Ordinary differential systems describing hysteresis phenomena and numerical simulation (Evolution Equations and Asymptotic Analysis of Solutions)

Okazaki, Takanobu

数理解析研究所講究録 (2004), 1358: 46-68

2004-02

http://hdl.handle.net/2433/25219

Departmental Bulletin Paper

Kyoto University
Ordinary differential systems describing hysteresis phenomena and numerical simulation

千葉大学・自然科学研究科 岡崎 貴宣 (Takanobu Okazaki)
Department of Mathematics, Graduate School of Science and Technology,
Chiba University

1 Introduction

In this paper we deal with a nonlinear ordinary differential system which describes hysteresis input-output relations. Let us consider a system of the following form:

\[ aw' + bu' + \partial I_u(w) \ni F(u, w) \quad \text{in} \quad (0, \infty), \quad (1.1) \]

\[ cw' + dw' = h(u, w) \quad \text{in} \quad (0, \infty), \quad (1.2) \]

subject to the initial conditions:

\[ u(0) = u_0, \ w(0) = w_0, \quad (1.3) \]

where \( a > 0, b < 0, c > 0, d > 0 \) are given constants, \( F, h : R \times R \rightarrow R \) are Lipschitz continuous functions, \( f_*, f^* : R \rightarrow R \) are non-decreasing Lipschitz continuous functions with \( f_* \leq f^* \), \( I_u(\cdot) \) is the indicator function of the closed interval \([f_*(u), f^*(u)]\), and \( \partial I_u(\cdot) \) is its subdifferential defined by

\[
\partial I_u(w) = \begin{cases} 
\emptyset & \text{for } w > f^*(u) \text{ or } w < f_*(u), \\
[0, +\infty) & \text{for } w = f^*(u) > f_*(u), \\
\{0\} & \text{for } f_*(u) < w < f^*(u), \\
(-\infty, 0] & \text{for } w = f_*(u) < f^*(u), \\
(-\infty, +\infty) & \text{for } w = f_*(u) = f^*(u). 
\end{cases} \quad (1.4)
\]

Equation (1.1) describes a lot of input-output relations \( u \rightarrow w \) which are physically relevant. For example, when \( b = 0 \) (resp. \(-1\)), \( a = 1 \) and \( F \equiv 0 \), the relation between \( w(t) \) and \( u(t) \) is called a play (resp. stop) operator. These operators are typical examples of hysteresis input-output relations, and are used to present various phase transition effects. Moreover, in the case when \( a = 1, b = 0, c = 1, d = 1, F \equiv 0, h \equiv 0 \), the system was studied by Visintin [5]. In the general case when \( a = a(u, w), b = b(u, w), c = c(u, w), d = d(u, w) \) are functions of \( u, w \) with \( a(u, w) > 0, c(u, w) > 0, d(u, w) > 0 \) and \( a(u, w)d(u, w) - b(u, w)c(u, w) > 0 \), the existence and uniqueness results of the system were obtained in [2].
Our main objective of this paper is to study the large time behaviour of solutions of our system. The behaviour of solutions of (1.1), (1.2) depends on the coefficients $a, b, c, d$ and the functions $F, h$. Under some conditions on $a, b, c, d, F, h$ and $f_*, f^*$, we investigate the precise behaviour of orbits of solutions of our system. At the same time, we give some numerical experiments for the connection with the behaviour of the orbits.

2 Preliminaries and main results

In this section, we mention the precise assumptions on the coefficients $a, b, c, d$ and the functions $F, h, f_*, f^*$, and a theoretical result on the behaviour of orbits of solutions of our system. Now we make the following assumptions:

(A1) $F := \alpha u + \beta w, \ h := \gamma u + \delta w, \ \alpha, \beta, \gamma, \delta \in \mathbb{R}$
and $c\alpha - a\gamma = d\beta - b\delta = 0, \ d\alpha - b\gamma > 0, \ c\beta - a\delta > 0$.

(A2) Functions $f_*, f^*$ are non-decreasing Lipschitz continuous functions of $C^2$-class such that $f_*(u) \leq f^*(u)$ for all $u \in \mathbb{R}$, and there are constants $f^\infty > 0, f_\infty < 0$ and $\kappa^*>0, \kappa_*<0$ such that $f_*(u) = f^*(u) \equiv f^\infty$ for all sufficiently large $u > 0$,
$f_*(0) < 0 < f^*(0),
 f_*(u) = f^*(u) \equiv f_\infty$ for all sufficiently small $u < 0$,
$f_*(u) = f^*(u)$ for $u \in (-\infty, \kappa_*) \cup [\kappa^*, +\infty)$.

(A3) The number of connected components of the sets
\{u \in \mathbb{R} | (a\delta - c\beta)f_*(u)f'_*(u) - (d\alpha - b\gamma)u = 0\} and
\{u \in \mathbb{R} | (a\delta - c\beta)f^*(u)f'^*(u) - (d\alpha - b\gamma)u = 0\} is finite.

Assumption (A1) means that if there is no subdifferential $\partial I_u(w)$ in our system, then the orbits of solutions are anticlockwise ellipse for all initial data (especially the orbits of solutions are anticlockwise circles when $d\alpha - b\gamma = c\beta - a\delta > 0$ hold). Assumptions (A2), (A3) are concerned with the geometry of the two curves $w = f^*$ and $w = f_*$.

Especially, assumption (A3) implies that the curves $w = f_*(u)$ and $w = f^*(u)$ have a finite number of circles with center $(0, 0)$ which are tangential to the curves $w = f_*(u)$ or $w = f^*(u)$.

Under these assumptions, we give the definition of a solution of our system.

Definition 2.1 A pair of functions $\{w, u\}$ is called a solution of the system (1.1), (1.2), and (1.3) if the following (1)-(4) are satisfied:
(1) $w, u \in W^{1,2}(0, T)$ for any finite $T > 0$,
(2) $aw' + bu' + \partial I_u(w) \ni \alpha u + \beta w$ a.e. on $(0, \infty)$,
(3) $cw' + du' = \gamma u + \delta w$ on $(0, \infty)$,
(4) $u(0) = u_0$, $w(0) = w_0$.

The following theorem holds true.

**Theorem 2.1** Under these assumptions, the system (1.1)-(1.3) possesses one and only one solution.

This theorem guarantees the existence and uniqueness of solutions and it is a special case of [2; Theorem 2.4].

The precise behaviour of solutions of our system is given in the following theorem.

**Theorem 2.2** Suppose that assumptions (A1),(A2) and (A3) are satisfied. Let $S = \{(u, w) \in R^2 | f_*(u) \leq w \leq f^*(u) \}$, and denote by $\{u, w\}$ the solution of our system with initial values $u_0, w_0$. Then $S$ is divided into the following three subsets $S_1, S_2$ and $S_3$, i.e. $S = S_1 \cup S_2 \cup S_3$, such that

(i) if $(u_0, w_0) \in S_1$, then $(u(t), w(t))$ reaches a periodic ellipse around the origin in a finite time;
(ii) if $(u_0, w_0) \in S_2$, then $(u(t), w(t))$ converges (as $t \to +\infty$) to a stationary point $(u_\infty, w_\infty)$ which satisfies

$$\{ \begin{align*}
\partial I_{u_\infty}(w_\infty) \ni \alpha u_\infty + \beta w_\infty \\
\gamma u_\infty + \delta w_\infty = 0;
\end{align*} \}
$$

(iii) if $(u_0, w_0) \in S_3$, then $(u(t), w(t))$ diverges to $(+\infty, f^\infty)$ or to $(-\infty, f_-\infty)$ as $t \to +\infty$.

Moreover, the sets $S_1, S_2$ and $S_3$ are determined by the geometries of the curves $w = f_*(u)$, $w = f^*(u)$ and the line $\gamma u + \delta w = 0$ and their expressions are given in the next section.

In order to prove Theorem 2.2, we prepare the following section.
3 Subsets $S_i$ ($i = 1, 2, 3$)

In this section, we consider how to describe the subsets $S_i(i = 1, 2, 3)$ of $S$ on $(u, w)$ plane. Now we use the following notations:

$$
\Gamma^* := \{(u, w) | w = f^*(u)\}, \quad \Gamma_* := \{(u, w) | w = f_*(u)\},
$$

$$
B(u, w) := \{(d\alpha - b\gamma)u^2 + (c\beta - a\delta)w^2\}^{\frac{1}{2}}, \quad l := \{(u, w) \in \mathbb{R}^2 | \gamma u + \delta w = 0\},
$$

$$
\Gamma^*(l) := \{(u, w) \in \mathbb{R}^2 | \Gamma^* \cap l \cap u > 0\}, \quad \Gamma_*(l) := \{(u, w) \in \mathbb{R}^2 | \Gamma_* \cap l \cap u < 0\},
$$

$$
r_0^* := \min \{B(u, w) | (u, w) \in \Gamma^*\}, \quad u^* := \min \{u | (u, w) \in \Gamma^*, B(u, w) = r_0^*\},
$$

$$
r_0_* := \min \{B(u, w) | (u, w) \in \Gamma_*\}, \quad u_* := \max \{u | (u, w) \in \Gamma_*, B(u, w) = r_0_*\},
$$

$$
r_1^* := \min \{B(u, w) | (u, w) \in \Gamma^*(l)\}, \quad R_1^* := \max \{B(u, w) | (u, w) \in \Gamma^*(l)\},
$$

$$
r_1_* := \min \{B(u, w) | (u, w) \in \Gamma_*(l)\}, \quad R_1_* := \max \{B(u, w) | (u, w) \in \Gamma_*(l)\},
$$

$$
A^+ := \{(u, w) | u_*w - f_*(u_*)u < 0 \text{ if } u < 0, \quad u_*w - f^*(u_*)u > 0 \text{ if } u > 0\},
$$

$$
A^- := \{(u, w) | u_*w - f_*(u_*)u < 0 \text{ if } u > 0, \quad u_*w - f^*(u_*)u > 0 \text{ if } u < 0\},
$$

$$
S_0 := \{(u, w) \in S | B(u, w) \leq r_0\} \text{ with } r_0 := \min \{r_0^*, r_0_*\}.\n$$

By our assumptions, we have

$$
r_0^* < r_1^* \leq R_1^* \text{ and } r_0_* < r_1_* \leq R_1\ast.
$$

As to the relationships of $r_0^*, r_1^*, R_1^*, r_0, r_1, r_0^* \text{ and } R_1^*$ there are the following 6 cases to be considered:

1. $r_0 \leq r_0^* < r_1^* \leq R_1^*$
2. $r_0^* < r_1^* \leq r_0_* \leq R_1^*$
3. $r_0 < r_1^* \leq R_1^* < r_0^*$
4. $r_0^* < r_1^* \leq r_0_* \leq R_1^*$
5. $r_0 \leq r_1^* \leq r_0_* \leq R_1^*$
6. $r_0 < r_1^* \leq R_1^* < r_0^*$

In the case of (1) we define

$$
S_1 := S_0 \cup S_1^+ \cup S_1^-,
$$

where

$$
S_1^+ := \{(u, w) \in S \cap A^+ | r_0 < B(u, w) < r_1^*\},
$$

$$
S_1^- := \{(u, w) \in S \cap A^- | r_0 < B(u, w) < r_1^*\},
$$

$$
S_2 := S_2^+ \cup S_2^-,
$$

where

$$
S_2^+ := \{(u, w) \in S \cap A^+ | r_1^* \leq B(u, w) \leq R_1^*\}.
$$
$S_2^- := \{(u, w) \in S \cap A^- | r_{1*} \leq B(u, w) \leq R_{1*}\}; \hspace{1cm} (3.6)$

$S_3 := S_3^+ \cup S_3^-,$ \hspace{1cm} (3.7)

where

$S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w)\}, \hspace{1cm} (3.8)$

$S_3^- := \{(u, w) \in S \cap A^- | R_{1*} < B(u, w)\}. \hspace{1cm} (3.9)$

In the case of (2) we define

$S_1 := S_0 \cup S_1^0,$ \hspace{1cm} (3.10)

where

$S_1^0 := \{(u, w) \in S | r_{0*} < B(u, w) < r_{1*}\}; \hspace{1cm} (3.11)$

$S_2 := S_2^+ \cup S_2^-,$ \hspace{1cm} (3.12)

where

$S_2^+ := \{(u, w) \in S \cap A^+ | r_{1*} \leq B(u, w) \leq R_{1*}\}, \hspace{1cm} (3.13)$

$S_2^- := \{(u, w) \in S \cap A^+ | r_{1*} > B(u, w) \leq R_{1*}\} \cup \{(u, w) \in S \cap A^+ | r_{1*} \leq B(u, w) \leq R_{1*}\}; \hspace{1cm} (3.14)$

$S_3 := S_3^+ \cup S_3^-,$ \hspace{1cm} (3.15)

where

$S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w)\}, \hspace{1cm} (3.16)$

$S_3^- := \{(u, w) \in S \cap A^- | R_{1*} < B(u, w)\}. \hspace{1cm} (3.17)$

In the case of (3) we define

$S_1 := S_0 \cup S_1^0,$ \hspace{1cm} (3.18)

where

$S_1^0 := \{(u, w) \in S | r_{0*} < B(u, w) < r_{1*}\}; \hspace{1cm} (3.19)$

$S_2 := S_2^+ \cup S_2^-,$ \hspace{1cm} (3.20)

where

$S_2^+ := \{(u, w) \in S \cap A^+ | r_{1*} \leq B(u, w) \leq R_{1*}\}, \hspace{1cm} (3.21)$
$S_2 := \{(u,w) \in S | r_{1*} \leq B(u,w) \leq R_{1*}\}; \quad (3.22)$

$S_3 := S_3^+ \cup S_3^-$, \quad (3.23)

where

$S_3^+ := \{(u,w) \in S \cap A^+ | R_1^* < B(u,w)\}$, \quad (3.24)

$S_3^- := \{(u,w) \in S \cap A^- | R_1^* < B(u,w)\}$; \quad (3.25)

In the case of (4) we define

$S_1 := S_0 \cup S_1^+ \cup S_1^-$,

where

$S_1^+ := \{(u,w) \in S \cap A^+ | r_0^* < B(u,w) < r_1^*\}$,

$S_1^- := \{(u,w) \in S \cap A^- | r_0^* < B(u,w) < r_1^*\}$;

$S_2 := S_2^+ \cup S_2^-$,

where

$S_2^+ := \{(u,w) \in S \cap A^+ | r_1^* \leq B(u,w) \leq R_1^*\}$,

$S_2^- := \{(u,w) \in S \cap A^- | r_1^* \leq B(u,w) \leq R_1^*\}$;

$S_3 := S_3^+ \cup S_3^-$,

where

$S_3^+ := \{(u,w) \in S \cap A^+ | R_1^* < B(u,w)\}$,

$S_3^- := \{(u,w) \in S \cap A^- | R_1^* < B(u,w)\}$.

In the case of (5) we define

$S_1 := S_0 \cup S_1^0$,

where

$S_1^0 := \{(u,w) \in S | r_0^* < B(u,w) < r_1^*\}$;

$S_2 := S_2^+ \cup S_2^-$,
where
\[ S^+_2 := \{(u, w) \in S \cap A^+ | r_1^* \leq B(u, w) \leq R_1^* \} \]
\[ \cup \{(u, w) \in S \cap A^- | r_1^* \leq B(u, w) < r_1* \}; \]
\[ S^-_2 := \{(u, w) \in S \cap A^- | r_1* \leq B(u, w) \leq R_1* \}; \]
\[ S_3 := S^+_3 \cup S^-_3, \]
where
\[ S^+_3 := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \leq R_1* \} \]
\[ \cup \{(u, w) \in S \cap A^- | R_1* < B(u, w) < r_1* \} \]
\[ S^-_3 := \{(u, w) \in S \cap A^- | R_1* < B(u, w) \}. \]

In the case of (6) we define
\[ S_1 := S_0 \cup S^+_1, \]
where
\[ S^+_1 := \{(u, w) \in S | r_0^* < B(u, w) < r_1^* \}; \]
\[ S_2 := S^+_2 \cup S^-_2, \]
where
\[ S^+_2 := \{(u, w) \in S | r_1^* \leq B(u, w) \leq R_1^* \}, \]
\[ S^-_2 := \{(u, w) \in S \cap A^- | r_1* \leq B(u, w) \leq R_1* \}; \]
\[ S_3 := S^+_3 \cup S^-_3, \]
where
\[ S^+_3 := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \leq R_1* \} \]
\[ \cup \{(u, w) \in S \cap A^- | R_1* < B(u, w) < r_1* \}, \]
\[ S^-_3 := \{(u, w) \in S \cap A^- | R_1* < B(u, w) \}. \]

In any cases of (1)-(6), when the initial data belong to any subset of $S_1, S_2$ and $S_3$, the orbits of the solutions satisfy the statements (i)-(iii) of Theorem 2.2. In the next section, we prepare some Lemmas in order to prove Theorem 2.2.
4 Local behaviour of orbits

In this section, we investigate the local behaviour of the orbit \((u(t), w(t))\), satisfying

\[
\begin{align*}
aw'(t) + bu'(t) + \partial I_{u(t)}(w(t)) &\ni \alpha u(t) + \beta w(t), \\
cw'(t) + du'(t) &\ni \gamma u(t) + \delta w(t)
\end{align*}
\]

for \(t \geq 0\). We only give proof of Lemma 4.3. Other Lemmas are shown without proofs.

**Lemma 4.1** Assume that \((u(t_1), w(t_1)), t_1 \geq 0\), is in the interior of \(S\). Then:

(a) if \(B(u(t_1), w(t_1)) \leq r_0\), then \(\{u, w\}\) satisfies

\[
\begin{align*}
\frac{d\alpha - b\gamma}{ad - bc}u &= -\frac{c\beta - a\delta}{ad - bc}w, \\
\frac{d\alpha - b\gamma}{ad - bc}u &= \frac{d\alpha - b\gamma}{ad - bc}u
\end{align*}
\]

for all \(t \geq t_1\), and hence the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(C_1 := \{(u, w)|B(u, w) = B(u(t_1), w(t_1))\}\) and it is periodic in time on \([t_1, +\infty)\).

(b) if \(B(u(t_1), w(t_1)) > r_0\), then \(\{u, w\}\) satisfies system \((4.1)\) on a compact interval \([t_1, t_2]\) with \(t_2 > t_1\), where \(t_2\) is the earliest time of all \(t > t_1\) at which \((u(t), w(t)) \in \Gamma_* \cup \Gamma^*\). Hence the orbit \((u(t), w(t))\) draws an anticlockwise arc on the ellipse \(C_1\) for \(t_1 \leq t \leq t_2\).

We note that the stationary problem of \((1.1)-(1.2)\) is of the form

\[
\partial I_u(w) \ni \alpha u + \beta w, \quad \gamma u + \delta w = 0.
\]

**Lemma 4.2** (a) Let \((\tilde{u}, \tilde{w})\) be an interior point of \(S\). Then \(\{\tilde{u}, \tilde{w}\}\) is a stationary solution of \((1.1)-(1.2)\) if and only if \(\tilde{u} = 0\) and \(\tilde{w} = 0\).

(b) Let \((\bar{u}, \bar{w})\) be a boundary point of \(S\). Then \(\{u, w\}\) is a stationary solution of \((1.1)-(1.2)\) if and only if \((\bar{u}, \bar{w}) \in \Gamma_* \cup \Gamma^*\).

**Lemma 4.3** Assume that \((u(t_1), w(t_1)), t_1 \geq 0\), is on \(\Gamma_*\) and \(w(t_1) < 0\). Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) > 0\) and if there exists \(\bar{u} > u(t_1)\) such that

\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v \leq \bar{u},
\]

and moreover if

\[
\frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f_*(u)} \leq f'_*(u) \text{ for } u(t_1) \leq v \leq \bar{u},
\]

(4.2)
then \( \{ u, w \} \) satisfies
\[
\frac{d}{dt} B(u(t), w(t)) \leq 0 \text{ on } [t_1, t_2].
\] (4.4)

(b) if \( \gamma u(t_1) + \delta w(t_1) > 0 \) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma_*(l)\) with \( \bar{u} > u(t_1) \) such that
\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v < \bar{u},
\]
then \( \{ u, w \} \) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves upward along the curve \(\Gamma_*\) and converges to \((\bar{u}, \bar{w})\) as \( t \to +\infty \);

(c) if \( \gamma u(t_1) + \delta w(t_1) < 0 \) and if there exists a stationary point \((u, w) \in \Gamma_*(l)\) with \( u < u(t_1) \) such that
\[
\gamma v + \delta f_*(v) < 0 \text{ for } u < v \leq u(t_1),
\]
then \( \{ u, w \} \) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \(\Gamma_*\) and converges to \((u, w)\) as \( t \to +\infty \);

(d) if \( \gamma v + \delta f_*(v) < 0 \) holds for all \( v \leq u(t_1) \), then \( \{ u, w \} \) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \( t \to +\infty \).

**Proof.** We prove (a). We put \((u_1, w_1) = (u(t_1), w(t_1))\); note that \( w_1 = f_*(u_1) \), since \((u_1, w_1) \in \Gamma_*\). We can find a positive constant \( M \) such that
\[
\gamma v + \delta f_*(v) \geq M \text{ for } u_1 \leq v \leq \bar{u}.
\] (4.5)

Now, consider the Cauchy problem
\[
\hat{u}'(t) = \frac{\gamma \hat{u}(t) + \delta f_*(\hat{u}(t))}{cf_*(\hat{u}(t)) + d}, \quad t_1 \leq t < t_1^*,
\] (4.6)
\[
\hat{u}(t_1) = u_1
\] (4.7)
where \( t_1^* \) is the supremum of positive number \( t_1' (> t_1) \) such that problem (4.6)-(4.7) has a solution on \([t_1, t_1']\). In fact, since the function \( v \mapsto \frac{\gamma v + \delta f_*(v)}{cf_*(v) + d} \) is Lipschitz continuous in a neighborhood of \( v = u_1 \), by the general theory of ODEs the problem (4.6)-(4.7) has a (unique) local (in time) solution \( \hat{u}(t) \). It is easy to see from (4.5) that \( \hat{u}(\cdot) \) is monotonically increasing and reaches the value \( \bar{u} \) in a finite time \( t_2 \in (t_1, t_1^*) \). Now,
putting $\hat{w}(t) = f_*(\hat{u}(t))$ on $[t_1, t_2]$, we have that $\{\hat{u}, \hat{w}\}$ satisfies our system (1.1) and (1.2) on $[t_1, t_2]$. In fact, it follows from (4.6) that

$$
cf'(\hat{u}(t))\hat{u}'(t) + d\hat{u}'(t) = \gamma\hat{u}(t) + \delta f_*(\hat{u}(t)),
$$

which implies $c\hat{w}'(t) + d\hat{u}'(t) = \gamma\hat{u}(t) + \delta \hat{w}(t)$ on $[t_1, t_2]$. Thus (1.2) is satisfied. Equation (1.1) is checked as follows. By assumption (A1) and (4.2), calculating $\alpha\hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}'$, we obtain

$$
\alpha\hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}' = \alpha\hat{u} + \beta f_*(\hat{u}) - \frac{\gamma\hat{u} + \delta f_*(\hat{u})}{cf_*(\hat{u}) + d}(af_*(\hat{u}) + b)
$$

$$
= \frac{(\alpha\hat{u} + \beta f_*(\hat{u}))(cf_*(\hat{u}) + d) - (\gamma\hat{u} + \delta f_*(\hat{u}))(af_*(\hat{u}) + b)}{cf_*(\hat{u}) + d}
$$

$$
= \frac{((c\alpha - a\gamma)\hat{u} + (c\beta - a\delta)f_*(\hat{u}))f_*(\hat{u}) - (b\gamma - da)\hat{u} - (b\delta - db)f_*(\hat{u})}{cf_*(\hat{u}) + d}
$$

$$
= \frac{(c\beta - a\delta)f_*(u)f_*(u) - (b\gamma - da)u}{cf_*(u) + d}
$$

$$
\leq 0
$$
on $[t_1, t_2]$. By the definition of subdifferentials (see (1.4)) we have $\partial I_\hat{u}(\hat{w}) = (-\infty, 0]$ for $\hat{w} = f_*(\hat{u})$. Therefore

$$
\alpha\hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}' \in \partial I_\hat{u}(\hat{w}) \text{ on } [t_1, t_2].
$$

Thus, by the uniqueness, $\{\hat{u}, \hat{w}\}$ must be the solution $\{u, w\}$ of (1.1)-(1.2) on $[t_1, t_2]$. Next we show (4.4). Since (4.2) and (4.3) hold on $[t_1, t_2]$, we obtain

$$
\frac{d}{dt}B(u, w) = \frac{u'}{B(u, w)}\{(c\beta - a\delta)f_*(u)f_*(u) - (b\gamma - da)u\}
$$

$$
\leq 0 \text{ on } [t_1, t_2].
$$

Next we prove (b). Let us recall that $\bar{u} < 0$, $f_*(\bar{u}) < 0$ by Lemma 4.2 (b). We obtain automatically

$$
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f'_*(v) \text{ for } u_1 \leq v \leq \bar{u}.
$$

(4.8)

Therefore, in the same way as in (a), $\{u, w\}$ satisfies (4.3) for a moment after the time $t_1$ and the orbit $(u(t), w(t))$ moves along the curve $\Gamma_*$ starting from $(u(t_1), w(t_1))$. We now
show that \((u(t), w(t))\) converges to \((\bar{u}, \bar{w})\) as \(t \to +\infty\). Let \(T\) be the supremum of all \(s \geq t_1\) such that
\[
u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{c f_*(u(t)) + d}, \quad w(t) = f_*(u(t)) \text{ for } \forall t \in [t_1, s].
\]
Then, just as in the case of (a), we see that \(T > t_1\). Since \(\nu\) is non-decreasing on \([t_1, T)\), \(\lim_{t \to T} \nu(t)\) exists. We want to see that \(\lim_{t \to T} \nu(t) = \bar{u}\). We show it by contradiction. Now, assume that \(\lim_{t \to T} \nu(t) < \bar{u}\). Then we consider the following statements:

(i) \(T = +\infty\), \(u_\infty := \lim_{t \to +\infty} \nu(t)\) and \(w_\infty := \lim_{t \to +\infty} w(t)\) give a pair of stationary solutions
or

(ii) \(T < \infty\) and \(\frac{(d\alpha - b\gamma)u(t)}{(a\delta - c\beta)f_*(u(t))} > f'_*(u(t))\) for some \(t > T\).

But these cases do not occur in our situations considered now. In fact, the case (i) yields that \(\nu(t_1) \leq u_\infty < \bar{u}\) and \(\gamma u_\infty + \delta f_*(u_\infty) = 0\), which contradicts our assumption. Also, the case (ii) yields a contradiction to (4.8).

Assertion (c) is similarly proved to (b).

Finally we prove (d). By the same argument as above, we have
\[
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f'_*(v) \text{ for all } v \leq u_1,
\]
and find a negative constant \(\tilde{M}\) such that
\[
\gamma v + \delta f_*(v) \leq \tilde{M} \text{ for all } v \leq u_1.
\]
Hence \(\{u, w\}\) satisfies (4.3) for all \(t \geq t_1\) and \(\nu()\) is monotonically decreasing on \([t_1, \infty)\).

By assumption (A2), \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \(t \to +\infty\).

**Lemma 4.4** Assume that \((u(t_1), w(t_1)), t_1 \geq 0\), is on \(\Gamma^*\) and \(w(t_1) > 0\). Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) < 0\) and if there exists \(\bar{u} < u(t_1)\) such that
\[
\gamma v + \delta f_*(v) < 0 \text{ for } \bar{u} \leq u(t_1) \leq v,
\]
and moreover if the following condition hold that
\[
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f'^*(v) \text{ for } \bar{u} \leq v \leq u(t_1),
\]

then \( \{u, w\} \) satisfies
\[
  u'(t) = \frac{\gamma u + \delta f^*(u)}{cf^*(u) + d}, \quad w'(t) = f^*(u)u'(t)
\]
on a compact interval \([t_1, t_2]\), where \( t_2 \) is the earliest time at which \( u(t_2) = \bar{u} \), and the orbit \((u(t), w(t))\) moves along \( \Gamma_* \) from \((u(t_1), w(t_1))\) to \((\bar{u}, f^*(\bar{u}))\) for \( t_1 \leq t \leq t_2 \). Moreover
\[
  \frac{d}{dt} B(u, w) \leq 0 \text{ on } [t_1, t_2].
\]

(b) if \( \gamma u(t_1) + \delta w(t_1) < 0 \) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma^*(l)\) with \( \bar{u} < u(t_1) \) such that
\[
  \gamma v + \delta f_*(v) < 0 \text{ for } \bar{u} < v \leq u(t_1),
\]
then \( \{u, w\} \) satisfies (4.4) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \( \Gamma^* \) and converges to \((\bar{u}, \bar{w})\) as \( t \to +\infty \).

(c) if \( \gamma u(t_1) + \delta w(t_1) > 0 \) and if there exists a stationary point \((u, w) \in \Gamma^*(l)\) with \( u > u(t_1) \) such that
\[
  \gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v < u,
\]
then \( \{u, w\} \) satisfies (4.4) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) moves upward along the curve \( \Gamma^* \) and converges to \((u, w)\) as \( t \to +\infty \).

(d) if \( \gamma v + \delta f_*(v) > 0 \) holds for all \( v \geq u(t_1) \), then \( \{u, w\} \) satisfies (4.4) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) diverges to \((\infty, f^\infty)\) as \( t \to +\infty \).

5 Large time behaviour of orbits

In this section, we prove Theorem 2.2 in the case (1) in section 3. Any other cases can be treated by a simple modification of them. We investigate the behaviour of the solution \( \{u, w\} \) when the initial data \((u_0, w_0)\) belong to each of \( S_0, S_1, S_2 \) and \( S_3 \).

In the case of \((u_0, w_0) \in S_0\)

When \((u_0, w_0) \in S_0\), we obtain \( B(u_0, w_0) \leq r_0 \). Therefore, by Lemma 4.1(a), we see that the orbit \((u(t), w(t))\) draws anticlockwise ellipse \( B(u, w) = B(u_0, w_0) \) for all \( t \geq 0 \), and is periodic in time.

In the case of \((u_0, w_0) \in S_1\)

First, we consider the case of \((u_0, w_0) \in S_1^-\) and \( w_0 \leq 0 \). Clearly \( B(u_0, w_0) > r_0 \).
By Lemma 4.1 (b), the orbit \((u(t), w(t))\) draws an anticlockwise ellipse on \(B(u, w) = B(u_0, w_0)\), untill it reaches \(\Gamma_*\), satisfying

\[
\begin{align*}
  u'(t) &= -\frac{c\beta - a\delta}{ad - bc} w(t), \quad 0 \leq t \leq t_1, \\
  w'(t) &= \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad 0 \leq t \leq t_1,
\end{align*}
\]

where \(t_1\) is the earliest time such that \((u(t_1), w(t_1)) \in \Gamma_*\). We have \(w(t_1) = f_*(u(t_1))\), \(B(u(t_1), w(t_1)) < r_{1*}\) and

\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v \leq u_*.
\]

Next, take the number \(u_2\) so that

\[
u_2 = \sup \left\{ \bar{u} \mid u(t_1) \leq \bar{u} \leq u_*, \frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f(u)} \leq f'_*(u) \text{ for } \forall u \in [u(t_1), \bar{u}] \right\}.
\]

Then we have the following three possibilities: (i) \(u(t_1) < u_2 < u_*\), (ii) \(u_2 = u(t_1)\), (iii) \(u_2 = u_*\).

In the case of (i), by Lemma 4.3 (a)

\[
\begin{align*}
  u'(t) &= \frac{\gamma u(t) + \delta f_*(u(t))}{cf'_*(u(t)) + d}, \quad w(t) = f_*(u(t)), \quad t \in [t_1, t_2]
\end{align*}
\]

where \(t_2\) is the earliest time such that \(u(t_2) = u_2\). We denote by \(C_2\) the ellipse \(B(u, w) = B(u_2, f_*(u_2)) =: r_2\). By assumption (A3) and the definition of \(u_2\), we see that an arc \(\{(u, w) \mid u_2 \leq u \leq \tilde{u}_3, \ B(u, w) = r_2\}\) on \(C_2\) is contained in \(S\). Now, denote by \(u_3\) the largest one of such numbers \(\tilde{u}_3\), we have \(u_3 > u_2\). Moreover, by Lemma 4.1 (b), \(\{u, w\}\) is given by

\[
\begin{align*}
  u'(t) &= -\frac{c\beta - a\delta}{ad - bc} w(t), \quad w'(t) = \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad t \in [t_2, t_3],
\end{align*}
\]

where \(t_3\) is the earliest time such that \(u(t_3) = u_3\). Our assumption (A3) guarantees that the orbit \((u(t), w(t))\) reaches \((u_*, f_*(u_*))\) at \(t = t_* (< \infty)\) by repeating finitely many times such behaviours as above. Here, after the time \(t_*\), the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(B(u, w) = r_0\) periodically in time (see Lemma 4.1 (a)).

In the case of (ii), it is the case that \(t_1 = t_2\) with the same notation as above, and the behaviour of \((u(t), w(t))\) is similar to the case of (i) after the time \(t_2\).

In the case of (iii), it is the case that \(t_2 = t_*\), and the behaviour of \((u(t), w(t))\) is the
anticlockwise ellipse $B(u, w) = r_0$ after the time $t_*$.

Next, consider the case of $(u_0, w_0) \in S_1^-$ with $w_0 > 0$. In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_1$. If $(u(s_1), w(s_1)) \in \Gamma_*$, then the behaviour of $(u(t), w(t))$ is exactly the same as in the previous case after time $s_1$. On the other hand, if $(u(s_1), w(s_1)) \in \Gamma^*$, then the orbit $(u(t), w(t))$ moves downward for a time interval $[s_1, s_2]$ with $s_1 \leq s_2$ along the curve $\Gamma^*$ by Lemma 4.4 (a) (in this step assumption (A3) regarding the function $f^*(\cdot)$ is used), where $s_2$ is the largest time of $s_2$ such that $(u(t), w(t)) \in \Gamma^*$ for $\forall t \in [s_1, s_2]$. It is easy to see that $w(s_2) > 0$ and $s_2 < +\infty$. After time $s_2$, the orbit $(u(t), w(t))$ draws an anticlockwise arc on $B(u, w) = B(u(s_2), w(s_2))$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_3$. Repeating such procedures finitely many times, the orbit $(u(t), w(t))$ arrives at $\Gamma_*$ at time $t = t_1$ in the last step. After time $t_1$, the behaviour of $(u(t), w(t))$ was already seen in the case of $(u_0, w_0) \in S_1^-$ with $w_0 \leq 0$.

Finally, we consider the case of $(u_0, w_0) \in S_1^+$. We have the following three cases:

(i) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) \geq r_0^*$

(ii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r_0^*$ and $w_0 \geq 0$

(iii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r_0^*$ and $w_0 < 0$

First, we consider the case (i). In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = r \in [r_0^*, r_1^*]$ and a part of $\Gamma^*$ alternately and reaches the point $(u^*, f^*(u^*))$ at a finite time $t = t^*$. Since $(u^*, f^*(u^*)) \in S_1^-$, the behaviour of $(u(t), w(t))$ after the time $t^*$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 > 0$.

In the second case (ii), the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ and reaches a point $(u_0, w_0) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the third case (iii), the orbit $(u(t), w(t))$ possibly draws an anticlockwise arc on the ellipse $B(u, w) = r \in (r_0^*, r_1^*]$ and a part of $\Gamma_*$ alternately and reaches a point $(u_1, w_1) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a finite time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the case of $(u_0, w_0) \in S_2$

We give a proof only in the case of $(u_0, w_0) \in S_2^-$, since the proof of the case of $(u_0, w_0) \in S_2^+$ is quite similar. In a way similar to that in the case of $(u_0, w_0) \in S_1$, we see that the orbit $(u(t), w(t))$, drawing an anticlockwise arc on the ellipse $B(u, w) = r \in [r_1^*, R_1^*]$, arrives at a point $(u_1, w_1) \in \Gamma_*$ at a certain finite time $t = t_1$. If $(u(t_1), w(t_1)) = (u_1, w_1) \in \Gamma_*(l)$, then $(u(t_1), w(t_1))$ is a stationary solution of (1.1)-(1.3) by Lemma 4.2 (b). If $(u(t_1), w(t_1)) \notin \Gamma_*(l)$, then we have the following two cases:
(i) $\gamma u(t_1) + \delta w(t_1) > 0,$
(ii) $\gamma u(t_1) + \delta w(t_1) < 0.$

Suppose now that (i) holds. Then there is a closed interval $[\overline{u}, \underline{u}] \subset (-\infty, 0)$ on the $u$-axis such that $\underline{u} < u(t_1) < \overline{u}, \gamma v + \delta f_{*}(v) > 0$ for all $v \in (\overline{u}, \underline{u})$ and $\gamma \underline{u} + \delta f_{*}(\underline{u}) = \gamma \overline{u} + \delta f_{*}(\overline{u}) = 0.$ Therefore, the orbit $(u(t), w(t))$ converges to $(\overline{u}, f_{*}(\overline{u})) \in \Gamma_{*}(l)$ as $t \to +\infty$ by Lemma 4.3 (b). On the other hand, when (ii) holds, the orbit $(u(t), w(t))$ converges to a stationary point as $t \to +\infty$, too.

**In the case of $(u_0, w_0) \in S_3$**

It is enough to consider only the case $(u_0, w_0) \in S_3^{-}$. In the same way as in the case of $(u_0, w_0) \in S_2$, the orbit $(u(t), w(t))$ reaches $\Gamma_{*}$ in a finite time $t_1$. Also, we obtain $B(u(t_1), w(t_1)) < R_{1*}$ and $\gamma v + \delta f_{*}(v) < 0$ for $v < u_1$. Therefore, by Lemma 4.3 (d), we see that $(u(t), w(t))$ diverges to $(-\infty, f_{\infty})$ as $t \to +\infty$. Similarly, in the case $(u_0, w_0) \in S_3^{+}$, we see that $(u(t), w(t))$ diverges to $(\infty, f_{\infty})$ as $t \to +\infty$.

**Remark 5.1** We have many cases about the stability around stationary points in $S_2$. If, for instance, we restrict our geometry of the curves $\Gamma_{*}$, $\Gamma^{*}$ and $l$ to the one as illustrated by the picture (Fig. 1), then stationary points are classified into the following three categories: Let $(u_{\infty}, w_{\infty})$ be any stationary point in $S_2$. Then one of the following cases happens.

![Fig. 1](image-url)

(1) $(u_{\infty}, w_{\infty})$ is stable. Namely, there is a neighborhood $U_1$ of $(u_{\infty}, w_{\infty})$ in $R^2$ such that the orbit $(\tilde{u}(t), \tilde{w}(t))$ stays in $U_1 \cap S$ for all $t \geq 0$ and converges to $(u_{\infty}, w_{\infty})$ as $t \to +\infty$, whenever $(\tilde{u}_0, \tilde{w}_0) = (\tilde{u}(0), \tilde{w}(0)) \in U_1 \cap S.$

(2) $(u_{\infty}, w_{\infty})$ is semistable. Namely, there is a neighborhood $U_2$ of $(u_{\infty}, w_{\infty})$ in $R^2$ such that the following properties (i) and (ii) are satisfied:

(i) For any initial point $(\tilde{u}_0, \tilde{w}_0) \in U_2 \cap S \cap K_{\infty}$, the orbit $(\tilde{u}(t), \tilde{w}(t))$ stays in $U_2 \cap S$ for all $t \geq 0$ and converges to $(u_{\infty}, w_{\infty})$ as $t \to +\infty$, whenever $(\tilde{u}_0, \tilde{w}_0) = (\tilde{u}(0), \tilde{w}(0)) \in U_2 \cap S.$

(ii) For any initial point $(\tilde{u}_0, \tilde{w}_0) \in U_2 \cap S \cap K_{\infty}$, the orbit $(\tilde{u}(t), \tilde{w}(t))$ gets out of $U_2$ after a certain time $t_1$. 
where $\mathcal{K}_{\infty} := \{(u, w)|B(u, w) \geq (u_{\infty}, w_{\infty})\}$.

(3) $(u_{\infty}, w_{\infty})$ is unstable. Namely, there is a neighborhood $U_{3}$ of $(u_{\infty}, w_{\infty})$ in $\mathbb{R}^{2}$ such that the following properties (iii) and (iv) are satisfied:

(iii) For any initial point $(\hat{u}_{0}, \hat{w}_{0}) \in U_{3} \cap S \cap C_{\infty}$, the orbit $(\hat{u}, \hat{w})$ stays in $U_{3} \cap S$ for all $t \geq 0$ and converges to $(u_{\infty}, w_{\infty})$ in a finite time $t_{1}$.

(iv) For any initial point $(\check{u}_{0}, \check{w}_{0}) \in U_{3} \cap S \cap C_{\infty}^{c}$, the orbit $(\check{u}(t), \check{w}(t))$ gets out of $U_{3}$ after a certain time $t_{1}$.

where $C_{\infty} := \{(u, w)|B(u, w) = B(u_{\infty}, w_{\infty})\}$.

6 Some numerical simulations

In this section, we give some numerical experiments to verify Theorem 2.2. In order to catch the behaviour of solutions, we simply take the coefficients $a, b, c, d$ and functions $F, h$ satisfying (A1) with $da - b\gamma = c\beta - a\delta > 0$ such that the orbits of solutions are anticlockwise circles without subdifferential term $\partial I_{u}(w)$. Now we fix the coefficients $a, b, c, d$ and functions $F, h$ as follows:

$$a = 1, \quad b = -1, \quad c = 1, \quad d = 1, \quad F(u, w) = u + w, \quad h(u, w) = u - w.$$  

In this case, our system is of the following form:

$$w' - u' + \partial I_{u}(w) \ni u + w, \quad 0 < t < T,$$

$$w' + u' = u - w, \quad 0 < t < T,$$

$$u(0) = u_{0}, \quad w(0) = w_{0}.$$  

Now let $\lambda$ and $\Delta t$ be small positive numbers, and $n$ be a large natural number. Then the difference scheme for our numerical simulation is of the form

$$\frac{w^{k+1} - w^{k}}{\Delta t} - \frac{u^{k+1} - u^{k}}{\Delta t} + \partial I_{u}^{\lambda}(w^{k+1}) = u^{k} + w^{k},$$

$$\frac{w^{k+1} - w^{k}}{\Delta t} + \frac{u^{k+1} - u^{k}}{\Delta t} = u^{k} - w^{k}, \quad k = 0, 1, 2, \ldots ,$$

$$u^{0} = u_{0}, \quad w^{0} = w_{0},$$

where

$$\partial I_{u}^{\lambda}(w^{k+1}) = \frac{[w^{k+1} - f^{\ast}(u^{k})]^{+}}{\lambda} - \frac{f_{\ast}(u^{k}) - w^{k+1}]^{+}}{\lambda}.$$  

The graphs of $I_{u}^{\lambda}$ and $\partial I_{u}^{\lambda}$ are illustrated in Figures 2 and 3, respectively.
In our actual computation
\[ \Delta t = \frac{1}{1000}, \quad \lambda = \frac{1}{1000}, \]
and we examine the following items:

- We define the subset \( S_i(i = 0, 1, 2, 3) \) by the geometries of the given functions \( f_*(u) \) and \( f^*(u) \) and the line \( \gamma u + \delta w = 0 \).

- By numerical simulations, we verify that the behaviour of solutions satisfies the statements of Theorem 2.2 when the initial data belong to each subset \( S_i(i = 0, 1, 2, 3) \).

**Experiment 1:**

We take the functions \( f_*(u), f^*(u) \) as follows:

\[
\begin{align*}
f_*(u) &= \begin{cases} 
-1 & \text{if } u \leq 0.4, \\
5u^2 - 4u - 0.2 & \text{if } 0.4 < u \leq 0.6, \\
2u - 2 & \text{if } 0.6 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
1 & \text{if } 1.6 < u,
\end{cases} \\
f^*(u) &= \begin{cases} 
-1 & \text{if } u \leq -1.6, \\
5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
-5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
1 & \text{if } -0.4 < u.
\end{cases}
\end{align*}
\]

\( f_*(u) \) and \( f^*(u) \) are symmetric with respect to origin. In this case, by our choice of \( f_*(u), f^*(u) \) and the line \( u + w = 0 \), we obtain that

\[ r_0 := r_0^* = \frac{2\sqrt{5}}{5}, \quad r_1 = r_1^* = R_1 = R_1^* = \sqrt{2} \]

and stationary points are \((1, 1), (0, 0)\) and \((-1, -1)\).

Therefore, subsets \( S_i(i = 0, 1, 2, 3) \) are defined by (3.1)-(3.9) and are illustrated by Figure 4. Now we take the initial data which belong to each subset \( S_i \) and numerical experiments are shown as follows:

<table>
<thead>
<tr>
<th>data</th>
<th>( u_0 )</th>
<th>( w_0 )</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 5</td>
<td>-0.2</td>
<td>-0.6</td>
<td>((u_0, w_0) \in S_0)</td>
</tr>
<tr>
<td>Fig. 6</td>
<td>-0.7</td>
<td>-0.8</td>
<td>((u_0, w_0) \in S_1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>( u_0 )</th>
<th>( w_0 )</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 7</td>
<td>-1.3</td>
<td>-0.8</td>
<td>((u_0, w_0) \in S_2)</td>
</tr>
<tr>
<td>Fig. 7</td>
<td>1.3</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_3)</td>
</tr>
</tbody>
</table>
When the initial data belong to $S_0$, the orbit draws anticlockwise circle from the initial point $(u_0, w_0)$ (Fig. 5). In the case when $(u_0, w_0) \in S_1$, the orbit draws an anticlockwise arc and a part of $\Gamma_*$ alternately and reaches a periodic circle $B(u, w) = r_0$ in a finite time (Fig. 6). On the other hand, in the case when $(u_0, w_0) \in S_3^-$ or $S_3^+$, the orbit diverges to $(-\infty, -1)$ or $(+\infty, 1)$ as $t \to +\infty$ (Fig. 7).

**Experiment 2:**

We take the functions $f_*(u), f^*(u)$ as follows

$$f_*(u) = \begin{cases} 
-1 & \text{if } u \leq -1, \\
3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\
-2 - 0.25 & \text{if } -0.75 < u \leq 0, \\
-0.25 & \text{if } 0 < u \leq 0.75, \\
4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
2u - 2 & \text{if } 1 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
1 & \text{if } 1.6 < u.
\end{cases}$$

$$f^*(u) = \begin{cases} 
-1 & u \leq -2, \\
u^2 + 4u + 3 & -2 < u \leq -1.5, \\
u + 0.75 & -1.5 < u \leq 0, \\
-u^2 + u + 0.75 & 0 < u \leq 0.5, \\
1 & 0.5 < u.
\end{cases}$$
In this case, we obtain that
\[
    r_0 := r_{0*} = \frac{1}{4}, \quad R_0 = \frac{3\sqrt{2}}{8}, \quad r_1* = \frac{\sqrt{2}}{2}, \quad R_1* = \sqrt{2}
\]
and

stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

Since \(r_{0*} < r_0^* < r_1* < R_1*\), \(S_i(i = 0, 1, 2, 3)\) are defined by (3.1)-(3.9) and are illustrated by Figure. 8. The initial data and the subsets \(S_i\) in which the initial data are given in this experiments are as follows

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 9</td>
<td>-1.1</td>
<td>0.4</td>
<td>((u_0, w_0) \in S_1)</td>
</tr>
<tr>
<td>Fig. 10</td>
<td>-1.4</td>
<td>-0.4</td>
<td>((u_0, w_0) \in S_2^-)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We also see that the behaviour of orbits of solutions for each initial data \((u_0, w_0) \in S_i(i = 0, 1, 2, 3)\) guarantee Theorem 2.2 (Fig. 9-11). Especially by Fig. 10 and 11, we
can recognize that the point \((-0.5, -0.5)\) is a semi stable stationary solution.

**Experiment 3:**

We take the functions \(f_\ast(u), f^\ast(u)\) as follows

\[
f_\ast(u) = \begin{cases} 
-1 & \text{if } u \leq -1, \\
3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\
-u^2 - 0.25 & \text{if } -0.75 < u \leq 0, \\
-0.25 & \text{if } 0 < u \leq 0.75, \\
4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
2u - 2 & \text{if } 1 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
1 & \text{if } 1.6 < u.
\end{cases}
\]

\[
f^\ast(u) = \begin{cases} 
-1 & \text{if } u \leq -1.6, \\
5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
-5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
1 & \text{if } -0.4 < u.
\end{cases}
\]

In this case, we obtain that

\[
r_0 := r_{0\ast} = \frac{1}{4}, \quad r_0^\ast = \frac{2\sqrt{5}}{5}, \quad r_{1\ast} = \frac{\sqrt{2}}{2}, \quad R_{1\ast} = \sqrt{2},
\]

and

stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

This implies that \(r_{0\ast} < r_{1\ast} < r_0^\ast < R_{1\ast}\). Therefore, \(S_i(i = 0, 1, 2, 3)\) are defined by (3.10)-(3.17) (Fig. 12). Given initial data, our experiments are the following (Fig. 13-15):

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 13</td>
<td>0.25</td>
<td>0.25</td>
<td>((u_0, w_0) \in S_1)</td>
</tr>
<tr>
<td>Fig. 14</td>
<td>0.8</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 15</td>
<td>-1.3</td>
<td>-0.8</td>
<td>((u_0, w_0) \in S_3)</td>
</tr>
<tr>
<td>Fig. 15</td>
<td>1.3</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_3)</td>
</tr>
</tbody>
</table>

![Fig. 12](image1.png)  
![Fig. 13](image2.png)
Note that the function $f_*(u)$ is the same as in experiment 2 but $f^*(u)$ is not. We see that the orbit starting from $(0.8,0.8)$ draws an anticlockwise arc and a part of $\Gamma^r$ alternately and reaches $\Gamma_*$ in a finite time, and then it goes to the semi stable stationary point $(-0.5,-0.5)$ as $t \to +\infty$.

**Experiment 4:**

We take the functions $f_*(u), f^*(u)$ as follows

$$f_*(u) = \begin{cases} 
-1 & \text{if } u \leq -1.25, \\
3u^2 + 7u + 3.6875 & \text{if } -1.25 < u \leq -1, \\
-u^2 - u - 0.3125 & \text{if } -1 < u \leq -0.25, \\
-0.25 & \text{if } -0.25 < u \leq -0.75, \\
4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
2u - 2 & \text{if } 1 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
1 & \text{if } 1.6 < u. 
\end{cases}$$

$$f^*(u) = \begin{cases} 
-1 & \text{if } u \leq -1.6, \\
5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
-5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
1 & \text{if } -0.4 < u.
\end{cases}$$

$f^*(u)$ is the same function as in experiment 3 and $f_*(u)$ slightly changes from the one in experiment 3. In this case, we obtain that

$$r_0 := r_{0*} = \frac{1}{16}, \quad r_0^* = \frac{4}{5}, \quad r_1 = R_{1*} = \frac{1}{8},$$

and stationary solutions are $(1,1)$, $(0,0)$ and $(-0.5,-0.5)$.

Since $r_{0*} < r_1 = R_{1*} < r_0^*$, subset $S_i (i = 0, 1, 2, 3)$ are defined by (3.18)-(3.24) (see Fig. 16). We take the initial data as follows:

<table>
<thead>
<tr>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 17</td>
<td>0.22</td>
<td>0.22</td>
<td>$(u_0, w_0) \in S_1$</td>
</tr>
<tr>
<td>Fig. 18</td>
<td>0.8</td>
<td>0.8</td>
<td>$(u_0, w_0) \in S_4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 19</td>
<td>1.3</td>
<td>0.8</td>
<td>$(u_0, w_0) \in S_3$</td>
</tr>
</tbody>
</table>

Then our experiments results are shown by Fig. 17-19.
These numerical experiments show that the subsets $S_i (i = 0, 1, 2, 3)$ are completely different from those in experiment 3. When the initial datum is $(0.8, 0.8)$, the orbit draws an anticlockwise arc and a part of $\Gamma^*$ alternately and reaches $\Gamma_*$ in a finite time, and moving along the curve $w = f_* (u)$ downward and diverges to $(-\infty, -1)$ as $t \to +\infty$.

References


