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TRANSITION LAYERS AND SPIKES FOR A BISTABLE REACTION-DIFFUSION EQUATION [1]

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1 Introduction

This talk is concerned with the following reaction-diffusion problem:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(x,u), & 0<x<1,\ t>0, \\ u_x(0,t) = u_x(1,t) = 0, & t>0, \\ u(x,0) = u_0(x), & 0<x<1. \end{cases} \tag{1.1}$$

Here ε is a positive parameter and $f(x,u)$ is given by

$$f(x,u) = u(1-u)(u-a(x)),$$

where a is a function of C^2-class which possesses the following properties:

(A1) $0 < a(x) < 1$ in $[0,1]$.

(A2) If Σ is defined by $\Sigma = \{x \in (0,1); a(x) = 1/2\}$, then Σ is a finite set and $a'(x) \neq 0$ at any $x \in \Sigma$.

(A3) $a'(x)^2 + a''(x)^2 > 0$ in $[0,1]$.

(A4) $a'(0) = a'(1) = 0$.

This problem is well known as an equation which describes a phase transition phenomenon.

We will mainly discuss the steady state problem associated with (1.1), which is written as follows:

$$\begin{cases} \varepsilon^2 u'' + f(x,u) = 0, & 0<x<1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.2}$$

[1] This is a joint work with KIMIE NAKASHIMA (*Tokyo University of Marine Science and Technology*) and YOSHIO YAMADA (*Waseda University*).

where ' $'$ ' denotes the derivative with respect to x. Angenent, Mallet-Paret and Peletier[1] proved the existence of stable solutions to (1.2) which possess transition layers. Here **transition layer** means a part of a solution where the value changes drastically from 0 to 1 or 1 to 0 in a very small interval. If u is a solution of (1.2) with transition layers, then it is called a **layered solution**. They discussed in detail profiles of layered solutions and their linearized stability. See also Hale and Sakamoto [2], where unstable layered solutions are studied. In the special case $\int_0^1 f(x,s)ds = 0$ for $x \in [0,1]$, Nakashima[3, 4] has shown the existence of layered solutions.

We will briefly explain the reason why layered solutions appear. Multiplying (1.1) by u_t and integrating it with respect to x over $(0,1)$ we get

$$\frac{d}{dt}I(u) \leq 0. \tag{1.3}$$

Here

$$I(u) = \int_0^1 \left[\frac{1}{2}\varepsilon^2 u_x^2 + W(x,u)\right] dx,$$

and

$$W(x,u) = -\int_{\phi_0}^{u} f(x,s)ds, \tag{1.4}$$

with

$$\phi_0 = \begin{cases} 0 & \text{if } a(x) \leq 1/2, \\ 1 & \text{if } a(x) > 1/2. \end{cases}$$

We call $I(u)$ an **energy function** and $W(x,u)$ a **bistable potential**. By (1.3) we see that $I(u(t))$ is monotone decreasing with respect to t. This implies that every solution of (1.1) behaves as the energy becomes small. Roughly speaking, if ε is sufficiently small, then $W(x,u)$ controls the energy. Hence the energy crucially depends on the potential $W(x,u)$. Here we should note the spatial inhomogeneity of $W(x,u)$. For each $x \in [0,1]$, if $a(x) < 1/2$, then the minimum of $W(x,u)$ is attained at $u = 1$, while, if $a(x) > 1/2$, then the minimum of $W(x,u)$ is attained at $u = 0$. Therefore, when $a(x)$ is very close to $1/2$ and $a'(x) \neq 0$, transition layers appear in order to make $W(x,u(x))$ small when ε is sufficiently small.

As mentioned above, the interaction of bistability and spatial inhomogeneity of $f(x,u)$ brings about many solutions of (1.2); so that the structure of the set of all solutions of (1.2) is very complicate. Among the existence results, Angenent. Mallet-Paret and Peletier [1] have proved the existence of layered solutions by the method of comparison principle. However, their method is not efficient to show the existence of unstable solutions of (1.2). See [2] for unstable layered solutions. Moreover, there exist some solutions with **spikes**.

To study such solutions u_ε with transition layers or spikes, we take account of the number of intersection points of u_ε and a. We introduce the notion of n-mode solution; u_ε is called an **n-mode solution** if u_ε has n intersecting points with a in $(0,1)$. Our main purpose is to study basic properties and profiles of n-mode solutions of (1.2). According to profiles of u_ε, we can show that $u_\varepsilon(x)$ is classified into the following three groups:

(N1) $u_\varepsilon(x)$ lies near 0 or 1,

(N2) $u_\varepsilon(x)$ forms transition layers,

(N3) $u_\varepsilon(x)$ forms spikes.

One of the most interesting and important problems for $u_\varepsilon \in S_{n,\varepsilon}$ is to know where its transition layers or spikes appear. At one of end-points of any transition layer, $u_\varepsilon(x)$ is very close to 0 or 1 when ε is sufficiently small. The situation is similar when we discuss a spike; if u_ε has a spike based on 1, then $u_\varepsilon(x)$ is very close to 1 at both end-points of the spike. Therefore, it will be important to study the asymptotic rate of $u_\varepsilon(x)$ and $1-u_\varepsilon(x)$ as $\varepsilon \to 0$ in a certain interval containing one local maximum point or local minimum point of u_ε. The analysis to get the asymptotic rate will be carried out by a kind of barrier method in Section 2.

In section 3, we will discuss the location of transition layers and spikes by using the information on the rate of asymptotic order obtained in Section 2. We will show that any transition layer appears only in a neighborhood of a point of Σ. Moreover, we will also prove that any spike appears in a neighborhood of a point of local maximum or minimum point of a.

Finally, it is interesting to investigate where multi-layers or multi-spikes appear. We will derive some satisfactory results on multi-layers and multi-spikes. These results help us to know their location.

Recently, Ai, Chen and Hastings[5] has obtained similar results concerning the location and multiplicity of layers and spikes. Moreover, they have discussed the Morse indices of such solutions of (1.2). However, their arguments are not so easy to follow and they do not give any results about the asymptotic rate. Our method is based on the asymptotic rate in Section 2; so that it is quite different from theirs.

2 Transition layers and spikes for n-mode solutions

Throughout this paper, we denote by $S_{n,\varepsilon}$ the set of all n-mode solutions of (1.2) and we fix $n \in \mathbb{N}$. Moreover, for $u_\varepsilon \in S_{n,\varepsilon}$, we define a set

$$\Xi = \{x \in [0,1]; u_\varepsilon(x) = a(x)\}.$$

For $u_\varepsilon \in S_{n,\varepsilon}$, it should be noted that by (A4) $u_\varepsilon(-x)$ is also a solution of (1.2) in $[-1,0]$ by extending a over $[-1,1]$ as an even function. In this manner, u_ε can be extended for all $x \in \mathbb{R}$ by the reflection.

In this section we will give some basic properties of solutions of (1.2).

Lemma 2.1 (Ai-Chen-Hastings [5]). *For $u \in S_{n,\varepsilon}$, it holds that*

$$\lim_{\varepsilon\to 0} \sup_{u_\varepsilon \in S_{n,\varepsilon}} \max_{x\in[0,1]} \left| u_\varepsilon(x)(1-u_\varepsilon(x)) \left[\frac{1}{2}\varepsilon^2 (u_\varepsilon'(x))^2 - W(x, u_\varepsilon(x)) \right] \right| = 0,$$

where $W(x,u)$ is defined by (1.4).

If ε is small enough, Lemma 2.1 implies that $u_\varepsilon(x)$, $1-u_\varepsilon(x)$ or $\varepsilon^2(u_\varepsilon'(x))^2/2 - W(x,u_\varepsilon(x))$ is very close to 0. If one of the first two assertions is valid in a certain interval, then $u_\varepsilon(x)$ approaches 0 or 1 in such an interval as $\varepsilon \to 0$. The last one gives information for the gradient of u_ε when $u_\varepsilon(x)$ is not very close to 0 or 1. For example, let ξ be any point in Ξ and consider $u_\varepsilon'(\xi)$. Note that there is a positive constant M_1 satisfying $u_\varepsilon(\xi)(1-u_\varepsilon(\xi)) = a(\xi)(1-a(\xi)) > M_1$. For any $\eta > 0$, Lemma 2.1 assures

$$\left| \frac{1}{2}\varepsilon^2 (u_\varepsilon'(\xi)) - W(\xi, a(\xi)) \right| < \eta$$

if ε is sufficiently small. Since $W(\xi, a(\xi)) > M_2$ with some $M_2 > 0$, we get $\varepsilon^2(u_\varepsilon'(\xi))^2 > M_2$ from the above inequality. Hence we see

$$|u_\varepsilon'(\xi)| > \frac{\sqrt{M_2}}{\varepsilon}$$

when ε is sufficiently small.

Moreover, since $a'(x)$ is bounded in $[0,1]$, we can get the following lemma.

Lemma 2.2. *For $u_\varepsilon \in S_{n,\varepsilon}$, set $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. If ε is sufficiently small, then u_ε' has exactly $(n-1)$ zero points $\{\zeta_k\}_{k=1}^{n-1}$ satisfying*

$$0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2 < \cdots < \xi_{n-1} < \zeta_{n-1} < \xi_n < 1.$$

Roughly speaking, Lemmas 2.1 and 2.2 imply that $u_\varepsilon(x)$ is classified into the three parts: (N1),(N2) and (N3).

Lemma 2.3. *For $u_\varepsilon \in S_{n,\varepsilon}$, let ξ^ε be any point in Ξ and define U_ε by $U_\varepsilon(t) = u_\varepsilon(\xi^\varepsilon + \varepsilon t)$. Then there exists a subsequence $\{\varepsilon_k\} \downarrow 0$ such that $\xi_k = \xi^{\varepsilon_k}$ and $U_k = U_{\varepsilon_k}$ satisfy*

$$\lim_{k\to\infty} \xi_k = \xi^* \quad \textit{and} \quad \lim_{k\to\infty} U_k = \phi \quad \textit{in} \ \ C^2_{loc}(\mathbb{R}).$$

Here ϕ satisfies one of the following properties:

(i) *If* $a(\xi^*) = 1/2$, *then* ϕ *is a unique solution to the following problem*:

$$\begin{cases} \phi'' + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(-\infty) = 0,\ \phi(+\infty) = 1, \quad (\textit{resp. } \phi(-\infty) = 1,\ \phi(+\infty) = 0) \\ \phi(0) = 1/2, \end{cases} \tag{2.1}$$

if $\phi'(0) > 0$ (*resp.* $\phi'(0) < 0$). *Moreover,* $\phi'(t) > 0$ *for* $t \in \mathbb{R}$ *if* $\phi'(0) > 0$, *while* $\phi'(t) < 0$ *for* $t \in \mathbb{R}$ *if* $\phi'(0) < 0$.

(ii) *If* $a(\xi^*) < 1/2$, *then* ϕ *is a unique solution to the following problem*:

$$\begin{cases} \phi'' + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(0) = a(\xi^*), \phi(\pm\infty) = 0. \end{cases} \tag{2.2}$$

Here ϕ *satisfies* $\sup_{x \in \mathbb{R}} \phi(x) > a(\xi^*)$.

(iii) *If* $a(\xi^*) > 1/2$, *then* ϕ *is a unique solution to the following problem*:

$$\begin{cases} \phi'' + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(0) = a(\xi^*), \phi(\pm\infty) = 1. \end{cases} \tag{2.3}$$

Here ϕ *satisfies* $\inf_{x \in \mathbb{R}} \phi(x) < a(\xi^*)$.

By Lemma 2.3, the profile of a transition layer is similar to the hetero-clinic solution of (2.1) and that of a spike is similar to the homo-clinic solution of (2.2) or (2.3).

Remark. Lemma 2.3 also tells us that if $\xi \in \Xi$ is away from a point of Σ, then there is another point of Ξ in a neighborhood of ξ.

By the above arguments, we see that every transition layer (spike) appears in a neighborhood of a point in Ξ. Therefore, we will study the location of points of Ξ instead of those points where transition layer or spike appears. Let ξ_1, ξ_2 be any adjacent points in Ξ and let (ξ_1, ξ_2) be any interval such that

$$u_\varepsilon(x) - a(x) > 0 \quad \text{in } (\xi_1, \xi_2). \tag{2.4}$$

Let $\zeta \in [0, 1]$ be a unique point satisfying $\xi_1 < \zeta$, $u'_\varepsilon(\zeta) = 0$ and $u'_\varepsilon(x) > 0$ in (ξ_1, ζ). The existence of such ζ is assured by Lemma 2.2.

We will establish asymptotic behavior of u_ε in (ξ_1, ξ_2) as $\varepsilon \to 0$. For this purpose, we will prepare

Lemma 2.4. *Let $g(v) = v(1-v)(v-a_0)$ with $a_0 \in (0,1)$. Then for $\sigma \in (a_0,1)$ and $M > 0$, there exists a unique solution of*

$$\begin{cases} v_{zz} + g(v) = 0 & \text{in } (-M,0), \\ v(-M) = \sigma,\ v_z(0) = 0, & \\ v > \sigma & \text{in } (-M,0). \end{cases} \tag{2.5}$$

Moreover, there exists a positive constant $\sigma^* \in (a_0,1)$ such that, if $\sigma > \sigma^*$, then

$$c_1 \exp(-RM) < 1 - v(0) < c_2 \exp(-rM),$$

where $r = \sqrt{-g'(\sigma)}$, $R = \sqrt{-g'(1)}$ and $c_1, c_2\, (0 < c_1 < c_2)$ are positive constants depending only on σ.

Theorem 2.5. *For $u_\varepsilon \in S_{n,\varepsilon}$, assume (2.4) and let $\zeta \in (\xi_1, \xi_2)$ satisfy $u_\varepsilon'(\zeta) = 0$. If $(\zeta - \xi_1)/\varepsilon \to +\infty$ as $\varepsilon \to 0$, then for sufficiently small $\varepsilon > 0$, there exist positive constants $C_1, C_2, r, R\, (0 < C_1 < C_2,\ 0 < r < R)$ such that*

$$C_1 \exp\left(-\frac{R(\zeta - \xi_1)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2 \exp\left(-\frac{r(x - \xi_1)}{\varepsilon}\right) \quad \text{for } x \in [\xi_1, \zeta]. \tag{2.6}$$

Proof. We begin with the proof of the right-hand-side inequality of (2.6). Let $\delta^* \in (0,1)$ be a constant which is close to 1 and take $a^* \in (0, \delta^*)$ such that $a^* > \max\{a(x); x \in [0,1]\}$. By the assumption and Lemma 2.3 we can find a point $\tilde{\xi}_1 \in (\xi_1, \zeta)$ such that $u_\varepsilon(\tilde{\xi}_1) = \delta^*$ and $u_\varepsilon(x) > \delta^*$ in $(\tilde{\xi}_1, \zeta)$ provided that ε is sufficiently small. Clearly, $\tilde{\xi}_1 - \xi_1 = O(\varepsilon)$ as $\varepsilon \to 0$; so $\zeta - \tilde{\xi}_1 > \varepsilon$.

Now take any $x^* \in (\tilde{\xi}_1 + \varepsilon, \zeta)$ and apply Lemma 2.4 with $a_0 = a^*, \sigma = \delta^*$ and $M = (x^* - \tilde{\xi}_1 - \varepsilon)/\varepsilon$ in order to construct $v(z)$ as the unique solution of (2.5). We use the change of variable $z = (x - x^*)/\varepsilon$ and define V by $V(x) = v((x - x^*)/\varepsilon)$; then

$$\begin{cases} \varepsilon^2 V'' + V(1-V)(V-a^*) = 0 & \text{in } (\tilde{\xi}_1 + \varepsilon, x^*), \\ V(\tilde{\xi}_1 + \varepsilon) = \delta^*,\ V'(x^*) = 0, & \\ V > \delta^* & \text{in } (\tilde{\xi}_1 + \varepsilon, x^*). \end{cases} \tag{2.7}$$

By virtue of Lemma 2.4, V satisfies

$$c_1 e^R \exp\left(-\frac{R(x^* - \tilde{\xi}_1)}{\varepsilon}\right) < 1 - V(x^*) < c_2 e^r \exp\left(-\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon}\right), \tag{2.8}$$

where c_1, c_2, r and R are positive constants depending only on a^* and δ^*.

We will show

$$V(x) \le u_\varepsilon(x) \quad \text{in} \quad (\tilde{\xi}_1 + \varepsilon, \zeta). \tag{2.9}$$

For this purpose, we introduce the following auxiliary function

$$h(x) = \frac{V(x) - a^*}{u_\varepsilon(x) - a^*} \quad \text{in } [\tilde{\xi}_1 + \varepsilon, x^*],$$

and show $h(x) \leq 1$ in $[\tilde{\xi}_1 + \varepsilon, x^*]$ by contradiction. Suppose that there exists an $x_1 \in [\tilde{\xi}_1 + \varepsilon, x^*]$ such that

$$h(x_1) = \max\{h(x); x \in [\tilde{\xi}_1 + \varepsilon, x^*]\} = \frac{1}{\eta} > 1.$$

Then

$$\begin{cases} V_\eta(x) \leq u_\varepsilon(x) & \text{in } [\tilde{\xi}_1 + \varepsilon, x^*], \\ V_\eta(x_1) = u_\varepsilon(x_1), \end{cases}$$

where

$$V_\eta(x) = \eta(V(x) - a^*) + a^*.$$

We will prove

$$V_\eta''(x_1) \leq u_\varepsilon''(x_1). \tag{2.10}$$

Clearly, $h(\tilde{\xi}_1 + \varepsilon) < 1$. Moreover, since $u_\varepsilon'(x^*) > 0$ and $V'(x^*) = 0$ (by (2.7)), it is easy to see $h'(x^*) < 0$. Therefore, x_1 must be an interior point in $(\tilde{\xi}_1 + \varepsilon, x^*)$. So

$$h'(x_1) = 0 \quad \text{and} \quad h''(x_1) \leq 0. \tag{2.11}$$

From the definition of h,

$$h(x)(u_\varepsilon(x) - a^*) = V(x) - a^*.$$

Differentiating the above identity two times with respect to x we get

$$u_\varepsilon''(x_1) + 2\eta u_\varepsilon'(x_1)h'(x_1) + \eta(u_\varepsilon(x_1) - a^*)h''(x_1) = \eta V''(x_1) = V_\eta''(x_1). \tag{2.12}$$

Then (2.11) and (2.12) imply (2.10).

We next use $f(x, V_\eta) > \eta V(1 - V)(V - a^*)$. Indeed, since $V > a^* > 1/2$, a simple calculation yields this assertion. Hence it follows from (2.7) that

$$\varepsilon^2 V_\eta'' + f(x, V_\eta) = \eta\varepsilon^2 V'' + f(x, V_\eta) > \eta\{\varepsilon^2 V'' + V(1 - V)(V - a^*)\} = 0.$$

Therefore, using (2.10) we have

$$0 = \varepsilon^2 u_\varepsilon''(x_1) + f(x_1, u_\varepsilon(x_1)) \geq \varepsilon^2 V_\eta''(x_1) + f(x_1, V_\eta(x_1)) > 0,$$

which is a contradiction. Thus we have shown (2.9).

Now (2.8) and (2.9) imply

$$1 - u_\varepsilon(x^*) \le 1 - V(x^*) < c_2 e^r \exp\left(-\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon}\right).$$

Here we should note that c_2 and r are independent of x^*. Recalling that $x^* \in (\tilde{\xi}_1 + \varepsilon, \zeta)$ is arbitrary, one can conclude that

$$1 - u_\varepsilon(x) < c_2 e^r \exp\left(-\frac{r(x - \tilde{\xi}_1)}{\varepsilon}\right) \tag{2.13}$$

is valid for $x \in (\tilde{\xi}_1 + \varepsilon, \zeta)$. Moreover, since $\tilde{\xi}_1 - \xi_1 = O(\varepsilon)$, we can extend (2.13) for all $x \in [\xi_1, \zeta]$ with $\tilde{\xi}_1$ replaced by ξ_1 (for $x = \zeta$, it is sufficient to use the x-continuity of u_ε).

The left-hand-side inequality of (2.6) is shown in a similar manner. For details, see [6]. □

Using the same method as the proof of Theorem 2.5 we can also prove the following result.

Theorem 2.6. *For $u_\varepsilon \in S_{n,\varepsilon}$, assume* (2.4) *and let $\zeta \in (\xi_1, \xi_2)$ satisfy $u'(\zeta) = 0$. If $(\xi_2 - \zeta)/\varepsilon \to +\infty$, then for sufficiently small $\varepsilon > 0$, there exist positive constants C_1', C_2', r', R' ($0 < C_1' < C_2'$, $0 < r' < R'$) such that*

$$C_1' \exp\left(-\frac{R'(\xi_2 - \zeta)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2' \exp\left(-\frac{r'(\xi_2 - x)}{\varepsilon}\right) \quad \text{for } x \in [\zeta, \xi_2]. \tag{2.14}$$

Remark. Theorems 2.5 and 2.6 treat the case when $\zeta \in (\xi_1, \xi_2)$ is a local maximum point of u_ε; i.e., the case when $u_\varepsilon(\zeta)$ is very close to 1. One can also derive analogous inequalities as (2.6) and (2.14) in case that ζ is a local minimum point of u_ε and $(\zeta - \xi_1)/\varepsilon \to \infty$ as $\varepsilon \to 0$; so that $u_\varepsilon(x)$ is bounded by exponential functions from above and below.

3 Location of transition layers and spikes

In this section we introduce the following set

$$\Lambda = \{x \in [0,1]\,;\, a'(x) = 0\}$$

in addition to $\Sigma = \{x \in [0,1]\,;\, a(x) = 1/2\}$. We will show that any transition layer appears only in a neighborhood of a point of Σ and any spike appears only in a neighborhood of a point of Λ.

Theorem 3.1. *Let ξ be any point in Ξ. Then ξ lies in a neighborhood of a point in $\Sigma \cup \Lambda$ when ε is sufficiently small. Moreover, if u_ε has a transition layer near a point $x_0 \in \Sigma \cup \Lambda$, then $x_0 \in \Sigma$, and if u_ε has a spike near a point $x_0 \in \Sigma \cup \Lambda$, then $x_0 \in \Lambda$.*

Proof. If u_ε has a transition layer near $x = \xi$, it is easy to see from (i) of Lemma 2.3 that ξ lies in a neighborhood of a point in Σ. So it is sufficient to show that, if u_ε has a spike near $x = \xi$, then ξ does not lie in an interval $I \subset \{x \in [0,1]\,;\, a(x) > 1/2 \text{ and } a'(x) > 0\}$.

We employ the contradiction method. Let $\xi_k \in \Xi$ satisfy $u_\varepsilon'(\xi_k) < 0$ and assume that $\xi_k \in \Xi$ belongs to I. Then, by Lemma 2.3, we see that there exists $\xi_{k+1} \in \Xi$ which satisfies $\xi_k < \xi_{k+1}$ and $\xi_{k+1} - \xi_k = O(\varepsilon)$. We can choose local maximum and minimum points of u_ε denoted by $\zeta_{k-1}, \zeta_k, \zeta_{k+1}$ as in Lemma 2.2.

For the sake of simplicity, we only consider the case when both ζ_{k-1} and ζ_{k+1} lie in I. We rewrite (1.2) as

$$\varepsilon^2 u_\varepsilon'' + f(\zeta_k, u_\varepsilon) = u_\varepsilon(1 - u_\varepsilon)(a(x) - a(\zeta_k)). \tag{3.1}$$

Multiplying (3.1) by u_ε' and integrating it over $(\zeta_{k-1}, \zeta_{k+1})$ with respect to x we get

$$W(u_\varepsilon(\zeta_{k-1})) - W(u_\varepsilon(\zeta_{k+1})) = \int_{\zeta_{k-1}}^{\zeta_{k+1}} u_\varepsilon(1 - u_\varepsilon)(a(x) - a(\zeta_k))u_\varepsilon' dx \tag{3.2}$$

where $W(u) = W(\zeta_k, u)$. Since a is monotone increasing in $(\zeta_{k-1}, \zeta_{k+1})$, the right-hand side of (3.2) is bounded from below by

$$\int_{\zeta_k+\varepsilon}^{\zeta_{k+1}} u_\varepsilon(1-u_\varepsilon)(a(x)-a(\zeta_k))u_\varepsilon' dx > (a(\zeta_k+\varepsilon)-a(\zeta_k)) \int_{\zeta_k+\varepsilon}^{\zeta_{k+1}} u_\varepsilon(1-u_\varepsilon)u_\varepsilon' dx > L\varepsilon$$

with some constant $L > 0$. Hence

$$W(u_\varepsilon(\zeta_{k-1})) - W(u_\varepsilon(\zeta_{k+1})) > L\varepsilon. \tag{3.3}$$

Here we have used the fact that $u_\varepsilon(\zeta_{k+1})$ is very close to 1.

We next investigate the left-hand-side of (3.2). By virtue of Chauchy's mean value theorem, there exists a constant $\theta_1 \in (u_\varepsilon(\zeta_{k-1}), u_\varepsilon(\zeta_{k+1}))$ such that

$$\frac{W(u_\varepsilon(\zeta_{k-1})) - W(u_\varepsilon(\zeta_{k+1}))}{(1 - u_\varepsilon(\zeta_{k-1}))^2 - (1 - u_\varepsilon(\zeta_{k+1}))^2} = \frac{f(\zeta_k, \theta_1)}{2(1 - \theta_1)}. \tag{3.4}$$

Since $f(\zeta_k, 1) = 0$, we use Chauchy's mean value theorem again to choose a constant $\theta_2 \in (\theta_1, 1)$ which satisfies

$$\frac{f(\zeta_k, \theta_1)}{2(1 - \theta_1)} = -\frac{f(\zeta_k, 1) - f(\zeta_k, \theta_1)}{2(1 - \theta_1)} = -\frac{1}{2} f_u(\zeta_k, \theta_2). \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$W(u_\varepsilon(\zeta_{k-1})) - W(u_\varepsilon(\zeta_{k+1})) = -\frac{1}{2}f_u(\zeta_k, \theta_2)\{(1 - u_\varepsilon(\zeta_{k-1}))^2 - (1 - u_\varepsilon(\zeta_{k+1}))^2\}.$$

Since θ_2 is very close to 1, there exists a positive constant M, which is independent of ε, such that

$$W(u_\varepsilon(\zeta_{k-1})) - W(u_\varepsilon(\zeta_{k+1})) < M(1 - u_\varepsilon(\zeta_{k-1}))^2. \tag{3.6}$$

Hence (3.3) and (3.6) imply that there is a positive constant κ such that

$$1 - u_\varepsilon(\zeta_{k-1}) > \kappa\sqrt{\varepsilon}. \tag{3.7}$$

Using (3.7) and Theorem 2.6 with $x = \zeta_{k-1}$ and replacing ξ_{k+1} by ξ_k, we obtain

$$\kappa\sqrt{\varepsilon} < C_2' \exp\left(-\frac{r'(\xi_k - \zeta_{k-1})}{\varepsilon}\right) \tag{3.8}$$

with some $C_2' > 0$ and $r' > 0$. Here we should note that there exists $\xi_{k-1} \in \Xi$ such that $u_\varepsilon(x) > a(x)$ for $x \in (\xi_{k-1}, \xi_k)$. Therefore, Theorem 2.5 together with (3.7) implies

$$\kappa\sqrt{\varepsilon} < C_2 \exp\left(-\frac{r(\zeta_{k-1} - \xi_{k-1})}{\varepsilon}\right). \tag{3.9}$$

Hence (3.8) and (3.9) implies

$$\xi_k - \xi_{k-1} < K\varepsilon|\log\varepsilon| \tag{3.10}$$

with some positive constant K. This fact implies that ξ_{k-1} belongs to I if ε is small.

When ξ_{k-1} lies in I, Lemma 2.3 tells us that there must be another $\xi_{k-2} \in \Xi \cap I$. If we repeat this procedure, we see that the number of points in Ξ increases in each process. This contradicts the definition of n-mode solutions. □

4 Multiplicity of transition layers and spikes

In this section we will discuss a cluster of multiple transition layers and spikes. By Theorem 3.1, such a cluster of multiple transition layers appears in a neighborhood of a point in Σ if it exists, while a cluster of spikes appears in a neighborhood a point in Λ if it exists.

Definition 4.1 (multi-layer). Let u_ε be a solution of (1.2). If u_ε has a cluster of multiple transition layers, then such a cluster is called a **multi-layer**.

Definition 4.2 (multi-spike). Let u_ε be a solution of (1.2). If u_ε has a cluster of multiple spikes, then such a cluster is called a **multi-spike**.

We introduce some notations to study multi-layers and multi-spikes.

$$\Sigma^+ = \{x^* \in \Sigma\,;\, a'(x^*) > 0\}, \quad \Sigma^- = \{x^* \in \Sigma\,;\, a'(x^*) < 0\},$$
$$\Lambda^+ = \{x^* \in \Lambda\,;\, a(x^*) < 1/2 \text{ and } a \text{ attains its local maximum at } x = x^*\},$$
$$\Lambda^- = \{x^* \in \Lambda\,;\, a(x^*) > 1/2 \text{ and } a \text{ attains its local minimum at } x = x^*\}.$$

We begin with the study of multi-layer. We only discuss the case where u_ε has a multi-layer in a neighborhood of $z_0 \in \Sigma^+$ because the analysis for the case $z_0 \in \Sigma^-$ is almost the same.

By virtue of Lemma 2.3, there exists one-to-one correspondence between a transition layer and a zero-point of $u_\varepsilon - a$. We can show the following lemma in the same way as the proof of (3.10).

Lemma 4.1. *For $z_0 \in \Sigma^+$, let $\xi_1, \xi_2 \in (z_0 - \delta, z_0 + \delta)$ be adjacent points in Ξ satisfying $u_\varepsilon'(\xi_1) < 0$ and $u_\varepsilon'(\xi_2) > 0$ (resp. $u_\varepsilon'(\xi_1) > 0$ and $u_\varepsilon'(\xi_2) < 0$) with some $\delta > 0$. Then there exitst another $\xi \in \Xi$ such that $z_0 - \delta < \xi < \xi_1$ and $u_\varepsilon'(\xi) > 0$ (resp. $\xi_2 < \xi < z_0 + \delta$ and $u_\varepsilon'(\xi) < 0$) provided that ε is sufficiently small.*

Lemma 4.1 enables us to derive information on the profile of a multi-layer.

Lemma 4.2. *Let $z_0 \in \Sigma^+$ and assume that u_ε has a multi-layer in $(z_0 - \delta, z_0 + \delta)$ with some $\delta > 0$. If ε is sufficiently small, then $\Xi \cap (z_0 - \delta, z_0 + \delta)$ consists of odd number of points. Moreover, if*

$$\Xi \cap (z_0 - \delta, z_0 + \delta) = \{\xi_l, \ldots, \xi_m\}$$

with some $l, m \in \mathbb{N}$ such that $m - l$ is even, then $u_\varepsilon'(\xi_l) > 0$ and $u_\varepsilon'(\xi_m) > 0$.

Let u_ε have a multi-layer in a neighborhood $V(z_0)$ of $z_0 \in \Sigma^+$. Set $\Xi \cap V(z_0) = \{\xi_l, \xi_{l+1}, \ldots, \xi_m\}$. By Lemma 2.2 u_ε has critical points $\zeta_{l-1}, \zeta_l, \ldots, \zeta_m$ such that $\zeta_{l-1} < \xi_l < \zeta_l < \cdots < \xi_m < \zeta_m$. Here we should note that $u_\varepsilon(\zeta_{l-1})$ is close to 0 and that $u_\varepsilon(\zeta_m)$ is close to 1. Such a multi-layer is called a multi-layer from 0 to 1.

In the same way, we can show that if there exists a multi-layer in a neighborhood of a point in Σ^-, it must be a multi-layer from 1 to 0.

So we get the following theorem.

Theorem 4.3. *Any multi-layer from 0 to 1 (resp. from 1 to 0) appears in a neighborhood of a point in Σ^+ (resp. Σ^-).*

One can also give some results on multi-spikes.

Theorem 4.4. *Any multi-spike based on 1 (resp. 0) appears in a neighborhood of a point in Λ^- (resp. Λ^+).*

For the proofs of Theorems 4.3 and 4.4, see [6].

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