数理解析研究所講究録 1358 巻 2004 年 24-33

Existence of Global Solutions for the Shigesada-Kawasaki-Teramoto Model with Cross-Diffusion [1]

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1 SKT model

This lecture is concerned with the initial boundary value problem for the following parabolic system with strongly coupled nonlinear diffusion

$$
\text{(P)} \quad \begin{cases} u_t = d_1\Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv) & \\ & \text{in } \Omega\times(0,\infty), \\ v_t = d_2\Delta[(1+\delta v)v] + bv(1-du-v) & \\ & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, Δ is the Laplacian, $d_1, d_2, a, b, c, d, \alpha, \gamma$ are positive constants, δ is a nonnegative constant, $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$ and u_0, v_0 are given nonnegative functions. In (P), α is called a cross-diffusion coefficient and γ, δ are called self-diffusion coefficients.

The above system was first introduced by Shigesada, Kawasaki and Teramoto [17] to describe the habitat segregation phenomena between two species which are competing in the same domain. Their model (SKT model) is described by the following system of parabolic equations:

$$
\begin{cases} u_t = d_1\Delta[(1+\rho_{11}u+\rho_{12}v)u] + au(1-u-cv), \\ v_t = d_2\Delta[(1+\rho_{21}u+\rho_{22}v)v] + bv(1-du-v), \end{cases} \tag{1.1}
$$

in full generality with homogeneous Neumann boundary conditions. In (1.1), u, v denote the population densities of two species, ρ_{11}, ρ_{22} are coefficients of self-diffusion and ρ_{12}, ρ_{21} are coefficients of cross-diffusion. Since the numerical simulations for (1.1) exhibit interesting pattern formations, the SKT model has attracted interests of many mathematicians.

[1] This is a joint work with Y.S. Choi (University of Connecticut) and R. Lui (Worcester Polytechnic Institute).

Mathematically, one of the most important problem for (1.1) is to establish the existence of global solutions. After Kim [8] showed the global existence in the one dimensional case, (1.1) and related systems have been discussed by a lot of mathematicians. However, the analysis is very hard because of the nonlinear diffusivity and the global existence for (1.1) is still an open problem for the full system. In case $\rho_{11} = \rho_{21} = \rho_{22} = 0$, the global existence result was shown without any restrictions on space dimensions and initial functions by Pozio-Tesei [15], Yamada [19] and Redlinger [16]. But their results are not valid for (1.1) because some restrictions are required for the reaction term; so that the standard reaction term like Lotka-Volterra type is excluded in their works. On the other hand, we have to put some restrictions on nonlinear diffusion coefficients in order to study the Lotka-Volterra reaction-term. In this direction, we refer to Yagi[18] or Ichikawa-Yamada[6], where it is assumed that self-diffusion coefficients are dominant over cross-diffusion coefficients in a sense.

In what follows, we will focus on the global solvability for (P), which is slightly simpler because the second equation does not contain a cross-diffusion term. In case $N = 2$, Yagi [18] proved that (P) has a unique global solution if $\alpha > 0, \gamma > 0$ and $\delta = 0$. This result has been extended by Lou, Ni and Wu [12] to the case where $N = 2, \alpha > 0, \gamma \geq 0$ and $\delta \geq 0$. Our purpose is to establish a sufficient condition for the existence of global solutions for (P) without any restrictions on the amplitude of initial data in the higher dimensional case ($N \geq 3$). We will prove two global existence results: Theorem 1 in case $\delta = 0$ and Theorem 2 in case $\delta > 0$ and $N \leq 5$. See the work of Choi, Lui and Yamada [3, 4]. Roughly speaking these theorems assert that (P) admits a unique global classical solution for any nonnegative smooth initial functions. Here we should say that similar global existence results are obtained by Le, L. V. Nguyen and T. T. Nguyen [5] via a different approach.

Finally, we will give some comments on the stationary problem associated with (P) or (1.1). Consider the following elliptic system:

$$\begin{cases} \Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u+\delta v)v] + bv(1-du-v) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

What we should do is to look for non-constant positive solutions for (1.2). In case $N = 1$, Mimura, Nishiura, Tesei and Tsujikawa [14] discussed non-constant positive solutions by singular perturbation method. See also Kan-on [7], where the stability of such non-constant solutions are studied. As in the non-stationary problem, the analysis of (1.2) for the higher dimensional case is difficult. To overcome the difficulty, Lou and Ni [10, 11] have proposed a kind of limiting system which can be derived by letting one of cross-diffusion coefficients to infinity. In this direction, we also refer to a recent work of Lou, Ni and Yotsutani [13], where the analysis of the limiting system is accomplished in case $N = 1$.

2 Global Existence Results

We will discuss (P) in the framework of classical solutions. So u_0 and v_0 are assumed to satisfy

$$\text{(A)} \qquad u_0 \geq 0, v_0 \geq 0 \quad \text{and} \quad u_0, v_0 \in C^{2+\lambda}(\overline{\Omega}) \quad \text{with} \quad \lambda > 0.$$

In what follows, we always assume

$$N \geq 2, \quad \alpha > 0 \quad \text{and} \quad \gamma > 0.$$

The first global result is concerned with the case where the diffusion in the second equation of (P) is linear.

Theorem 2.1. *For* $\delta = 0$, *assume that* u_0, v_0 *satisfy* (A). *Then* (P) *admits a unique solution* $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0, \infty))$.

The second result is concerned with the case where the diffusion in the second equation is nonlinear.

Theorem 2.2. *Let* $\delta > 0$ *and* $N \leq 5$. *If* u_0, v_0 *satisfy* (A), *then* (P) *admits a unique solution* $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0, \infty)$.

Remark. In [5], the same restriction $N \leq 5$ is also imposed to derive the global existence result.

Although complete proofs of these theorems are stated in our work [4] (see also [3]), we will briefly explain the idea of the essential parts of the proofs.

First of all, we will prepare two local existence results for (P).

Theorem 2.3. [1, 2] *If* $u_0, v_0 \in W_p^1(\Omega)$ *with* $p > N$, *then* (P) *admits a unique solution* u, v *in* $C([0,T); W_p^1(\Omega)) \cap C((0,T); W_p^2(\Omega)) \cap C^1((0,T); L_p(\Omega))$, *where* T *is a maximal existence time.*

Theorem [2] is valid if we work in the framework of $L^p(\Omega)$ spaces. If classical solutions of (P) are concerned, then we have to use the following result (see [9]):

Theorem 2.4. *Assume that* u_0, v_0 *satisfy* (A). *Then* (P) *possesses a unique solution* u, v *in* $C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0, T])$ *with some* $T > 0$.

By virtue of Theorems 2.3 and 2.4, it is sufficient to show some suitable a priori estimates of u, v in order to establish the global existence. We will explain how to derive such a priori estimates in the subsequent sections.

3 A priori Estimates

We begin with the following lemma.

Lemma 3.1. *Let u, v be a solution of* (P) *in* $[0, T]$. *Then*

$$u \geq 0 \quad \textit{and} \quad m \geq v \geq 0 \quad \textit{in} \ Q_T,$$

where $Q_T = \overline{\Omega} \times [0, T]$ *and* $m = \max\{1, \| v_0 \|_\infty\}$.

Proof. The first equation in (P) is expressed as

$$u_t = d_1(1 + \alpha v + 2\gamma u)\Delta u + 2d_1(\alpha \nabla v + \gamma \nabla u) \cdot \nabla u + \{\alpha d_1 \Delta v + a(1 - u - cv)\}u \tag{3.1}$$

and the second one is written as

$$v_t = d_2(1 + 2\delta v)\Delta v + 2\delta d_2 \nabla v \cdot \nabla v + b(1 - du - v)v. \tag{3.2}$$

Then application of the maximum principle for (3.1) and (3.2) yields the nonnegativity of u and v. Applying the maximum principle to (3.2) again one can also show the boundedness of v. □

Lemma 3.2. *Let u, v be a solution of* (P) *in* $[0, T]$. *Then*

$$\sup_{0 \leq t \leq T} \| u(t) \|_{L^1(\Omega)} \leq C_T \quad \textit{and} \quad \| u \|_{L^2(Q_T)} \leq C_T$$

with some $C_T > 0$.

Proof. Integration of the first equation in (P) with respect to x gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega u \, dx &= d_1 \int_\Omega \Delta[(1 + \alpha v + \gamma u)u]dx + a\int_\Omega (1 - u - cv)u \, dx \\ &= d_1 \int_{\partial\Omega} \frac{\partial}{\partial \nu}[(1 + \alpha v + \gamma u)u]d\sigma + a\int_\Omega (1 - u - cv)u \, dx \\ &\leq a\int_\Omega u \, dx - a\int_\Omega u^2 \, dx. \end{aligned}$$

Hence Gronwall's inequality yields

$$\| u(t) \|_{L^1} + a\int_0^t \| u(s) \|_{L^2}^2 \, ds \leq \| u_0 \|_{L^1} + a\int_0^t \| u(s) \|_{L^1} \, ds \leq \| u_0 \|_{L^1} e^{at}.$$

□

We are now ready to prove the following fundamental L^q estimates.

Proposition 3.3. *Let u, v be a solution of* (P) *in* $[0,T]$. *If* $\delta = 0$, *then*

$$\| u \|_{L^q(Q_T)} \leq C_T \quad \textit{for any} \quad q > 1$$

and, if $\delta > 0$, *then*

$$\| u \|_{L^q(Q_T)} \leq C_T \qquad \textit{for any} \quad 1 < q < \frac{2(N+1)}{N-2}.$$

Moreover, $\| \nabla u \|_{L^2(Q_T)} \leq C_T$.

Proposition 3.3 plays a very important role in the proofs of Theorems 2.1 and 2.2. We will briefly explain the procedure to accomplish the proof in case $\delta = 0$. The proof in case $\delta > 0$ can be carried out in a similar manner with some modification. The complete proofs can be found in [4].

(i) L^q estimates of v_t and Δv.

For $\delta = 0$, (3.2) is written as

$$v_t = d_2 \Delta v + bv(1 - du - v). \tag{3.3}$$

Since $f := bv(1 - du - v) \in L^q(Q_T)$ by Lemma 3.1 and Proposition 3.3, the maximal regularity result for (3.3) yields $L^q(Q_T)$ estimates of v_t and Δv.

(ii) Hölder continuity of v and ∇v.

Since the estimates of (i) imply $v \in W_q^{2,1}(Q_T)$, the embedding theorem ([9]) assures the Hölder continuity of v and ∇v with respect to $(x,t) \in Q_T$.

(iii) L^∞ estimate of u.

The idea is to write (3.1) as a linear parabolic equation in the divergence form:

$$u_t = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (a_i(x,t)u) + b(x,t)u, \tag{3.4}$$

where

$$a_{ij} = d_1(1 + \alpha v + 2\gamma u)\delta_{ij}, \quad a_i = d_1 \alpha \frac{\partial v}{\partial x_i} \quad \text{and} \quad b = a(1 - u - cv).$$

Since u can be regarded as a generalized solution of (3.4), one can apply the maximum principle in [9, p.181] to get $L^\infty(Q_T)$ boundedness of u

(iv) Hölder continuity of u.

By (ii) and (iii), all a_{ij}, a_i and b appearing in (3.4) are bounded functions. Therefore, using the regularity theory for a weak solution of (3.4) one can derive the Hölder continuity of u with respect $(x,t) \in Q_T$.

(v) Hölder continuity of v_t and Δv.

We go back to (3.3), where $f = bv(1-du-v)$ is Hölder continuous with respect x, t by (ii) and (iv). Hence the famous Schauder estimate implies the Hölder continuity of v_t and Δv for $(x,t) \in Q_T$.

(vi) Hölder continuity of u_t and Δu.

By (3.1), u satisfies

$$u_t = d_1(1+\alpha v + 2\gamma u)\Delta u + 2d_1 \nabla(\alpha v + \gamma u)\cdot \nabla u + b^* u,$$

where $b^* = d_1 \alpha \Delta v + a(1-u-cv)$. Since all the coefficients are Hölder continuous for $(x,t) \in Q_T$, the Hölder continuity of u_t and Δu comes from the Schauder estimate.

4 Proof of Proposition 3.3

We will give the proof of Proposition 3.3 in case $\delta = 0$. For the proof in case $\delta > 0$, see [4].

We first multiply the first equation in (P) by u^{q-1}:

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega u^q\,dx &= \int_\Omega u^{q-1}u_t\,dx \\
&= d_1 \int_\Omega u^{q-1}\nabla[(1+\alpha v + 2\gamma u)\nabla u]\,dx + d_1\alpha\int_\Omega u^{q-1}\nabla[u\nabla v]\,dx \\
&\qquad + a\int_\Omega u^q(1-u-cv)\,dx \\
&= -(q-1)d_1\int_\Omega (1+\alpha v+2\gamma u)u^{q-2}|\nabla u|^2 dx - (q-1)d_1\alpha\int_\Omega u^{q-1}\nabla u\cdot\nabla v\,dx \\
&\qquad + a\int_\Omega u^q(1-u-cv)\,dx \\
&=: -(q-1)d_1 I_1 + (q-1)d_1\alpha I_2 + aI_3.
\end{aligned}$$

Since u and v are positive, it is easy to see

$$\begin{aligned}
I_1 &\geq 2\gamma\int_\Omega u^{q-1}|\nabla u|^2\,dx = \frac{8\gamma}{(q+1)^2}\int_\Omega \left|\nabla\left(u^{(q+1)/2}\right)\right|^2\,dx, \\
I_3 &\leq \int_\Omega u^q(1-u)\,dx \leq |\Omega|,
\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω. We also note

$$qI_2 = -\int_\Omega \nabla(u^q)\cdot\nabla v\,dx = \int_\Omega u^q\Delta v\,dx.$$

Therefore, one can deduce the following inequality after integration with respect to t:

$$\| u(t) \|_{L^q}^q + c_0 \| \nabla(u^{(q+1)/2}) \|_{L^2(Q_t)}^2 \leq \| u_0 \|_{L^q}^q + C_1 + C_2 \int_{Q_T} u^q \Delta v \, dxdt \tag{4.1}$$

with some positive constants c_0, C_1, C_2. By Hölder's inequality

$$\left| \int_{Q_T} u^q \Delta v \, dxdt \right| \leq \| u \|_{L^{q+1}(Q_T)}^q \| \Delta v \|_{L^{q+1}(Q_T)} .$$

The maximal regularity for (3.3) implies

$$\begin{aligned} \| v_t \|_{L^{q+1}(Q_T)} + \| \Delta v \|_{L^{q+1}(Q_T)} &\leq M \left(\| v_0 \|_{W^2_{q+1}} + \| v(1 - du - v) \|_{L^{q+1}(Q_T)} \right) \\ &\leq C_3 \left(1 + \| u \|_{L^{q+1}(Q_T)} \right) \end{aligned}$$

with some positive numbers M and C_3. Here we have used (A) and Lemma 3.1. Hence it follows from these inequalities that

$$\left| \int_{Q_T} u^q \Delta v \, dxdt \right| \leq C_4 \left(1 + \| u \|_{L^{q+1}(Q_T)}^{q+1} \right) . \tag{4.2}$$

The substitution of (4.2) into (4.1) leads to

$$\sup_{0 \leq t \leq T} \| u(t) \|_{L^q}^q + \| \nabla(u^{(q+1)/2)} \|_{L^2(Q_T)}^2 \leq C_5 \left(1 + \| u \|_{L^{q+1}(Q_T)}^{q+1} \right) . \tag{4.3}$$

We introduce $w = u^{(q+1)/2}$; then (4.3) leads to get

$$E_T := \sup_{0 \leq t \leq T} \| w(t) \|_{L^{2q/(q+1)}}^{2q/(q+1)} + \| \nabla w \|_{L^2(Q_T)}^2 \leq C_5 \left(1 + \| w \|_{L^2(Q_T)}^2 \right) . \tag{4.4}$$

Recall that Lemma 3.2 implies $u \in L^2(Q_T)$; so

$$\| w \|_{L^{4/(q+1)}(Q_T)} \leq C_6 .$$

Let q^* be any number greater than 2. Then we see from Hölder's inequality

$$\| w \|_{L^2(Q_T)}^2 \leq \| w \|_{L^{q^*}(Q_T)}^{2(1-\lambda)} \| w \|_{L^{4/(q+1)}(Q_T)}^{2\lambda} \leq C_6^{2\lambda} \| w \|_{L^{q^*}(Q_T)}^{2(1-\lambda)}, \tag{4.5}$$

where

$$\lambda = \left(\frac{1}{2} - \frac{1}{q^*} \right) / \left(\frac{q+1}{4} - \frac{1}{q^*} \right) .$$

Here we also use Gagliardo-Nireberg's inequality; for any $q^* \in [\frac{2q}{q+1}, \frac{2N}{N-2}]$

$$\| w \|_{L^{q^*}} \leq C_7 \left(\| \nabla w \|_{L^2}^\theta \| w \|_{L^{2q/(q+1)}}^{1-\theta} + \| w \|_{L^1} \right) , \tag{4.6}$$

where

$$\theta = \left(\frac{q+1}{2q} - \frac{1}{q^*} \right) / \left(\frac{1}{N} + \frac{1}{2q} \right).$$

Setting $w = w(t)$ in (4.6) and integrating it with respect to t one can prove

$$\| w \|_{L^{q^*}(Q_T)}^{q^*} \leq C_8 \left(\int_0^T \| \nabla w(t) \|_{L^2}^{q^* \theta} \cdot \| w(t) \|_{L^{2q/(q+1)}}^{q^*(1-\theta)} dt + 1 \right) \tag{4.7}$$

with some $C_8 > 0$. So it follows from (4.7) that

$$\| w \|_{L^{q^*}(Q_T)}^{q^*} \leq C_8 \left(\sup_{0 \leq t \leq T} \| w(t) \|_{L^{2q/(q+1)}}^{q^*(1-\theta)} \cdot \int_0^T \| \nabla w(t) \|_{L^2}^{q^* \theta} dt + 1 \right). \tag{4.8}$$

Choose q^* such that $q^* \theta = 2$; $q^* = 2 + 4q/\{(q+1)N\}$. Recalling the definition of E_T we get

$$\| w \|_{L^{q^*}(Q_T)}^{q^*} \leq C_9 \left(E_T^{(N+2)/N} + 1 \right). \tag{4.9}$$

Then it follow from (4.4), (4.5) and (4.9) that

$$E_T \leq C_{10} \left(1 + E_T^{\mu} \right) \tag{4.10}$$

with

$$\mu = \frac{2(1-\lambda)(N+2)}{N q^*} < 1.$$

Thus (4.10) implies

$$\sup_{0 \leq t \leq T} \| w(t) \|_{L^{2q/(q+1)}}^{2q/(q+1)} \leq E_T \leq C$$

with some $C > 0$; so that

$$\sup_{0 \leq t \leq T} \| u(t) \|_{L^q} = \sup_{0 \leq t \leq T} \| w(t) \|_{L^{2q/(q+1)}} \leq C$$

and the proof is complete.

5 Open Problems

We will give some open problems for SKT model.

1. In Theorem 2.2, we have imposed the restriction on the space dimension. It still remains an open problem to establish the existence of global solutions of (P) in case $\delta > 0$ and $N \geq 6$.

2. In the proofs of Theorems 2.1 and 2.2, the positivity of self-diffusion coefficients is crucial. Especially, our proof of Proposition 3.3 depend on the positivity of γ. It will be interesting if we can prove the global existence result in case $\gamma = 0$.

3. Theorems 2.1 and 2.2 give us no information on the uniform boudedeness of solutions u, v as $t \to \infty$. In order to study the asymptotic behavior of u, v as $t \to \infty$, we have to establish the uniform boundedness of global solutions.

4. The most difficult problem is to show the existence of global solutions for the following full SKT model:

$$\begin{cases} u_t = d_1\Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv) \\ v_t = d_2\Delta[(1+\beta u+\delta v)v] + bv(1-du-v). \end{cases}$$

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