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Ordinary differential systems describing hysteresis phenomena and numerical simulation

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1 Introduction

In this paper we deal with a nonlinear ordinary differential system which describes hysteresis input-output relations. Let us consider a system of the following form:

\begin{align}
aw' + bu' + \partial I_u(w) &\ni F(u, w) \text{ in } (0, \infty), \\
cw' + du' &= h(u, w) \text{ in } (0, \infty),
\end{align}

subject to the initial conditions:

\begin{align}
u(0) = u_0, \ w(0) = w_0,
\end{align}

where \(a > 0, b < 0, c > 0, d > 0\) are given constants, \(F, h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) are Lipschitz continuous functions, \(f_*, f^* : \mathbb{R} \to \mathbb{R}\) are non-decreasing Lipschitz continuous functions with \(f_* \leq f^*\), \(I_u(\cdot)\) is the indicator function of the closed interval \([f_*(u), f^*(u)]\), and \(\partial I_u(\cdot)\) is its subdifferential defined by

\[\partial I_u(w) = \begin{cases}
\emptyset & \text{for } w > f^*(u) \text{ or } w < f_*(u), \\
[0, +\infty) & \text{for } w = f^*(u) > f_*(u), \\
\{0\} & \text{for } f_*(u) < w < f^*(u), \\
(-\infty, 0] & \text{for } w = f_*(u) < f^*(u), \\
(-\infty, +\infty) & \text{for } w = f_*(u) = f^*(u).
\end{cases}\]

Equation (1.1) describes a lot of input-output relations \(u \to w\) which are physically relevant. For example, when \(b = 0\) (resp. \(-1\)), \(a = 1\) and \(F \equiv 0\), the relation between \(w(t)\) and \(u(t)\) is called a play (resp. stop) operator. These operators are typical examples of hysteresis input-output relations, and are used to present various phase transition effects. Moreover, in the case when \(a = 1, b = 0, c = 1, d = 1, F \equiv 0, h \equiv 0\), the system was studied by Visintin [5]. In the general case when \(a = a(u, w), b = b(u, w), c = c(u, w), d = d(u, w)\) are functions of \(u, w\) with \(a(u, w) > 0, c(u, w) > 0, d(u, w) > 0\) and \(a(u, w)d(u, w) - b(u, w)c(u, w) > 0\), the existence and uniqueness results of the system were obtained in [2].
Our main objective of this paper is to study the large time behaviour of solutions of our system. The behaviour of solutions of (1.1),(1.2) depends on the coefficients $a,b,c,d$ and the functions $F, h$. Under some conditions on $a,b,c,d,F,h$ and $f_*, f^*$, we investigate the precise behaviour of orbits of solutions of our system. At the same time, we give some numerical experiments for the connection with the behaviour of the orbits.

2 Preliminaries and main results

In this section, we mention the precise assumptions on the coefficients $a,b,c,d$ and the functions $F,h,f_*, f^*$, and a theoretical result on the behaviour of orbits of solutions of our system. Now we make the following assumptions:

\begin{enumerate}
\item[(A1)] $F := \alpha u + \beta w$, $h := \gamma u + \delta w$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$
\quad and $c\alpha - a\gamma = d\beta - b\delta = 0$, $d\alpha - b\gamma > 0$, $c\beta - a\delta > 0$.
\item[(A2)] Functions $f_*, f^*$ are non-decreasing Lipschitz continuous functions
\quad of $C^2$-class such that $f_*(u) \leq f^*(u)$ for all $u \in \mathbb{R}$, and there are
\quad constants $f^\infty > 0, f_\infty < 0$ and $\kappa^*>0,\kappa_*<0$ such that
\quad $f_*(u) = f^*(u) \equiv f^\infty$ for all sufficiently large $u > 0$,
\quad $f_*(0) < 0 < f^*(0)$,
\quad $f_*(u) = f^*(u) \equiv f_\infty$ for all sufficiently small $u < 0$,
\quad $f_*(u) = f^*(u)$ for $u \in (-\infty, \kappa_*] \cup [\kappa^*, +\infty)$.
\item[(A3)] The number of connected components of the sets
\quad $\{u \in \mathbb{R} | (a\delta - c\beta)f_*(u)f'_*(u) - (d\alpha - b\gamma)u = 0\}$ and
\quad $\{u \in \mathbb{R} | (a\delta - c\beta)f^*(u)f'_*(u) - (d\alpha - b\gamma)u = 0\}$ is finite.
\end{enumerate}

Assumption (A1) means that if there is no subdifferential $\partial I_u(w)$ in our system, then the orbits of solutions are anticlockwise ellipse for all initial data (especially the orbits of solutions are anticlockwise circles when $d\alpha - b\gamma = c\beta - a\delta > 0$ hold). Assumptions (A2), (A3) are concerned with the geometry of the two curves $w = f^*$ and $w = f_*$. Especially, assumption (A3) implies that the curves $w = f_*(u)$ and $w = f^*(u)$ have a finite number of circles with center $(0,0)$ which are tangential to the curves $w = f_*(u)$ or $w = f^*(u)$.

Under these assumptions, we give the definition of a solution of our system.

\textbf{Definition 2.1} A pair of functions $\{w,u\}$ is called a solution of the system (1.1), (1.2), and (1.3) if the following (1)-(4) are satisfied:
The following theorem holds true.

**Theorem 2.1** Under these assumptions, the system (1.1)-(1.3) possesses one and only one solution.

This theorem guarantees the existence and uniqueness of solutions and it is a special case of [2; Theorem 2.4].

The precise behaviour of solutions of our system is given in the following theorem.

**Theorem 2.2** Suppose that assumptions (A1),(A2) and (A3) are satisfied. Let $S = \{(u, w) \in \mathbb{R}^2| f_*(u) \leq w \leq f^*(u)\}$, and denote by $\{u, w\}$ the solution of our system with initial values $u_0, w_0$. Then $S$ is divided into the following three subsets $S_1, S_2$ and $S_3$, i.e., $S = S_1 \cup S_2 \cup S_3$, such that

(i) if $(u_0, w_0) \in S_1$, then $(u(t), w(t))$ reaches a periodic ellipse around the origin in a finite time;

(ii) if $(u_0, w_0) \in S_2$, then $(u(t), w(t))$ converges (as $t \to +\infty$) to a stationary point $(u_\infty, w_\infty)$ which satisfies

\[
\begin{align*}
\partial I_{u_\infty}(w_\infty) &\ni \alpha u_\infty + \beta w_\infty \\
\gamma u_\infty + \delta w_\infty & = 0;
\end{align*}
\]

(iii) if $(u_0, w_0) \in S_3$, then $(u(t), w(t))$ diverges to $(+\infty, f^\infty)$ or to $(-\infty, f^\infty)$ as $t \to +\infty$.

Moreover, the sets $S_1, S_2$ and $S_3$ are determined by the geometries of the curves $w = f_*(u), w = f^*(u)$ and the line $\gamma u + \delta w = 0$ and their expressions are given in the next section.

In order to prove Theorem 2.2, we prepare the following section.
3 Subsets $S_i$ ($i = 1, 2, 3$)

In this section, we consider how to describe the subsets $S_i (i = 1, 2, 3)$ of $S$ on $(u, w)$ plane. Now we use the following notations:

\[\Gamma^{*} := \{(u, w)|w = f^{*}(u)\}, \quad \Gamma_{*} := \{(u, w)|w = f_{*}(u)\},\]

\[B(u, w) := \{(d\alpha - b\gamma)u^{2} + (c\beta - a\delta)w^{2}\}^{\frac{1}{2}}, \quad l := \{(u, w) \in R^{2}|\gamma u + \delta w = 0\},\]

\[\Gamma^{*}(l) := \{(u, w) \in \Gamma^{*} \cap l|u > 0\}, \quad \Gamma_{*}(l) := \{(u, w) \in \Gamma_{*} \cap l|u < 0\},\]

\[r_{0}^{*} := \min\{B(u, w)|(u, w) \in \Gamma^{*}\}, \quad u^{*} := \min\{u|(u, w) \in \Gamma^{*}, B(u, w) = r_{0}^{*}\},\]

\[r_{0*} := \min\{B(u, w)|(u, w) \in \Gamma_{*}\}, \quad u_{*} := \max\{u|(u, w) \in \Gamma_{*}, B(u, w) = r_{0*}\},\]

\[r_{1}^{*} := \min\{B(u, w)|(u, w) \in \Gamma^{*}(l)\}, \quad R_{1}^{*} := \max\{B(u, w)|(u, w) \in \Gamma^{*}(l)\},\]

\[\Gamma^{*}(l):= \Gamma^{*} \cap l, \quad \Gamma_{*}(l):= \Gamma_{*} \cap l,\]

\[A^{+} := \{(u, w)|u^{*}w - f_{*}(u_{*})u < 0 \mathrm{if} u < 0, \quad u^{*}w - f^{*}(u^{*})u < 0 \mathrm{if} u > 0\},\]

\[A^{-} := \{(u, w)|u_{*}w - f_{*}(u_{*})u < 0 \mathrm{if} u < 0, \quad u_{*}w - f^{*}(u_{*})u < 0 \mathrm{if} u > 0\},\]

\[S_{0} := \{(u, w) \in S|B(u, w) \leq r_{0}\} \quad \text{with} \quad r_{0} := \min\{r_{0}^{*}, r_{0*}\}.\]

By our assumptions, we have

\[r_{0}^{*} < r_{1}^{*} \leq R_{1}^{*} \quad \text{and} \quad r_{0*} < r_{1*} \leq R_{1*}.\]

As to the relationships of $r_{0}^{*}, r_{1}^{*}, R_{1}^{*}, r_{0*}, r_{1*}$ and $R_{1*}$ there are the following 6 cases to be considered:

1. \[r_{0}^{*} \leq r_{0} < r_{1*} \leq R_{1*}\]
2. \[r_{0} < r_{1*} \leq R_{1*} \leq r_{0}^{*}\]
3. \[r_{0} < r_{1*} \leq R_{1*} \leq r_{0}^{*}\]
4. \[r_{0}^{*} \leq r_{0*} < r_{1}^{*} \leq R_{1}^{*}\]
5. \[r_{0}^{*} < r_{1}^{*} \leq r_{0*} \leq R_{1}^{*}\]
6. \[r_{0}^{*} < r_{1}^{*} \leq R_{1}^{*} \leq r_{0*}\]

In the case of (1) we define

\[S_{1} := S_{0} \cup S_{1}^{+} \cup S_{1}^{-},\] (3.1)

where

\[S_{1}^{+} := \{(u, w) \in S \cap A^{+}|r_{0*} < B(u, w) < r_{1}^{*}\},\] (3.2)

\[S_{1}^{-} := \{(u, w) \in S \cap A^{-}|r_{0*} < B(u, w) < r_{1*}\};\] (3.3)

\[S_{2} := S_{2}^{+} \cup S_{2}^{-},\] (3.4)

where

\[S_{2}^{+} := \{(u, w) \in S \cap A^{+}|r_{1}^{*} \leq B(u, w) \leq R_{1}^{*}\},\] (3.5)
$S_2^- := \{(u, w) \in S \cap A^- \mid r_{1*} \leq B(u, w) \leq R_{1*}\}$; \hspace{1cm} (3.6)

$S_3 := S_3^+ \cup S_3^-$, \hspace{1cm} (3.7)

where

$S_3^+ := \{(u, w) \in S \cap A^+ \mid r_{1}^* \leq B(u, w) \leq R_{1}^*\}$, \hspace{1cm} (3.8)

$S_3^- := \{(u, w) \in S \cap A^- \mid R_{1*} < B(u, w)\}$. \hspace{1cm} (3.9)

In the case of (2) we define

$S_1 := S_0 \cup S_1^0$, \hspace{1cm} (3.10)

where

$S_1^0 := \{(u, w) \in S \mid r_{0*} < B(u, w) < r_{1*}\}$; \hspace{1cm} (3.11)

$S_2 := S_2^+ \cup S_2^-$, \hspace{1cm} (3.12)

where

$S_2^+ := \{(u, w) \in S \cap A^+ \mid r_{1}^* \leq B(u, w) \leq R_{1}^*\}$, \hspace{1cm} (3.13)

$S_2^- := \{(u, w) \in S \cap A^+ \mid r_{1*} \leq B(u, w) \leq r_{1*}\}$ \hspace{0.5cm} \cup \hspace{0.5cm} \{(u, w) \in S \cap A^- \mid r_{1*} \leq \text{B}(u, w) \leq R_{1*}\};$ \hspace{1cm} (3.14)

$S_3 := S_3^+ \cup S_3^-$, \hspace{1cm} (3.15)

where

$S_3^+ := \{(u, w) \in S \cap A^+ \mid r_{1}^* \leq B(u, w) \leq R_{1}^*\}$, \hspace{1cm} (3.16)

$S_3^- := \{(u, w) \in S \cap A^- \mid R_{1*} < B(u, w)\}$. \hspace{1cm} (3.17)

In the case of (3) we define

$S_1 := S_0 \cup S_1^0$, \hspace{1cm} (3.18)

where

$S_1^0 := \{(u, w) \in S \mid r_{0*} < B(u, w) < r_{1*}\}$; \hspace{1cm} (3.19)

$S_2 := S_2^+ \cup S_2^-$, \hspace{1cm} (3.20)

where

$S_2^+ := \{(u, w) \in S \cap A^+ \mid r_{1}^* \leq B(u, w) \leq R_{1}^*\}$, \hspace{1cm} (3.21)
$S_2 := \{(u, w) \in S \mid r_1 \leq B(u, w) \leq R_1\}; \quad (3.22)
$ $S_3 := S_3^+ \cup S_3^-,$ \quad (3.23)

where

$S_3^+ := \{(u, w) \in S \cap A^+ \mid R_1^* < B(u, w)\}$, \quad (3.24)

$S_3^- := \{(u, w) \in S \cap A^- \mid R_1^* < B(u, w)\}$; \quad (3.25)

In the case of (4) we define

$S_1 := S_0 \cup S_1^+ \cup S_1^-$,

where

$S_1^+ := \{(u, w) \in S \cap A^+ \mid r_0^* < B(u, w) < r_1^*\}$,

$S_1^- := \{(u, w) \in S \cap A^- \mid r_0^* < B(u, w) < r_1^*\}$,

$S_2 := S_2^+ \cup S_2^-$,

where

$S_2^+ := \{(u, w) \in S \cap A^+ \mid r_1^* \leq B(u, w) \leq R_1^*\}$,

$S_2^- := \{(u, w) \in S \cap A^- \mid r_1^* \leq B(u, w) \leq R_1^*\}$,

$S_3 := S_3^+ \cup S_3^-$,

where

$S_3^+ := \{(u, w) \in S \cap A^+ \mid R_1^* < B(u, w)\}$,

$S_3^- := \{(u, w) \in S \cap A^- \mid R_1^* < B(u, w)\}$.

In the case of (5) we define

$S_1 := S_0 \cup S_1^0$,

where

$S_1^0 := \{(u, w) \in S \mid r_0^* < B(u, w) < r_1^*\}$;

$S_2 := S_2^+ \cup S_2^-$. 


where

\[
S_2^+ := \{(u, w) \in S \cap A^+ | r_1^* \leq B(u, w) \leq R_1^* \} \\
\quad \cup \{(u, w) \in S \cap A^- | r_1^* \leq B(u, w) < r_1* \},
\]

\[
S_2^- := \{(u, w) \in S \cap A^- | r_1* \leq B(u, w) \leq R_1* \};
\]

\[
S_3 := S_2^+ \cup S_2^-,
\]

where

\[
S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \leq R_1* \} \\
\quad \cup \{(u, w) \in S \cap A^- | R_1* < B(u, w) < r_1* \}.
\]

In the case of (6) we define

\[
S_1 := S_0 \cup S_1^0,
\]

where

\[
S_1^0 := \{(u, w) \in S | r_0^* < B(u, w) < r_1^* \};
\]

\[
S_2 := S_2^+ \cup S_2^-,
\]

where

\[
S_2^+ := \{(u, w) \in S | r_1^* \leq B(u, w) \leq R_1^* \},
\]

\[
S_2^- := \{(u, w) \in S \cap A^- | r_1* \leq B(u, w) \leq R_1* \};
\]

\[
S_3 := S_2^+ \cup S_2^-,
\]

where

\[
S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \leq R_1* \} \\
\quad \cup \{(u, w) \in S \cap A^- | R_1* < B(u, w) < r_1* \},
\]

\[
S_3^- := \{(u, w) \in S \cap A^- | R_1* < B(u, w) \}.
\]

In any cases of (1)-(6), when the initial data belong to any subset of \( S_1, S_2 \) and \( S_3 \), the orbits of the solutions satisfy the statements (i)-(iii) of Theorem 2.2. In the next section, we prepare some Lemmas in order to prove Theorem 2.2.
4 Local behaviour of orbits

In this section, we investigate the local behaviour of the orbit \((u(t), w(t))\), satisfying
\[
aw'(t) + bu'(t) + \partial I_{u(t)}(w(t)) \ni \alpha u(t) + \beta w(t),
\]
\[
w'(t) + du'(t) = \gamma u(t) + \delta w(t)
\]
for \(t \geq 0\). We only give proof of Lemma 4.3. Other Lemmas are shown without proofs.

**Lemma 4.1** Assume that \((u(t_1), w(t_1))\), \(t_1 \geq 0\), is in the interior of \(S\). Then:

(a) if \(B(u(t_1), w(t_1)) \leq r_0\), then \(\{u, w\}\) satisfies
\[
\begin{aligned}
u'(t) &= -\frac{c\beta - a\delta}{ad - bc}w, \\
w'(t) &= \frac{d\alpha - b\gamma}{ad - bc}u.
\end{aligned}
\] (4.1)

for all \(t \geq t_1\), and hence the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(C_1 := \{(u, w)|B(u, w) = B(u(t_1), w(t_1))\}\) and it is periodic in time on \([t_1, +\infty)\).

(b) if \(B(u(t_1), w(t_1)) > r_0\), then \(\{u, w\}\) satisfies system (4.1) on a compact interval \([t_1, t_2]\) with \(t_2 > t_1\), where \(t_2\) is the earliest time of all \(t > t_1\) at which \((u(t), w(t)) \in \Gamma_* \cup \Gamma^\ast\). Hence the orbit \((u(t), w(t))\) draws an anticlockwise arc on the ellipse \(C_1\) for \(t_1 \leq t \leq t_2\).

We note that the stationary problem of (1.1)-(1.2) is of the form
\[\partial I_w(u) \ni \alpha u + \beta w, \gamma u + \delta w = 0.\]

**Lemma 4.2** (a) Let \((\tilde{u}, \tilde{w})\) be an interior point of \(S\). Then \(\{\tilde{u}, \tilde{w}\}\) is a stationary solution of (1.1)-(1.2) if and only if \(\tilde{u} = 0\) and \(\tilde{w} = 0\).

(b) Let \((\tilde{u}, \tilde{w})\) be a boundary point of \(S\). Then \(\{\tilde{u}, \tilde{w}\}\) is a stationary solution of (1.1)-(1.2) if and only if \((\tilde{u}, \tilde{w}) \in \Gamma_\ast(l) \cup \Gamma^\ast(l)\).

**Lemma 4.3** Assume that \((u(t_1), w(t_1))\), \(t_1 \geq 0\), is on \(\Gamma_\ast\) and \(w(t_1) < 0\). Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) > 0\) and if there exists \(\bar{u} > u(t_1)\) such that
\[\gamma v + \delta f_\ast(v) > 0 \text{ for } u(t_1) \leq v \leq \bar{u},\]

and moreover if
\[
\frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f_\ast(u)} \leq f_\ast'(u) \text{ for } u(t_1) \leq v \leq \bar{u},
\] (4.2)
then \{u, w\} satisfies
\[
    u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{cf'_*(u(t)) + d}, \quad w'(t) = f'_*(u(t))u'(t)
\]  
(4.3)
on a compact interval \([t_1, t_2]\), where \(t_2\) is the earliest time at which \(u(t_2) = \bar{u}\), and the orbit \((u(t), w(t))\) moves along \(\Gamma_*\) from \((u(t_1), w(t_1))\) to \((\bar{u}, f_*(\bar{u}))\) for \(t_1 \leq t \leq t_2\). Moreover
\[
    \frac{d}{dt}B(u(t), w(t)) \leq 0 \text{ on } [t_1, t_2].
\]  
(4.4)
(b) if \(\gamma u(t_1) + \delta w(t_1) > 0\) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma_*(l)\) with \(\bar{u} > u(t_1)\) such that
\[
    \gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v < \bar{u},
\]then \{u, w\} satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves upward along the curve \(\Gamma_*\) and converges to \((\bar{u}, \bar{w})\) as \(t \to +\infty\);
(c) if \(\gamma u(t_1) + \delta w(t_1) < 0\) and if there exists a stationary point \((\underline{u}, \underline{w}) \in \Gamma_*(l)\) with \(\underline{u} < u(t_1)\) such that
\[
    \gamma v + \delta f_*(v) < 0 \text{ for } \underline{u} < v \leq u(t_1),
\]then \{u, w\} satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \(\Gamma_*\) and converges to \((\underline{u}, \underline{w})\) as \(t \to +\infty\).
(d) if \(\gamma v + \delta f_*(v) < 0\) holds for all \(v \leq u(t_1)\), then \{u, w\} satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \(t \to +\infty\).

\textbf{Proof. } We prove (a). We put \((u_1, w_1) = (u(t_1), w(t_1))\); note that \(w_1 = f_*(u_1)\), since \((u_1, w_1) \in \Gamma_*\). We can find a positive constant \(M\) such that
\[
    \gamma v + \delta f_*(v) \geq M \text{ for } u_1 \leq v \leq \bar{u}.
\]  
(4.5)
Now, consider the Cauchy problem
\[
    \hat{u}'(t) = \frac{\gamma \hat{u}(t) + \delta f_*(\hat{u}(t))}{cf'_*(\hat{u}(t)) + d}, \quad t_1 \leq t < t_1^*,
\]  
(4.6)
\[
    \hat{u}(t_1) = u_1
\]  
(4.7)
where \(t_1^*\) is the supremum of positive number \(t_1^* (> t_1)\) such that problem (4.6)-(4.7) has a solution on \([t_1, t_1^*]\). In fact, since the function \(v \mapsto \frac{\gamma v + \delta f_*(v)}{cf'_*(v) + d}\) is Lipschitz continuous in a neighborhood of \(v = u_1\), by the general theory of ODEs the problem (4.6)-(4.7) has a (unique) local (in time) solution \(\hat{u}(t)\). It is easy to see from (4.5) that \(\hat{u}(\cdot)\) is monotonically increasing and reaches the value \(\bar{u}\) in a finite time \(t_2 \in (t_1, t_1^*)\). Now,
putting $\hat{w}(t) = f_{*}(\hat{u}(t))$ on $[t_1, t_2]$, we have that $\{\hat{u}, \hat{w}\}$ satisfies our system (1.1) and (1.2) on $[t_1, t_2]$. In fact, it follows from (4.6) that
\[
cf_{*}'(\hat{u}(t))\hat{u}'(t) + d\hat{u}'(t) = \gamma \hat{u}(t) + \delta f_{*}(\hat{u}(t)),
\]
which implies $c\hat{w}'(t) + d\hat{u}'(t) = \gamma \hat{u}(t) + \delta \hat{w}(t)$ on $[t_1, t_2]$. Thus (1.2) is satisfied. Equation (1.1) is checked as follows. By assumption (A1) and (4.2), calculating $\alpha \hat{u} + \beta \hat{w} - a \hat{w}' - b \hat{u}'$, we obtain
\[
\begin{align*}
\alpha \hat{u} + \beta \hat{w} - a \hat{w}' - b \hat{u}' &= \alpha \hat{u} + \beta f_{*}(\hat{u}) - \frac{\gamma \hat{u} + \delta f_{*}(\hat{u})}{cf_{*}'(\hat{u}) + d} (af_{*}'(\hat{u}) + b) \\
&= \frac{(\alpha \hat{u} + \beta f_{*}(\hat{u}))(cf_{*}'(\hat{u}) + d) - (\gamma \hat{u} + \delta f_{*}(\hat{u}))(af_{*}'(\hat{u}) + b)}{cf_{*}'(\hat{u}) + d} \\
&= \frac{(c\beta - a\delta)f_{*}(u)f_{*}'(u) - (b\gamma - d\alpha)u}{cf_{*}'(u) + d}
\end{align*}
\]
\[
\leq 0
\]
on $[t_1, t_2]$. By the definition of subdifferentials (see (1.4)) we have $\partial I_{\hat{u}}(\hat{w}) = (-\infty, 0]$ for $\hat{w} = f_{*}(\hat{u})$. Therefore
\[
\alpha \hat{u} + \beta \hat{w} - a \hat{w}' - b \hat{u}' \in \partial I_{\hat{u}}(\hat{w}) \text{ on } [t_1, t_2].
\]
Thus, by the uniqueness, $\{\hat{u}, \hat{w}\}$ must be the solution $\{u, w\}$ of (1.1)-(1.2) on $[t_1, t_2]$. Next we show (4.4). Since (4.2) and (4.3) hold on $[t_1, t_2]$, we obtain
\[
\frac{d}{dt}B(u, w) = \frac{u'}{B(u, w)} \{ (c\beta - a\delta)f_{*}'(u)f_{*}(u) - (b\gamma - d\alpha)u \} \\
\leq 0 \text{ on } [t_1, t_2].
\]
Next we prove (b). Let us recall that $\bar{u} < 0$, $f_{*}(\bar{u}) < 0$ by Lemma 4.2 (b). We obtain automatically
\[
\frac{(d\alpha - b\gamma)\nu}{(a\delta - c\beta)f_{*}'(\nu)} \leq f_{*}'(\nu) \text{ for } u_1 \leq v \leq \bar{u}.
\]
(4.8)
Therefore, in the same way as in (a), $\{u, w\}$ satisfies (4.3) for a moment after the time $t_1$ and the orbit $(u(t), w(t))$ moves along the curve $\Gamma_{*}$ starting from $(u(t_1), w(t_1))$. We now
show that \((u(t), w(t))\) converges to \((\bar{u}, \bar{w})\) as \(t \to +\infty\). Let \(T\) be the supremum of all \(s(\geq t_1)\) such that

\[
    u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{cf_*(u(t)) + d}, \quad w(t) = f_*(u(t)) \quad \text{for } \forall t \in [t_1, s].
\]

Then, just as in the case of (a), we see that \(T > t_1\). Since \(u\) is non-decreasing on \([t_1, T)\), \(\lim_{t \uparrow T} u(t)\) exists. We want to see that \(\lim_{t \uparrow T} u(t) = \bar{u}\). We show it by contradiction. Now, assume that \(\lim_{t \uparrow T} u(t) < \bar{u}\). Then we consider the following statements:

(i) \(T = +\infty, \ u_\infty := \lim_{t \to +\infty} u(t)\) and \(w_\infty := \lim_{t \to +\infty} w(t)\) give a pair of stationary solutions

or

(ii) \(T < +\infty\) and \(\frac{(a\alpha - b\gamma)u(t)}{(a\delta - c\beta)f_*(u(t))} > f'_*(u(t))\) for some \(t > T\).

But these cases do not occur in our situations considered now. In fact, the case (i) yields that \(u(t_1) \leq u_\infty < \bar{u}\) and \(\gamma u_\infty + \delta f_*(u_\infty) = 0\), which contradicts our assumption. Also, the case (ii) yields a contradiction to (4.8).

Assertion (c) is similarly proved to (b).

Finally, we prove (d). By the same argument as above, we have

\[
    \frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f'_*(v) \quad \text{for } \forall v \leq u_1,
\]

and find a negative constant \(\tilde{M}\) such that

\[
    \gamma v + \delta f_*(v) \leq \tilde{M} \quad \text{for } \forall v \leq u_1.
\]

Hence \(\{u, w\}\) satisfies (4.3) for all \(t \geq t_1\) and \(u(\cdot)\) is monotonically decreasing on \([t_1, \infty)\). By assumption (A2), \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \(t \to +\infty\). \(\blacksquare\)

**Lemma 4.4** Assume that \((u(t_1), w(t_1)), t_1 \geq 0,\) is on \(\Gamma^*\) and \(w(t_1) > 0\). Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) < 0\) and if there exists \(\bar{u} < u(t_1)\) such that

\[
    \gamma v + \delta f^*_*(v) < 0 \quad \text{for } \bar{u} \leq u(t_1) \leq v,
\]

and moreover if the following condition hold that

\[
    \frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f^*_*(v)} \leq f''_*(v) \quad \text{for } \bar{u} \leq v \leq u(t_1),
\]
then \( \{u, w\} \) satisfies

\[
    u'(t) = \frac{\gamma u + \delta f^*(u)}{cf^*(u) + d}, \\
    w'(t) = f^*(u)u'(t)
\]

on a compact interval \([t_1, t_2]\), where \( t_2 \) is the earliest time at which \( u(t_2) = \bar{u} \), and the orbit \((u(t), w(t))\) moves along \( \Gamma_* \) from \((u(t_1), w(t_1))\) to \((\bar{u}, f^*(\bar{u}))\) for \( t_1 \leq t \leq t_2 \). Moreover

\[
    \frac{d}{dt}B(u, w) \leq 0 \text{ on } [t_1, t_2].
\]

(b) if \( \gamma u(t_1) + \delta w(t_1) < 0 \) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma^*(l)\) with \( \bar{u} < u(t_1) \) such that

\[
    \gamma v + \delta f_*(v) < 0 \text{ for } \bar{u} < v \leq u(t_1),
\]

then \( \{u, w\} \) satisfies \((4.4)\) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \( \Gamma^* \) and converges to \((\bar{u}, \bar{w})\) as \( t \to +\infty \).

(c) if \( \gamma u(t_1) + \delta w(t_1) > 0 \) and if there exists a stationary point \((\underline{u}, \underline{w}) \in \Gamma^*(l)\) with \( \underline{u} > u(t_1) \) such that

\[
    \gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v < \underline{u},
\]

then \( \{u, w\} \) satisfies \((4.4)\) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) moves upward along the curve \( \Gamma^* \) and converges to \((\underline{u}, \underline{w})\) as \( t \to +\infty \).

(d) if \( \gamma v + \delta f_*(v) > 0 \) holds for all \( v \geq u(t_1) \), then \( \{u, w\} \) satisfies \((4.4)\) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) diverges to \((\infty, f^\infty)\) as \( t \to +\infty \).

5 Large time behaviour of orbits

In this section, we prove Theorem 2.2 in the case (1) in section 3. Any other cases can be treated by a simple modification of them. We investigate the behaviour of the solution \( \{u, w\} \) when the initial data \((u_0, w_0)\) belong to each of \( S_0, S_1, S_2 \) and \( S_3 \).

In the case of \((u_0, w_0) \in S_0\)

When \((u_0, w_0) \in S_0\), we obtain \( B(u_0, w_0) \leq r_0 \). Therefore, by Lemma 4.1(a), we see that the orbit \((u(t), w(t))\) draws anticlockwise ellipse \( B(u, w) = B(u_0, w_0) \) for all \( t \geq 0 \), and is periodic in time.

In the case of \((u_0, w_0) \in S_1\)

First, we consider the case of \((u_0, w_0) \in S_1^- \) and \( w_0 \leq 0 \). Clearly \( B(u_0, w_0) > r_0 \).
By Lemma 4.1 (b), the orbit \((u(t), w(t))\) draws an anticlockwise ellipse on \(B(u, w) = B(u_0, w_0)\), until it reaches \(\Gamma_*\), satisfying
\[
\begin{align*}
u'(t) &= -\frac{c\beta - a\delta}{ad - bc} w(t), \quad 0 \leq t \leq t_1, \\
w'(t) &= \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad 0 \leq t \leq t_1, \\
u(0) &= u_0, \quad w(0) = w_0,
\end{align*}
\]
where \(t_1\) is the earliest time such that \((u(t_1), w(t_1)) \in \Gamma_*\). We have \(w(t_1) = f_*(u(t_1))\), \(B(u(t_1), w(t_1)) < r_{1*}\) and
\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v \leq u_*.
\]

Next, take the number \(u_2\) so that
\[
u_2 = \sup \left\{ \tilde{u} \mid u(t_1) \leq \tilde{u} \leq u_*, \quad \frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f(u)} \leq f'_*(u) \text{ for } \forall u \in [u(t_1), \tilde{u}] \right\}.
\]

Then we have the following three possibilities: (i) \(u(t_1) < u_2 < u_*\), (ii) \(u_2 = u(t_1)\), (iii) \(u_2 = u_*\).

In the case of (i), by Lemma 4.3 (a)
\[
u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{cf'_*(u(t)) + d}, \quad w(t) = f_*(u(t)), \quad t \in [t_1, t_2]
\]
where \(t_2\) is the earliest time such that \(u(t_2) = u_2\). We denote by \(C_2\) the ellipse \(B(u, w) = B(u_2, f_*(u_2)) =: r_2\). By assumption (A3) and the definition of \(u_2\), we see that an arc \(\{(u, w)|u_2 \leq u \leq \tilde{u}_3, B(u, w) = r_2\}\) on \(C_2\) is contained in \(S\). Now, denote by \(u_3\) the largest one of such numbers \(\tilde{u}_3\), we have \(u_3 > u_2\). Moreover, by Lemma 4.1 (b), \(\{u, w\}\) is given by
\[
u'(t) = -\frac{c\beta - a\delta}{ad - bc} w(t), \quad w'(t) = \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad t \in [t_2, t_3],
\]
where \(t_3\) is the earliest time such that \(u(t_3) = u_3\). Our assumption (A3) guarantees that the orbit \((u(t), w(t))\) reaches \((u_*, f_*(u_*))\) at \(t = t_*(< \infty)\) by repeating finitely many times such behaviours as above. Here, after the time \(t_*\), the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(B(u, w) = r_0\) periodically in time (see Lemma 4.1 (a)).

In the case of (ii), it is the case that \(t_1 = t_2\) with the same notation as above, and the behaviour of \((u(t), w(t))\) is similar to the case of (i) after the time \(t_2\).

In the case of (iii), it is the case that \(t_2 = t_*\), and the behaviour of \((u(t), w(t))\) is the
anticlockwise ellipse $B(u, w) = r_0$ after the time $t_*$.  

Next, consider the case of $(u_0, w_0) \in S_1^+$ with $w_0 > 0$. In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_1$. If $(u(s_1), w(s_1)) \in \Gamma_*$, then the behaviour of $(u(t), w(t))$ is exactly the same as in the previous case after time $s_1$. On the other hand, if $(u(s_1), w(s_1)) \in \Gamma^*$, then the orbit $(u(t), w(t))$ moves downward for a time interval $[s_1, s_2]$ with $s_1 \leq s_2$ along the curve $\Gamma^*$ by Lemma 4.4 (a) (in this step assumption (A3) regarding the function $f^*(\cdot)$ is used), where $s_2$ is the largest time of $s_2$ such that $(u(t), w(t)) \in \Gamma^*$ for all $t \in [s_1, s_2]$. It is easy to see that $w(s_2) > 0$ and $s_2 < +\infty$. After time $s_2$, the orbit $(u(t), w(t))$ draws an anticlockwise arc on $B(u, w) = B(u(s_2), w(s_2))$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_3$. Repeating such procedures finitely many times, the orbit $(u(t), w(t))$ arrives at $\Gamma_*$ at time $t = t_1$ in the last step. After time $t_1$, the behaviour of $(u(t), w(t))$ was already seen in the case of $(u_0, w_0) \in S_1^-$ with $w_0 \leq 0$.

Finally, we consider the case of $(u_0, w_0) \in S_1^+$. We have the following three cases:

(i) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) \geq r^*_0$,

(ii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r^*_0$ and $w_0 \geq 0$,

(iii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r^*_0$ and $w_0 < 0$.

First, we consider the case (i). In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = r \in [r^*_0, r^*_1]$ and a part of $\Gamma^*$ alternately and reaches the point $(u^*, f^*(u^*))$ at a finite time $t = t^*$. Since $(u^*, f^*(u^*)) \in S_1^-$, the behaviour of $(u(t), w(t))$ after the time $t^*$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 > 0$.

In the second case (ii), the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ and reaches a point $(u_1, w_1) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the third case (iii), the orbit $(u(t), w(t))$ possibly draws an anticlockwise arc on the ellipse $B(u, w) = r \in (r^*_0, r^*_1)$ and a part of $\Gamma^*$ alternately and reaches a point $(u_1, w_1) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a finite time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the case of $(u_0, w_0) \in S_2$

We give a proof only in the case of $(u_0, w_0) \in S_2^-$, since the proof of the case of $(u_0, w_0) \in S_2^+$ is quite similar. In a way similar to that in the case of $(u_0, w_0) \in S_1$, we see that the orbit $(u(t), w(t))$, drawing an anticlockwise arc on the ellipse $B(u, w) = r \in [r_{1*}, R_{1*}]$, arrives at a point $(u_1, w_1) \in \Gamma_*$ at a certain finite time $t = t_1$. If $(u(t_1), w(t_1)) = (u_1, w_1) \in \Gamma_*(l)$, then $(u(t_1), w(t_1))$ is a stationary solution of (1.1)-(1.3) by Lemma 4.2 (b). If $(u(t_1), w(t_1)) \notin \Gamma_*(l)$, then we have the following two cases:
(i) $\gamma u(t_1) + \delta w(t_1) > 0$,

(ii) $\gamma u(t_1) + \delta w(t_1) < 0$.

Suppose now that (i) holds. Then there is a closed interval $[u, \bar{u}] \subset (-\infty, 0)$ on the $u$-axis such that $u < u(t_1) < \bar{u}$, $\gamma v + \delta f_*(v) > 0$ for all $v \in (u, \bar{u})$ and $\gamma \bar{u} + \delta f_*(\bar{u}) = \gamma \bar{u} + \delta f_*(\bar{u}) = 0$. Therefore, the orbit $(u(t), w(t))$ converges to $(\bar{u}, f_*(\bar{u})) \in \Gamma_*(l)$ as $t \to +\infty$ by Lemma 4.3 (b). On the other hand, when (ii) holds, the orbit $(u(t), w(t))$ converges to a stationary point as $t \to +\infty$, too.

In the case of $(u_0, w_0) \in S_3$

It is enough to consider only the case $(u_0, w_0) \in S_3^-$. In the same way as in the case of $(u_0, w_0) \in S_2$, the orbit $(u(t), w(t))$ reaches $\Gamma_*$ in a finite time $t_1$. Also, we obtain $B(u(t_1), w(t_1)) < R_*$ and $\gamma v + \delta f_*(v) < 0$ for $v < u_1$. Therefore, by Lemma 4.3 (d), we see that $(u(t), w(t))$ diverges to $(-\infty, f_\infty)$ as $t \to +\infty$. Similarly, in the case $(u_0, w_0) \in S_3^+$, we see that $(u(t), w(t))$ diverges to $(\infty, f_\infty)$ as $t \to +\infty$.

**Remark 5.1** We have many cases about the stability around stationary points in $S_2$. If, for instance, we restrict our geometry of the curves $\Gamma_*, \Gamma^*$ and $l$ to the one as illustrated by the picture (Fig. 1), then stationary points are classified into the following three categories: Let $(u_\infty, w_\infty)$ be any stationary point in $S_2$. Then one of the following cases happens.

(1) $(u_\infty, w_\infty)$ is stable. Namely, there is a neighborhood $U_1$ of $(u_\infty, w_\infty)$ in $R^2$ such that the orbit $(\hat{u}(t), \hat{w}(t))$ stays in $U_1 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ as $t \to +\infty$, whenever $(\hat{u}_0, \hat{w}_0) = (\hat{u}(0), \hat{w}(0)) \in U_1 \cap S$.

(2) $(u_\infty, w_\infty)$ is semistable. Namely, there is a neighborhood $U_2$ of $(u_\infty, w_\infty)$ in $R^2$ such that the following properties (i) and (ii) are satisfied:

(i) For any initial point $(\hat{u}_0, \hat{w}_0) \in U_2 \cap S \cap K_\infty$, the orbit $(\hat{u}(t), \hat{w}(t))$ stays in $U_2 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ as $t \to +\infty$, whenever $(\hat{u}_0, \hat{w}_0) = (\hat{u}(0), \hat{w}(0)) \in U_2 \cap S$.

(ii) For any initial point $(\hat{u}_0, \hat{w}_0) \in U_2 \cap S \cap K_\infty$, the orbit $(\hat{u}(t), \hat{w}(t))$ gets out of $U_2$ after a certain time $t_1$. 

![Fig. 1](image-url)
where $\mathcal{K}_{\infty} := \{(u, w) | B(u, w) \geq (u_\infty, w_\infty)\}$.

(3) $(u_\infty, w_\infty)$ is unstable. Namely, there is a neighborhood $U_3$ of $(u_\infty, w_\infty)$ in $\mathbb{R}^2$ such that the following properties (iii) and (iv) are satisfied:

(iii) For any initial point $(\tilde{u}_0, \tilde{w}_0) \in U_3 \cap S \cap C_\infty$, the orbit $(\tilde{u}, \tilde{w})$ stays in $U_3 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ in a finite time $t_1$.

(iv) For any initial point $(\check{u}_0, \check{w}_0) \in U_3 \cap S \cap C_\infty^c$, the orbit $(\check{u}(t), \check{w}(t))$ gets out of $U_3$ after a certain time $t_1$.

where $C_\infty := \{(u, w) | B(u, w) = B(u_\infty, w_\infty)\}$.

6 Some numerical simulations

In this section, we give some numerical experiments to verify Theorem 2.2. In order to catch the behaviour of solutions, we simply take the coefficients $a, b, c, d$ and functions $F, h$ satisfying (A1) with $d\alpha - b\gamma = c\beta - a\delta > 0$ such that the orbits of solutions are anticlockwise circles without subdifferential term $\partial I_u(w)$. Now we fix the coefficients $a, b, c, d$ and functions $F, h$ as follows:

\[ a = 1, \ b = -1, \ c = 1, \ d = 1, \ F(u, w) = u + w, \ h(u, w) = u - w. \]

In this case, our system is of the following form:

\[ w' - u' + \partial I_u(w) \ni u + w, \ 0 < t < T, \]

\[ w' + u' = u - w, \ 0 < t < T, \]

\[ u(0) = u_0, \ w(0) = w_0. \]

Now let $\lambda$ and $\Delta t$ be small positive numbers, and $n$ be a large natural number. Then the difference scheme for our numerical simulation is of the form

\[ \frac{w^{k+1} - w^k}{\Delta t} - \frac{u^{k+1} - u^k}{\Delta t} + \partial I_u^\lambda(w^{k+1}) = u^k + w^k, \]

\[ \frac{w^{k+1} - w^k}{\Delta t} + \frac{u^{k+1} - u^k}{\Delta t} = u^k - w^k, \ k = 0, 1, 2, \ldots, \]

\[ u^0 = u_0, \ w^0 = w_0, \]

where

\[ \partial I_u^\lambda(w^{k+1}) = \frac{[w^{k+1} - f^*(u^k)]^+}{\lambda} - \frac{[f_*(u^k) - w^{k+1}]^+}{\lambda}. \]

The graphs of $I_u^\lambda$ and $\partial I_u^\lambda$ are illustrated in Figures 2 and 3, respectively.
In our actual computation

$$\Delta t = \frac{1}{1000}, \quad \lambda = \frac{1}{1000},$$

and we examine the following items:

- We define the subset $S_i (i = 0, 1, 2, 3)$ by the geometries of the given functions $f_*(u)$ and $f^*(u)$ and the line $\gamma u + \delta w = 0$.

- By numerical simulations, we verify that the behaviour of solutions satisfies the statements of Theorem 2.2 when the initial data belong to each subset $S_i (i = 0, 1, 2, 3)$.

Experiment 1:
We take the functions $f_*(u), f^*(u)$ as follows:

$$f_*(u) = \begin{cases} 
-1 & \text{if } u \leq 0.4, \\
5u^2 - 4u - 0.2 & \text{if } 0.4 < u \leq 0.6, \\
2u - 2 & \text{if } 0.6 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.8, \\
1 & \text{if } u > 1.8.
\end{cases}$$

and

$$f^*(u) = \begin{cases} 
-1 & \text{if } u \leq -1.6, \\
5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
-5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq 0.4, \\
1 & \text{if } u > 0.4.
\end{cases}$$

$f_*(u)$ and $f^*(u)$ are symmetric with respect to origin. In this case, by our choice of $f_*(u), f^*(u)$ and the line $u + w = 0$, we obtain that

$$r_0 := r_{0*} = r^*_0 = \frac{2\sqrt{5}}{5}, \quad r_{1*} = r^*_1 = R_{1*} = R^*_1 = \sqrt{2}$$

and

stationary points are $(1, 1), (0, 0)$ and $(-1, -1)$.

Therefore, subsets $S_i (i = 0, 1, 2, 3)$ are defined by (3.1)-(3.9) and are illustrated by Figure 4. Now we take the initial data which belong to each subset $S_i$ and numerical experiments are shown as follows:

<table>
<thead>
<tr>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 5</td>
<td>-0.2</td>
<td>-0.6</td>
<td>$(u_0, w_0) \in S_0$</td>
<td>Fig. 7</td>
<td>-1.3</td>
<td>-0.8</td>
<td>$(u_0, w_0) \in S_3^-$</td>
</tr>
<tr>
<td>Fig. 6</td>
<td>-0.7</td>
<td>-0.8</td>
<td>$(u_0, w_0) \in S_1$</td>
<td>Fig. 7</td>
<td>1.3</td>
<td>0.8</td>
<td>$(u_0, w_0) \in S_3^+$</td>
</tr>
</tbody>
</table>
When the initial data belong to \( S_0 \), the orbit draws anticlockwise circle from the initial point \((u_0, w_0)\) (Fig. 5). In the case when \((u_0, w_0)\) \( \in S_1 \), the orbit draws an anticlockwise arc and a part of \( \Gamma_\ast \) alternately and reaches a periodic circle \( B(u, w) = r_0 \) in a finite time (Fig. 6). On the other hand, in the case when \((u_0, w_0)\) \( \in S_3^- \) or \( S_3^+ \), the orbit diverges to \((-\infty, -1)\) or \((+\infty, 1)\) as \( t \rightarrow +\infty \) (Fig. 7).

**Experiment 2:**
We take the functions \( f_\ast(u), f^*(u) \) as follows

\[
\begin{align*}
    f_\ast(u) &= \begin{cases} 
        -1, & \text{if } u \leq -1, \\
        3u^2 + 6u + 2, & \text{if } -1 < u \leq -0.75, \\
        -u^2 - 0.25, & \text{if } -0.75 < u \leq 0, \\
        -0.25, & \text{if } 0 < u \leq 0.75, \\
        4u^2 - 6u + 2, & \text{if } 0.75 < u \leq 1, \\
        2u - 2, & \text{if } 1 < u \leq 1.4, \\
        -5u^2 + 16u - 11.8, & \text{if } 1.4 < u \leq 1.6, \\
        1 & \text{if } 1.6 < u.
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    f^*(u) &= \begin{cases} 
        -1, & \text{if } u \leq -2, \\
        u^2 + 4u + 3, & \text{if } -2 < u \leq -1.5, \\
        u + 0.75, & \text{if } -1.5 < u \leq 0, \\
        -u^2 + u + 0.75, & \text{if } 0 < u \leq 0.5, \\
        1 & \text{if } 0.5 < u.
    \end{cases}
\end{align*}
\]
In this case, we obtain that
\[ r_0 := r_0^* = \frac{1}{4}, \quad r_0^* = \frac{3\sqrt{2}}{8}, \quad r_1^* = \frac{\sqrt{2}}{2}, \quad R_1^* = \sqrt{2}, \quad R_1^* = \sqrt{2}. \]
and stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

Since \(r_0^* < r_1^* < R_1^*\), \(S_i(i = 0, 1, 2, 3)\) are defined by (3.1)-(3.9) and are illustrated by Figure 8. The initial data and the subsets \(S_i\) in which the initial data are given in this experiments are as follows

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 9</td>
<td>-1.1</td>
<td>0.4</td>
<td>((u_0, w_0) \in S_i)</td>
</tr>
<tr>
<td>Fig. 10</td>
<td>-1.4</td>
<td>-0.4</td>
<td>((u_0, w_0) \in S_i)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 11</td>
<td>-1.4</td>
<td>-0.8</td>
<td>((u_0, w_0) \in S_i)</td>
</tr>
<tr>
<td>Fig. 11</td>
<td>1.4</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_i)</td>
</tr>
</tbody>
</table>

We also see that the behaviour of orbits of solutions for each initial data \((u_0, w_0) \in S_i(i = 0, 1, 2, 3)\) guarantee Theorem 2.2 (Fig. 9-11). Especially by Fig. 10 and 11, we
can recognize that the point \((-0.5, -0.5)\) is a semi stable stationary solution.

**Experiment 3:**
We take the functions \(f_*(u), f^*(u)\) as follows
\[
    f_*(u) = \begin{cases} 
        -1 & \text{if } u \leq -1, \\
        3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\
        -u^2 - 0.25 & \text{if } -0.75 < u \leq 0, \\
        -0.25 & \text{if } 0 < u \leq 0.75, \\
        4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
        2u - 2 & \text{if } 1 < u \leq 1.4, \\
        -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
        1 & \text{if } 1.6 < u. 
    \end{cases}
\]

\[
    f^*(u) = \begin{cases} 
        -1 & \text{if } u \leq -1.6, \\
        5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
        2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
        -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
        1 & \text{if } -0.4 < u. 
    \end{cases}
\]

In this case, we obtain that
\[
    r_0 := r_{0*} = \frac{1}{4}, \quad r_0^* = \frac{2\sqrt{5}}{5}, \quad r_{1*} = \frac{\sqrt{2}}{2}, \quad R_{1*} = \sqrt{2},
\]
and the stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

This implies that \(r_{0s} < r_{1s} < r_0^* < R_{1s}\). Therefore, \(S_i(i = 0, 1, 2, 3)\) are defined by (3.10)-(3.17) (Fig. 12). Given initial data, our experiments are the following (Fig. 13-15):

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 13</td>
<td>0.25</td>
<td>0.25</td>
<td>((u_0, w_0) \in S_1)</td>
</tr>
<tr>
<td>Fig. 14</td>
<td>0.8</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_2^{-})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 15</td>
<td>-1.3</td>
<td>-0.8</td>
<td>((u_0, w_0) \in S_4^{-})</td>
</tr>
<tr>
<td>Fig. 15</td>
<td>1.3</td>
<td>0.8</td>
<td>((u_0, w_0) \in S_4^{+})</td>
</tr>
</tbody>
</table>
Note that the function $f_*(u)$ is the same as in experiment 2 but $f^*(u)$ is not. We see that the orbit starting from $(0.8, 0.8)$ draws an anticlockwise arc and a part of $\Gamma^*$ alternately and reaches $\Gamma_*$ in a finite time, and then it goes to the semi stable stationary point $(-0.5, -0.5)$ as $t \to +\infty$.

**Experiment 4:**
We take the functions $f_*(u), f^*(u)$ as follows

\[
\begin{align*}
    f_*(u) &= \begin{cases} 
        -1 & \text{if } u \leq -1.25, \\
        3u^2 + 7u + 3.6875 & \text{if } -1.25 < u \leq -1, \\
        -u^2 - u - 0.3125 & \text{if } -1 < u \leq -0.25, \\
        -0.25 & \text{if } -0.25 < u \leq 0.75, \\
        4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
        2u - 2 & \text{if } 1 < u \leq 1.4, \\
        -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
        1 & \text{if } 1.6 < u.
    \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
    f^*(u) &= \begin{cases} 
        -1 & \text{if } u \leq -1.6, \\
        5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
        2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
        -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
        0 & \text{if } -0.4 < u.
    \end{cases} \\
\end{align*}
\]

$f^*(u)$ is the same function as in experiment 3 and $f_*(u)$ slightly changes from the one in experiment 3. In this case, we obtain that

\[
r_0 := r_{0*} = \frac{1}{16}, \quad r_0^* = \frac{4}{5}, \quad r_{1*} = R_{1*} = \frac{1}{8},
\]

and stationary solutions are $(1, 1), \ (0, 0)$ and $(-0.5, -0.5)$.

Since $r_{0*} < r_{1*} = R_{1*} < r_0^*$, subset $S_i (i = 0, 1, 2, 3)$ are defined by (3.18)-(3.24) (see Fig. 16). We take the initial data as follows:

<table>
<thead>
<tr>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
<th>data</th>
<th>$u_0$</th>
<th>$w_0$</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 17</td>
<td>0.22</td>
<td>0.22</td>
<td>$S_1$</td>
<td>Fig. 19</td>
<td>1.3</td>
<td>0.8</td>
<td>$S_3^-$</td>
</tr>
<tr>
<td>Fig. 18</td>
<td>0.8</td>
<td>0.8</td>
<td>$S_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then our experiments results are shown by Fig. 17-19.
These numerical experiments show that the subsets $S_i (i=0,1,2,3)$ are completely different from those in experiment 3. When the initial datum is $(0.8,0.8)$, the orbit draws an anticlockwise arc and a part of $\Gamma^*$ alternately and reaches $\Gamma_*$ in a finite time, and moving along the curve $w = f_*(u)$ downward and diverges to $(-\infty, -1)$ as $t \to +\infty$.

References


