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STABILITY CRITERION FOR A SYSTEM OF DELAY-DIFFERENTIAL EQUATIONS

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Abstract. We analyze a system of linear differential equations with delays and establish necessary and sufficient conditions concerned with the absolutely stable for the system.

1. Introduction

Consider a system of ordinary differential equations with delay effect described by

$$u_j'(t) + \sum_{k=1}^{n} \{a_{jk}u_k(t) + b_{jk}u_k(t-\tau_{jk})\} = 0 \tag{1.1}$$

for $1 \le j \le n$. Here, $u(t) = (u_1, \cdots, u_n)^T(t)$ denotes unknown functions for $t \ge 0$, the coefficients a_{jk} and b_{jk} are real numbers, and time delay τ_{jk} is a nonnegative numbers for $1 \le j, k \le n$.

Our purpose is constructing the condition to derive the asymptotic stability for the system (1.1). The stability phenomenon of the system (1.1) is determined completely by the roots of the associated characteristic equations. The characteristic equation for the system (1.1) is expressed by

$$\det G(\lambda) = 0 \tag{1.2}$$

with

$$G(\lambda) := \begin{pmatrix} \lambda + d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & \lambda + d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & \lambda + d_{nn} \end{pmatrix},$$

where $d_{jk} := a_{jk} + b_{jk}e^{-\lambda\tau_{jk}}$ for $1 \le j, k \le n$. Then, $\lambda \in \mathbb{C}$ denotes a corresponding characteristic root called an eigenvalue. It is well known that the solution of the system (1.1) is asymptotically stable if and only if all of our eigenvalues lie in the left half of the complex plane (see, e.g., [2, 3, 8]). Consequently, our main goal is to establish the necessary and sufficient conditions that the real parts of all of the eigenvalues are negative.

Here, we define the absolute stability and the conditional stability introduced by Ruan [7].

Definition 1.1. *The equilibrium point of the system* (1.1) *is said to be absolutely stable if it is locally asymptotically stable for all delays* τ_{jk} *for* j, k *with* $1 \le j, k \le n$*. Furthermore, the equilibrium point of the system* (1.1) *is said to be conditionally stable if it is locally asymptotically stable for* τ_{jk} *for* j, k *with* $1 \le j, k \le n$ *in some intervals, but not necessarily for all delays.*

We introduce the necessary and sufficient condition for the absolute stability of the system (1.1). To this end, we prepare some notations. For $n \times n$ square matrix $X = (x_{jk})_{1 \le j,k \le n}$, we define the matrices $\widehat{X}$ and $\widetilde{X}$ as

$$\widehat{X} := \begin{pmatrix} x_{11} & -|x_{12}| & \cdots & -|x_{1n}| \\ -|x_{21}| & x_{22} & \cdots & -|x_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ -|x_{n1}| & -|x_{n2}| & \cdots & x_{nn} \end{pmatrix}, \qquad \widetilde{X} := \begin{pmatrix} |x_{11}| & \cdots & |x_{1n}| \\ \vdots & \ddots & \vdots \\ |x_{n1}| & \cdots & |x_{nn}| \end{pmatrix}.$$

Furthermore, to mention Stability Condition, we define principal minors (cf. Leslie [4]).

Definition 1.2. *Let M be a $n \times n$ square matrix. Let μ be a nonempty set of row indices and ν a nonempty set of column indices. A submatrix of M is a matrix $M[\mu, \nu]$ obtained by choosing the entries of M, which lie in rows μ and columns ν. A principal submatrix of M is a submatrix of the form $M[\mu, \mu]$. A principal minor is the determinant of a principal submatrix.*

Now, we define the constant matrices A and B as

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix},$$

and introduce Stability Condition (SC) as follows.

Stability Condition (SC): The coefficient matrices of (1.1) satisfy the following conditions.
(i) $\det(A + B) \neq 0$,
(ii) $a_{jj} - |b_{jj}| > 0$ or $a_{jj} = b_{jj} > 0$ for all j with $1 \le j \le n$,
(iii) all principal minors of $\widehat{A} - \widetilde{B}$ are nonnegative definite.

Since Stability Condition (SC), we derive the following theorem.

Theorem 1.3. *If the system* (1.1) *satisfies Stability Condition* (SC)*, then the equilibrium point is absolutely stable.*

Furthermore, under the condition that the matrix A in (1.1) is diagonal, that is, $A = diag(a_{11}, \cdots, a_{nn})$, we also obtain the following theorem.

Theorem 1.4. *Suppose $A = diag(a_{11}, \cdots, a_{nn})$. If the equilibrium point of the system* (1.1) *is absolutely stable, then the system* (1.1) *satisfies Stability Condition* (SC)*.*

Consequently, the simple combination of Theorem 1.3 and Theorem 1.4 gives the following corollary.

Corollary 1.5. *Suppose $A = diag(a_{11}, \cdots, a_{nn})$. The system* (1.1) *satisfies Stability Condition* (SC) *if and only if the equilibrium point is absolutely stable.*

2. Sufficient condition

To prove Theorem 1.3, we start from the definition of the irreducible matrix (cf. Lancaster and Tismenetsky [6]),

Definition 2.1. *Let M be a $n \times n$ square matrix. The matrix M is said to be reducible if there is an permutation matrix P of order n such that*

$$P^{-1}MP = \begin{pmatrix} M_{11} & O \\ M_{21} & M_{22} \end{pmatrix}, \tag{2.1}$$

where M_{11} and M_{22} are square matrices of order less than n and O is a zero matrix. If no such P exists, then M is irreducible.

We introduce the following two lemmas to show Theorem 1.3.

Lemma 2.2. (cf. Fiedler [5]) *Let M be a $n \times n$ real matrix whose off-diagonal entries are nonpositive and all principal minors are nonnegative. If M is irreducible, then there is a vector $v > 0$ such that $Mv \geq 0$.*

Here, $v > 0$ or $v \geq 0$ means that all components of the vector v are positive or nonnegative, respectively.

Lemma 2.3. *Let $Q = (\alpha_{jk} + \beta_{jk})_{1 \leq j,k \leq n}$ be a $n \times n$ square matrix, where α_{jk} and β_{jk} are complex numbers for $1 \leq j,k \leq n$. Then every eigenvalue of Q lies in at least one of the disks*

$$\Big\{ z \in \mathbb{C} \ ; \ |z - \alpha_{jj}| \leq \sum_{k=1,k\neq j}^{n} |\alpha_{jk}| + \sum_{k=1}^{n} |\beta_{jk}| \Big\} \tag{2.2}$$

for $1 \leq j \leq n$ in the complex z-plane.

Proof. Let λ be an eigenvalue of Q with the associated eigenvector w with $w = (w_1, \cdots, w_n)^T$. Since $Qw = \lambda w$, we have

$$\sum_{k=1}^{n} (\alpha_{jk} + \beta_{jk}) w_k = \lambda w_j$$

for $1 \leq j \leq n$. This means

$$(\lambda - \alpha_{jj}) w_j = \sum_{k=1,k\neq j}^{n} \alpha_{jk} w_k + \sum_{k=1}^{n} \beta_{jk} w_k.$$

Let p be a natural number which satisfies $|w_p| = \max_j |w_j|$. Then the p-th equation gives

$$\begin{aligned} |\lambda - \alpha_{pp}||w_p| &= \Big| \sum_{k=1,k\neq p}^{n} \alpha_{pk} w_k + \sum_{k=1}^{n} \beta_{pk} w_k \Big| \\ &\leq \sum_{k=1,k\neq p}^{n} |\alpha_{pk}||w_k| + \sum_{k=1}^{n} |\beta_{pk}||w_k| \\ &\leq \Big(\sum_{k=1,k\neq p}^{n} |\alpha_{pk}| + \sum_{k=1}^{n} |\beta_{pk}| \Big) |w_p|. \end{aligned}$$

Because of $w \neq 0$, we have $|w_p| \neq 0$. Consequently, we obtain

$$|\lambda - \alpha_{pp}| \leq \sum_{k=1,k\neq p}^{n} |\alpha_{pk}| + \sum_{k=1}^{n} |\beta_{pk}|,$$

and this estimate gives the conclusion of Lemma 2.3. $\square$

Remark 1. *If we suppose that $\beta_{jk} = 0$ for $1 \le j, k \le n$ in Lemma 2.3, this lemma becomes Geršgorin's theorem* (*cf. Lancaster and Tismenetsky* [6]).

Proof of Theorem 1.3. We suppose that $\widehat{A} - \widetilde{B}$ is irreducible. By employing Theorem 2.2, for an irreducible matrix $\widehat{A} - \widetilde{B}$ whose off-diagonal entries are nonpositive and all principal minors are nonnegative, there is a vector $v > 0$ such that $(\widehat{A} - \widetilde{B})v \ge 0$. Namely, there exists $v_j > 0$ such that

$$-a_{jj}v_j + \sum_{k=1,k\neq j}^{n} |a_{jk}|v_k + \sum_{k=1}^{n} |b_{jk}|v_k \le 0 \tag{2.3}$$

for $1 \le j \le n$.

We suppose that there exists a root λ_0 of (1.2) satisfying $\mathrm{Re}\lambda_0 \ge 0$. Then, we introduce the square matrix $E := -(a_{jk} + b_{jk}e^{-\lambda_0\tau_{jk}})_{1\le j,k\le n}$. We remark that λ_0 is an eigenvalue of E because of (1.2). On the other hand, we define $F := -(v_j^{-1}(a_{jk} + b_{jk}e^{-\lambda_0\tau_{jk}})v_k)_{1\le j,k\le n}$. Then, every eigenvalue of E is equivalent to every eigenvalue of F. Indeed, since $F = V^{-1}EV$, where $V := diag(v_1, \cdots, v_n)$, we obtain

$$\det(\lambda I - F) = \det(\lambda I - V^{-1}EV) = \det(V^{-1})\det(\lambda I - E)\det V = \det(\lambda I - E).$$

Namely, λ_0 is an eigenvalue of F.

We apply Lemma 2.3 to the matrix F and derive the following. For every eigenvalue of F, there exists p such that the eigenvalue lies within the disk

$$D_p := \Big\{ z \in \mathbb{C} \ ; \ |z + a_{pp}| \le \sum_{k=1,k\neq p}^{n} |a_{pk}|\frac{v_k}{v_p} + \sum_{k=1}^{n} |b_{pk}e^{-\lambda_0\tau_{pk}}|\frac{v_k}{v_p} \Big\}.$$

From $\mathrm{Re}\lambda_0 \ge 0$ and (2.3), we compute

$$\begin{aligned}\sum_{k=1,k\neq p}^{n} |a_{pk}|\frac{v_k}{v_p} + \sum_{k=1}^{n} |b_{pk}e^{-\lambda_0\tau_{pk}}|\frac{v_k}{v_p} &= \sum_{k=1,k\neq p}^{n} |a_{jk}|\frac{v_k}{v_p} + \sum_{k=1}^{n} |b_{pk}|e^{-\mathrm{Re}\lambda_0\tau_{pk}}\frac{v_k}{v_p} \\ &\le \sum_{k=1,k\neq p}^{n} |a_{jk}|\frac{v_k}{v_p} + \sum_{k=1}^{n} |b_{pk}|\frac{v_k}{v_p} \le a_{pp}.\end{aligned}$$

This estimate gives that $D_p \subseteq \{z \in \mathbb{C}; |z + a_{pp}| \le a_{pp}\}$. Therefore, we conclude that, for every eigenvalue of F, there is p such that the eigenvalue lies within the disk $\{z \in \mathbb{C}; |z + a_{pp}| \le a_{pp}\}$.

Consequently, there exists the natural number p such that $\lambda_0 \in \{z \in \mathbb{C}; |z + a_{pp}| \le a_{pp}\}$. Furthermore, the assumption $\mathrm{Re}\lambda_0 \ge 0$ gives $\lambda_0 = 0$. However, since $\det(A+B) \neq 0$, λ_0 must satisfy $\lambda_0 \neq 0$. Eventually, this fact is a contradiction and we conclude that the real part of the eigenvalues of (1.2) must be negative under Stability Condition (SC).

In the case of a reducible matrix $\widehat{A} - \widetilde{B}$, we can rewrite $\widehat{A} - \widetilde{B}$ into a lower block triangular matrix with irreducible blocks along the diagonal similar to (2.1) by using a suitable permutation matrix. Furthermore, this translation does not affect the principal minors of $\widehat{A} - \widetilde{B}$. Thus, the result follows by applying the previous argument to each irreducible diagonal block. □

3. Necessary condition

Next, we show Theorem 1.4. To prove this theorem, we derive three lemmas. For this purpose, we prepare the following theorem (cf. Bellman and Cooke [1]).

Theorem 3.1. (*Rouché's theorem*) *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|f(z)| > |g(z)|$ for each point on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.*

Then we will obtain the following lemmas.

Lemma 3.2. *Suppose* $\det(A+B)=0$. *Then,* (1.2) *has a zero eigenvalue.*

Lemma 3.3. *Suppose $a_{pp} < |b_{pp}|$ or $a_{pp} = b_{pp} = 0$ for some p with $1 \leq p \leq n$, Then, there exist τ_{jk} $(1 \leq j,k \leq n)$ such that* (1.2) *has a root λ with* $\mathrm{Re}\lambda > 0$.

Lemma 3.4. *Suppose that $a_{jj} > 0$ for any j with $1 \leq j \leq n$ and some principal minors of $A - \widetilde{B}$ is negative. Then, there exist τ_{jk} $(1 \leq j,k \leq n)$ such that* (1.2) *has a root λ with* $\mathrm{Re}\lambda > 0$.

From these lemmas, it is easy to prove Theorem 1.4. At the rest of this section, we give a proof of Lemma 3.2, 3.3 and 3.4.

Proof of Lemma 3.2. Because of $\det(A+B)=0$, $\lambda=0$ satisfies (1.2) and we complete the proof. □

Proof of Lemma 3.3. We introduce the matrix

$$G_{11}^{l}(\lambda) := \begin{pmatrix} \lambda + d_{ll} & b_{ll+1}e^{-\lambda\tau_{ll+1}} & \cdots & b_{ln}e^{-\lambda\tau_{ln}} \\ b_{l+1l}e^{-\lambda\tau_{l+1l}} & \lambda + d_{l+1l+1} & \cdots & b_{l+1n}e^{-\lambda\tau_{l+1n}} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nl}e^{-\lambda\tau_{nl}} & b_{nl+1}e^{-\lambda\tau_{nl+1}} & \cdots & \lambda + d_{nn} \end{pmatrix}$$

for $1 \leq l \leq n-1$, where $d_{jj} = a_{jj} + b_{jj}e^{-\lambda\tau_{jj}}$ for $1 \leq j \leq n$. Then, because of $A = diag(a_{11}, \cdots, a_{nn})$, we have $G(\lambda) = G_{11}^{1}(\lambda)$. Applying the cofactor expansion to $\det G(\lambda)$, we obtain

$$\det G(\lambda) = (\lambda + d_{11}) \det G_{11}^{2}(\lambda) + \sum_{j=2}^{n} (-1)^{j-1} b_{j1} e^{-\lambda\tau_{j1}} \det G_{j1}^{2}(\lambda), \tag{3.1}$$

where $G_{jk}^{2}(\lambda)$ is a submatrix of $G(\lambda)$ obtained by removeing j-th row and k-th column from $G(\lambda)$. Similarly, we apply the cofactor expansion to $\det G_{11}^{h}(\lambda)$, and get

$$\det G_{11}^{l}(\lambda) = (\lambda + d_{ll}) \det G_{11}^{l+1}(\lambda) + \sum_{j=l+1}^{n} (-1)^{j-l} b_{jl} e^{-\lambda\tau_{jl}} \det G_{j-l+11}^{l+1}(\lambda) \tag{3.2}$$

for $1 \leq l \leq n-1$, where $G_{jk}^{l+1}(\lambda)$ is also a submatrix of $G_{11}^{l}(\lambda)$ obtained by striking out j-th row and k-th column from $G_{11}^{l}(\lambda)$. Therefore, using (3.1) and (3.2), we obtain the expansion for $\det G(\lambda)$ that

$$
\begin{aligned}
\det G(\lambda) &= (\lambda + d_{11})(\lambda + d_{22}) \det G^3_{11}(\lambda) + \sum_{j=2}^{n} (-1)^{j-1} b_{j1} e^{-\lambda\tau_{j1}} \det G^2_{j1}(\lambda) \\
&\quad + (\lambda + d_{22}) \sum_{j=3}^{n} (-1)^{j-2} b_{j2} e^{-\lambda\tau_{j2}} \det G^3_{j-11}(\lambda) \\
&= \prod_{j=1}^{n} (\lambda + d_{jj}) + \sum_{j=2}^{n} (-1)^{j-1} b_{j1} e^{-\lambda\tau_{j1}} \det G^2_{j1}(\lambda) \\
&\quad + \sum_{k=2}^{n-1} \prod_{h=2}^{k} (\lambda + d_{jj}) \sum_{j=h+1}^{n} (-1)^{j-h} b_{jh} e^{-\lambda\tau_{jh}} \det G^{h+1}_{j-h+11}(\lambda).
\end{aligned} \tag{3.3}
$$

Here, the last term in (3.5) is neglected if $n = 2$. Summarizing the above, we define

$$
\begin{aligned}
f(\lambda) &:= \prod_{j=1}^{n} (\lambda + a_{jj} + b_{jj} e^{-\lambda\tau_{jj}}), \\
g(\lambda) &:= \sum_{j=2}^{n} (-1)^{j-1} b_{j1} e^{-\lambda\tau_{j1}} \det G^2_{j1}(\lambda) \\
&\quad + \sum_{k=2}^{n-1} \prod_{h=2}^{k} (\lambda + a_{jj} + b_{jj} e^{-\lambda\tau_{jj}}) \sum_{j=h+1}^{n} (-1)^{j-h} b_{jh} e^{-\lambda\tau_{jh}} \det G^{h+1}_{j-h+11}(\lambda),
\end{aligned} \tag{3.4}
$$

and obtain $\det G(\lambda) = f(\lambda) + g(\lambda)$.

Next, we consider $f(\lambda) = 0$ to indicate $f(\lambda)$ has the zero-solution in the right half of the complex plane. Because of $f(\lambda) = 0$, we find the eigenvalue λ which satisfies

$$
\lambda + a_{pp} + b_{pp} e^{-\lambda\tau} = 0, \tag{3.5}
$$

where $\tau = \tau_{pp}$. Now, we show that (3.5) has a purely imaginary root $\lambda = i\omega_1$ with some delay term $\tau = \tau_1$. Substituting $\lambda = i\omega$ with $\omega \geq 0$ into (3.5) yields

$$
i\omega + a_{pp} + b_{pp} e^{-i\omega\tau} = 0.
$$

Then, we put $f_1(\omega) := i\omega + a_{pp}$, and obtain $|f_1(0)| = |a_{pp}|$ and $\lim_{\omega\to\infty} |f_1(\omega)| \to \infty$. Therefore, under the assumption $|a_{pp}| < |b_{pp}|$, there is a positive number ω_1 such that $|f_1(\omega_1)| = |b_{pp}|$ by the intermediate value theorem. This tells us that there exists a positive number θ such that $f_1(\omega_1) = -b_{pp} e^{-i\theta}$, and we get

$$
i\omega_1 + a_{pp} + b_{pp} e^{-i\theta} = 0.
$$

Thus, choosing τ_1 such that $\omega_1\tau_1 = \theta + 2\pi m$ with $m \in \mathbb{N}_0$, the pair (ω_1, τ_1) satisfies (3.5). We note that τ_1 can be taken suitably large.

The next purpose is to show that (3.5) has a root λ with $\mathrm{Re}\lambda > 0$. We put $h(\lambda, \tau) := \lambda + a_{pp} + b_{pp} e^{-\lambda\tau}$. Then, using (3.5), we have

$$
\begin{aligned}
h_\lambda(\lambda, \tau) &= 1 - b_{pp} \tau e^{-\lambda\tau} = 1 + \tau(\lambda + a_{pp}), \\
h_\tau(\lambda, \tau) &= -b_{pp} \lambda e^{-\lambda\tau} = \lambda(\lambda + a_{pp}).
\end{aligned}
$$

By the implicit function theorem, we obtain a solution $\lambda(\tau)$ of (3.5) around τ_1. Furthermore the equality

$$\lambda'(\tau_1) = -\frac{h_\tau(i\omega_1, \tau_1)}{h_\lambda(i\omega_1, \tau_1)} = -\frac{i\omega_1(i\omega_1 + a_{pp})}{1 + \tau_1(i\omega_1 + a_{pp})}$$

gives us that

$$\mathrm{Re}\lambda'(\tau_1) = \frac{\omega_1^2}{(1 + \tau_1 a_{pp})^2 + (\tau_1\omega_1)^2} > 0.$$

Therefore there exists $\tau_2 > \tau_1$ such that $\mathrm{Re}\lambda(\tau_2) > 0$.

On the other hand, under the assumption $a_{pp} \leq -|b_{pp}|$, we put $\lambda := x \in \mathbb{R}$ and find the solution of

$$x + a_{pp} + b_{pp}e^{-x\tau} = 0. \tag{3.6}$$

We put that $f_1(x) := x + a_{pp}$ and $f_2(x) := -b_{pp}e^{-x\tau}$. We consider graphs of $y = f_1(x)$ and $y = f_2(x)$. Then there exists an intersection at $x > 0$ of these two graphs for suitably large τ. Hence, (3.6) has positive solution for large τ.

Finally, we go back to (3.4) and apply Rouché's theorem to $\det G(\lambda) = f(\lambda) + g(\lambda)$. We put $C := \{\lambda \in \mathbb{C}; \mathrm{Re}\lambda > 0\}$ and $C_\varepsilon := \{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < \varepsilon\}$. Then, there exists $\varepsilon_0 > 0$ such that $\bar{C}_{\varepsilon_0} \subset C$ holds, where $\bar{C}_\varepsilon$ is a closure of C_ε. Since all terms of $g(\lambda)$ contain $e^{-\lambda\tau_{jk}}$ which τ_{jk} is not τ_{pp}, there exists τ_0 such that $|f(\lambda)| > |g(\lambda)|$ on $\partial C_{\varepsilon_0}$ provided $\tau_{jk} > \tau_0$ except for $\tau_{jk} = \tau_{pp}$. Here, ∂C_ε denotes a boundary of C_ε. Therefore, we can apply Rouché's theorem and conclude that $f(\lambda) + g(\lambda) = 0$ has at least one solution in $\bar{C}_{\varepsilon_0}$. This means $\det G(\lambda) = 0$ has a root which real part is positive.

Furthermore, in the case that $a_{pp} = b_{pp} = 0$ for some p with $1 \leq p \leq n$, we consider the approximation and derive the desired result. Thus we complete the proof. □

Proof of Lemma 3.4. We modify the proof of Lemma 3.3. Since some principal minors of $A - \widetilde{B}$ is negative, there exists r with $1 \leq r \leq n$ such that

$$\det \begin{pmatrix} a_{rr} - |b_{rr}| & \cdots & -|b_{rn}| \\ \vdots & \ddots & \vdots \\ -|b_{nr}| & \cdots & a_{nn} - |b_{nn}| \end{pmatrix} < 0. \tag{3.7}$$

To show that (1.2) has a root λ with $\mathrm{Re}\lambda > 0$, we consider the following function

$$\gamma_\kappa(z) := \det \begin{pmatrix} \kappa z + a_{rr} + b_{rr}e^{-z\eta_{rr}} & \cdots & b_{rn}e^{-z\eta_{rn}} \\ \vdots & \ddots & \vdots \\ b_{nr}e^{-z\eta_{nr}} & \cdots & \kappa z + a_{nn} + b_{nn}e^{-z\eta_{nn}} \end{pmatrix},$$

where

$$\eta_{jk} := \begin{cases} 1/2 & (b_{jk} \geq 0), \\ 1 & (b_{jk} < 0), \end{cases}$$

for $r \leq j, k \leq n$. We show that $\gamma_\kappa(z) = 0$ has the zero-solution in the right half of the complex plane. For $z = x + 2\pi i$ with $x \in \mathbb{R}$, we obtain $\delta(x) := \gamma_0(x + 2\pi i)$ and

$$\delta(x) = \det \begin{pmatrix} a_{rr} - |b_{rr}|e^{-x\eta_{rr}} & \cdots & -|b_{rn}|e^{-x\eta_{rn}} \\ \vdots & \ddots & \vdots \\ -|b_{nr}|e^{-x\eta_{nr}} & \cdots & a_n - |b_{nn}|e^{-x\eta_{nn}} \end{pmatrix}.$$

Because of (3.7), we have $\delta(0) < 0$. On the other hand, under the assumption $a_{jj} > 0$ for all j with $1 \leq j \leq n$, we derive $\lim_{x\to\infty} \delta(x) = a_{rr} \cdots a_{nn} > 0$. Therefore, by the intermediate value theorem, there exists $x_0 > 0$ such that $\delta(x_0) = 0$. Namely, $z_0 = x_0 + 2\pi i$ is a solution of $\gamma_0(z)$.

Because of this fact and similar argument as in Lemma 3.3, we prove Lemma 3.4 and complete the proof. □

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