Appl. Anal., **3**(2003), 381-390.

[5] S.Kosugi, Y.Morita and S.Yotsutani, A complete bifurcation diagram of the Ginzburg-Landau equation with periodic boundary condition, preprint.

[6] Y.Lou,W.-M.Ni and S.Yotsutani, On a Limiting System in the Lotka-Voltera Competition with Cross-Diffusion, Discrete Contin. Dyn.Syst. **10**(2004), 435-458.