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Ordinary differential systems describing hysteresis phenomena and numerical simulation

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1 Introduction

In this paper we deal with a nonlinear ordinary differential system which describes hysteresis input-output relations. Let us consider a system of the following form:

\begin{align}
aw' + bu' + \partial I_u(w) & \ni F(u, w) \quad \text{in} \quad (0, \infty), \\
\partial I_u(w) & \ni F(u, w)
\end{align}

subject to the initial conditions:

\begin{align}
u(0) = u_0, \quad w(0) = w_0,
\end{align}

where \(a > 0, b < 0, c > 0, d > 0\) are given constants, \(F, h : R \times R \rightarrow R\) are Lipschitz continuous functions, \(f_*, f^* : R \rightarrow R\) are non-decreasing Lipschitz continuous functions with \(f_* \leq f^*\), \(I_u(\cdot)\) is the indicator function of the closed interval \([f_*(u), f^*(u)]\), and \(\partial I_u(\cdot)\) is its subdifferential defined by

\begin{align}
\partial I_u(w) = \begin{cases}
\emptyset & \text{for } w > f^*(u) \text{ or } w < f_*(u), \\
[0, +\infty) & \text{for } w = f^*(u) > f_*(u), \\
\{0\} & \text{for } f_*(u) < w < f^*(u), \\
(-\infty, 0] & \text{for } w = f_*(u) < f^*(u), \\
(-\infty, +\infty) & \text{for } w = f_*(u) = f^*(u).
\end{cases}
\end{align}

Equation (1.1) describes a lot of input-output relations \(u \rightarrow w\) which are physically relevant. For example, when \(b = 0 \) (resp. \(-1\)), \(a = 1\) and \(F \equiv 0\), the relation between \(w(t)\) and \(u(t)\) is called a play (resp. stop) operator. These operators are typical examples of hysteresis input-output relations, and are used to present various phase transition effects. Moreover, in the case when \(a = 1, b = 0, c = 1, d = 1, F \equiv 0, h \equiv 0\), the system was studied by Visintin [5]. In the general case when \(a = a(u, w), b = b(u, w), c = c(u, w), d = d(u, w)\) are functions of \(u, w\) with \(a(u, w) > 0, c(u, w) > 0, d(u, w) > 0\) and \(a(u, w)d(u, w) - b(u, w)c(u, w) > 0\), the existence and uniqueness results of the system were obtained in [2].
Our main objective of this paper is to study the large time behaviour of solutions of our system. The behaviour of solutions of (1.1), (1.2) depends on the coefficients $a, b, c, d$ and the functions $F, h$. Under some conditions on $a, b, c, d, F, h$ and $f_*, f^*$, we investigate the precise behaviour of orbits of solutions of our system. At the same time, we give some numerical experiments for the connection with the behaviour of the orbits.

2 Preliminaries and main results

In this section, we mention the precise assumptions on the coefficients $a, b, c, d$ and the functions $F, h, f_*, f^*$, and a theoretical result on the behaviour of orbits of solutions of our system. Now we make the following assumptions:

(A1) $F := \alpha u + \beta w$, $h := \gamma u + \delta w$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \gamma - a \beta = d \delta - b \gamma = 0$, $d \alpha - b \gamma > 0$, $c \beta - a \delta > 0$.

(A2) Functions $f_*, f^*$ are non-decreasing Lipschitz continuous functions of $C^2$-class such that $f_*(u) \leq f^*(u)$ for all $u \in \mathbb{R}$, and there are constants $f_\infty > 0$, $f_\infty < 0$ and $\kappa^* > 0$, $\kappa_* < 0$ such that $f_*(u) = f^*(u) \equiv f_\infty$ for all sufficiently large $u > 0$, $f_*(0) < 0 < f^*(0)$, $f_*(u) = f^*(u) \equiv f_\infty$ for all sufficiently small $u < 0$, $f_*(u) = f^*(u)$ for $u \in (-\infty, \kappa_*) \cup [\kappa^*, +\infty)$.

(A3) The number of connected components of the sets

$\{u \in \mathbb{R} | (a \delta - c \beta) f_*(u) f'_*(u) - (d \alpha - b \gamma) u = 0\}$ and

$\{u \in \mathbb{R} | (a \delta - c \beta) f^*(u) f'_*(u) - (d \alpha - b \gamma) u = 0\}$ is finite.

Assumption (A1) means that if there is no subdifferential $\partial I_u(w)$ in our system, then the orbits of solutions are anticlockwise ellipse for all initial data (especially the orbits of solutions are anticlockwise circles when $d \alpha - b \gamma = c \beta - a \delta > 0$ hold). Assumptions (A2), (A3) are concerned with the geometry of the two curves $w = f^*$ and $w = f_*$.

Especially, assumption (A3) implies that the curves $w = f_*(u)$ and $w = f^*(u)$ have a finite number of circles with center $(0, 0)$ which are tangential to the curves $w = f_*(u)$ or $w = f^*(u)$.

Under these assumptions, we give the definition of a solution of our system.

Definition 2.1 A pair of functions $\{w, u\}$ is called a solution of the system (1.1), (1.2), and (1.3) if the following (1)-(4) are satisfied:
The following theorem holds true.

**Theorem 2.1** Under these assumptions, the system (1.1)-(1.3) possesses one and only one solution.

This theorem guarantees the existence and uniqueness of solutions and it is a special case of [2; Theorem 2.4].

The precise behaviour of solutions of our system is given in the following theorem.

**Theorem 2.2** Suppose that assumptions (A1),(A2) and (A3) are satisfied. Let \( S = \{(u, w) \in \mathbb{R}^2 | f_*(u) \leq w \leq f^*(u)\} \), and denote by \( \{u, w\} \) the solution of our system with initial values \( u_0, w_0 \). Then \( S \) is divided into the following three subsets \( S_1, S_2 \) and \( S_3 \), i.e. \( S = S_1 \cup S_2 \cup S_3 \), such that

(i) if \( (u_0, w_0) \in S_1 \), then \( (u(t), w(t)) \) reaches a periodic ellipse around the origin in a finite time;

(ii) if \( (u_0, w_0) \in S_2 \), then \( (u(t), w(t)) \) converges (as \( t \to +\infty \)) to a stationary point \( (u_\infty, w_\infty) \) which satisfies

\[
\begin{align*}
\partial I_{u_\infty}(w_\infty) &\ni \alpha u_\infty + \beta w_\infty \\
\gamma u_\infty + \delta w_\infty &\equiv 0;
\end{align*}
\]

(iii) if \( (u_0, w_0) \in S_3 \), then \( (u(t), w(t)) \) diverges to \((+\infty, f^\infty)\) or to \((-\infty, f^\infty)\) as \( t \to +\infty \).

Moreover, the sets \( S_1, S_2 \) and \( S_3 \) are determined by the geometries of the curves \( w = f_*(u) \), \( w = f^*(u) \) and the line \( \gamma u + \delta w = 0 \) and their expressions are given in the next section.

In order to prove Theorem 2.2, we prepare the following section.
3 Subsets $S_i$ ($i = 1, 2, 3$)

In this section, we consider how to describe the subsets $S_i$ ($i = 1, 2, 3$) of $S$ on $(u, w)$ plane. Now we use the following notations:

$$\Gamma^* := \{(u, w) | w = f^*(u)\}, \quad \Gamma_* := \{(u, w) | w = f_*(u)\},$$

$$B(u, w) := \{(dx - br)u^2 + (c\beta - a\delta)w^2\}^{\frac{1}{2}}, \quad l := \{(u, w) \in R^2 | \gamma u + \delta w = 0\},$$

$$\Gamma^*(l) := \{(u, w) \in \Gamma^* \cap l | u > 0\}, \quad \Gamma_*(l) := \{(u, w) \in \Gamma_* \cap l | u < 0\},$$

$$r_0^* := \min \{B(u, w) | (u, w) \in \Gamma^*\}, \quad u^* := \min \{u | (u, w) \in \Gamma^*, B(u, w) = r_0^*\},$$

$$r_0_* := \min \{B(u, w) | (u, w) \in \Gamma_*\}, \quad u_* := \max \{u | (u, w) \in \Gamma_*, B(u, w) = r_0_*\},$$

$$r_1^* := \min \{B(u, w) | (u, w) \in \Gamma^*(l)\}, \quad R_1^* := \max \{B(u, w) | (u, w) \in \Gamma^*(l)\},$$

$$r_1_* := \min \{B(u, w) | (u, w) \in \Gamma_*(l)\}, \quad R_1_* := \max \{B(u, w) | (u, w) \in \Gamma_*(l)\},$$

$$A^+ := \{(u, w) | u^*w - f^*(u^*)u < 0 \text{ if } u < 0 \} \quad \text{and} \quad u_*w - f_*(u_*)u > 0 \text{ if } u > 0 \},$$

$$A^- := \{(u, w) | u^*w - f^*(u^*)u > 0 \text{ if } u > 0 \} \quad \text{and} \quad u_*w - f_*(u_*)u < 0 \text{ if } u < 0 \},$$

$$S_0 := \{(u, w) \in S | B(u, w) \leq r_0\} \text{ with } r_0 := \min \{r_0^*, r_0_*\}.$$

By our assumptions, we have

$$r_0^* < r_1^* \leq R_1^* \text{ and } r_0_* < r_1_* \leq R_1_*.$$

As to the relationships of $r_0^*, r_1^*, R_1^*, r_0_*, r_1_*$ and $R_1_*$ there are the following 6 cases to be considered:

1. $r_0^* \leq r_0^* < r_1_* \leq R_1_*$
2. $r_0^* < r_1^* \leq r_0^* < R_1^*$
3. $r_0^* < r_1^* \leq R_1^* < r_0_*$
4. $r_0^* \leq r_0^* \leq r_1^* \leq R_1^*$
5. $r_0^* < r_1^* \leq r_0_* \leq R_1^*$
6. $r_0^* < r_1^* \leq R_1^* < r_0_*$

In the case of (1) we define

$$S_1 := S_0 \cup S_1^+ \cup S_1^-,$$

where

$$S_1^+ := \{(u, w) \in S \cap A^+ | r_0^* < B(u, w) < r_1^*\},$$

$$S_1^- := \{(u, w) \in S \cap A^- | r_0^* < B(u, w) < r_1^-\},$$

$$S_2 := S_2^+ \cup S_2^-,$$
\[ S_2^- := \{(u, w) \in S \cap A^-| r_{1*} \leq B(u, w) \leq R_{1*}\} \]  \hfill (3.6)

\[ S_3 := S_3^+ \cup S_3^- \]  \hfill (3.7)

where

\[ S_3^+ := \{(u, w) \in S \cap A^+| R_{1*} < B(u, w)\} \]  \hfill (3.8)

\[ S_3^- := \{(u, w) \in S \cap A^-| R_{1*} < B(u, w)\} \]  \hfill (3.9)

In the case of (2) we define

\[ S_1 := S_0 \cup S_1^0 \]  \hfill (3.10)

where

\[ S_1^0 := \{(u, w) \in S|r_{0*} < B(u, w) < r_{1*}\} \]  \hfill (3.11)

\[ S_2 := S_2^+ \cup S_2^- \]  \hfill (3.12)

where

\[ S_2^+ := \{(u, w) \in S \cap A^+| r_{1*} \leq B(u, w) \leq R_{1*}\} \]  \hfill (3.13)

\[ S_2^- := \{(u, w) \in S \cap A^-| r_{1*} \leq B(u, w) \leq R_{1*}\} \] \hfill (3.14)

\[ S_3 := S_3^+ \cup S_3^- \]  \hfill (3.15)

where

\[ S_3^+ := \{(u, w) \in S \cap A^+| r_{1*} \leq B(u, w) \leq R_{1*}\} \]  \hfill (3.16)

\[ S_3^- := \{(u, w) \in S \cap A^-| r_{1*} \leq B(u, w) \leq R_{1*}\} \]  \hfill (3.17)

In the case of (3) we define

\[ S_1 := S_0 \cup S_1^0 \]  \hfill (3.18)

where

\[ S_1^0 := \{(u, w) \in S|r_{0*} < B(u, w) < r_{1*}\} \]  \hfill (3.19)

\[ S_2 := S_2^+ \cup S_2^- \]  \hfill (3.20)

where

\[ S_2^+ := \{(u, w) \in S \cap A^+| r_{1*} \leq B(u, w) \leq R_{1*}\} \]  \hfill (3.21)
\[ S_2 := \{(u, w) \in S | r_1^* \leq B(u, w) \leq R_1^* \}; \]  
(3.22)

\[ S_3 := S_3^+ \cup S_3^-, \]  
(3.23)

where

\[ S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \}, \]  
(3.24)

\[ S_3^- := \{(u, w) \in S \cap A^- | R_1^* < B(u, w) \}; \]  
(3.25)

In the case of (4) we define

\[ S_1 := S_0 \cup S_1^+ \cup S_1^- \],

where

\[ S_1^+ := \{(u, w) \in S \cap A^+ | r_0^* < B(u, w) < r_1^* \}, \]

\[ S_1^- := \{(u, w) \in S \cap A^- | r_0^* < B(u, w) < r_1^* \}; \]

\[ S_2 := S_2^+ \cup S_2^- \],

where

\[ S_2^+ := \{(u, w) \in S \cap A^+ | r_1^* \leq B(u, w) \leq R_1^* \}, \]

\[ S_2^- := \{(u, w) \in S \cap A^- | r_1^* \leq B(u, w) \leq R_1^* \}; \]

\[ S_3 := S_3^+ \cup S_3^- \],

where

\[ S_3^+ := \{(u, w) \in S \cap A^+ | R_1^* < B(u, w) \}, \]

\[ S_3^- := \{(u, w) \in S \cap A^- | R_1^* < B(u, w) \}. \]

In the case of (5) we define

\[ S_1 := S_0 \cup S_1^0 \],

where

\[ S_1^0 := \{(u, w) \in S | r_0^* < B(u, w) < r_1^* \}; \]

\[ S_2 := S_2^+ \cup S_2^- \],
where

\[ S_2^+ = \{(u,w) \in S \cap A^+ | r_1^* \leq B(u,w) \leq R_1^* \} \]
\[ \cup \{(u,w) \in S \cap A^- | r_1^* \leq B(u,w) < r_1* \}; \]

\[ S_2^- := \{(u,w) \in S \cap A^- | r_1* \leq B(u,w) \leq R_1* \}; \]

\[ S_3 := S_3^+ \cup S_3^-, \]

where

\[ S_3^+ := \{(u,w) \in S \cap A^+ | R_1^* < B(u,w) \leq R_1* \}; \]

\[ S_3^- := \{(u,w) \in S \cap A^- | R_1* < B(u,w) < r_1* \}. \]

In the case of (6) we define

\[ S_1 := S_0 \cup S_1^0, \]

where

\[ S_1^0 := \{(u,w) \in S | r_0.* < B(u,w) < r_1^* \}; \]

\[ S_2 := S_2^+ \cup S_2^-, \]

where

\[ S_2^+ := \{(u,w) \in S | r_1^* \leq B(u,w) \leq R_1^* \} \]

\[ S_2^- := \{(u,w) \in S \cap A^- | r_1* \leq B(u,w) \leq R_1* \}; \]

\[ S_3 := S_3^+ \cup S_3^-, \]

where

\[ S_3^+ := \{(u,w) \in S \cap A^+ | R_1^* < B(u,w) \leq R_1* \} \]
\[ \cup \{(u,w) \in S \cap A^- | ?_1* \leq B(u,w) < r_1* \}; \]

\[ S_3^- := \{(u,w) \in S \cap A^- | R_1* < B(u,w) < r_1* \}. \]

In any cases of (1)-(6), when the initial data belong to any subset of \( S_1, S_2 \) and \( S_3 \), the orbits of the solutions satisfy the statements (i)-(iii) of Theorem 2.2. In the next section, we prepare some Lemmas in order to prove Theorem 2.2.
4 Local behaviour of orbits

In this section, we investigate the local behaviour of the orbit \((u(t), w(t))\), satisfying
\[
\begin{align*}
aw'(t) + bu'(t) + \partial I_u(w(t)) \ni \alpha u(t) + \beta w(t), \\
cw'(t) + dw'(t) = \gamma u(t) + \delta w(t)
\end{align*}
\]
for \(t \geq 0\). We only give proof of Lemma 4.3. Other Lemmas are shown without proofs.

**Lemma 4.1** Assume that \((u(t_1), w(t_1))\), \(t_1 \geq 0\), is in the interior of \(S\). Then:

(a) if \(B(u(t_1), w(t_1)) \leq r_0\), then \(\{u, w\}\) satisfies
\[
\begin{align*}
&u'(t) = -\frac{c\beta - a\delta}{ad - bc}w, \\
&w'(t) = \frac{d\alpha - b\gamma}{ad - bc}u.
\end{align*}
\] (4.1)

for all \(t \geq t_1\), and hence the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(C_1 := \{(u, w)|B(u, w) = B(u(t_1), w(t_1))\}\) and it is periodic in time on \([t_1, +\infty)\).

(b) if \(B(u(t_1), w(t_1)) > r_0\), then \(\{u, w\}\) satisfies system (4.1) on a compact interval \([t_1, t_2]\) with \(t_2 > t_1\), where \(t_2\) is the earliest time of all \(t > t_1\) at which \((u(t), w(t)) \in \Gamma_* \cup \Gamma^*\). Hence the orbit \((u(t), w(t))\) draws an anticlockwise arc on the ellipse \(C_1\) for \(t_1 \leq t \leq t_2\).

We note that the stationary problem of (1.1)-(1.2) is of the form
\[
\partial I_u(w) \ni \alpha u + \beta w, \quad \gamma u + \delta w = 0.
\]

**Lemma 4.2** (a) Let \((\tilde{u}, \tilde{w})\) be an interior point of \(S\). Then \(\{\tilde{u}, \tilde{w}\}\) is a stationary solution of (1.1)-(1.2) if and only if \(\tilde{u} = 0\) and \(\tilde{w} = 0\).

(b) Let \((\tilde{u}, \tilde{w})\) be a boundary point of \(S\). Then \(\{\tilde{u}, \tilde{w}\}\) is a stationary solution of (1.1)-(1.2) if and only if \((\tilde{u}, \tilde{w}) \in \Gamma_*(l) \cup \Gamma^*(l)\).

**Lemma 4.3** Assume that \((u(t_1), w(t_1)), \ t_1 \geq 0,\) is on \(\Gamma_*\) and \(w(t_1) < 0.\) Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) > 0\) and if there exists \(\bar{u} > u(t_1)\) such that
\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v \leq \bar{u},
\]

and moreover if
\[
\frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f_*(u)} \leq f_*(u) \text{ for } u(t_1) \leq v \leq \bar{u},
\] (4.2)
then \(\{u, w\}\) satisfies

\[
u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{cf'_*(u(t)) + d}, \quad w'(t) = f'_*(u(t))u'(t) \tag{4.3}\]

on a compact interval \([t_1, t_2]\), where \(t_2\) is the earliest time at which \(u(t_2) = \bar{u}\), and the orbit \((u(t), w(t))\) moves along \(\Gamma_*\) from \((u(t_1), w(t_1))\) to \((\bar{u}, f_*(\bar{u}))\) for \(t_1 \leq t \leq t_2\). Moreover

\[
\frac{d}{dt} B(u(t), w(t)) \leq 0 \text{ on } [t_1, t_2]. \tag{4.4}\]

(b) if \(\gamma u(t_1) + \delta w(t_1) > 0\) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma_*(l)\) with \(\bar{u} > u(t_1)\) such that

\[
\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v < \bar{u},
\]

then \(\{u, w\}\) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves upward along the curve \(\Gamma_*\) and converges to \((\bar{u}, \bar{w})\) as \(t \to +\infty\);

(c) if \(\gamma u(t_1) + \delta w(t_1) < 0\) and if there exists a stationary point \((u, w) \in \Gamma_*(l)\) with \(\bar{u} < u(t_1)\) such that

\[
\gamma v + \delta f_*(v) < 0 \text{ for } \bar{u} < v \leq u(t_1),
\]

then \(\{u, w\}\) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \(\Gamma_*\) and converges to \((u, w)\) as \(t \to +\infty\);

(d) if \(\gamma v + \delta f_*(v) < 0\) holds for all \(v \leq u(t_1)\), then \(\{u, w\}\) satisfies (4.3) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \(t \to +\infty\).

**Proof.** We prove (a). We put \((u_1, w_1) = (u(t_1), w(t_1))\); note that \(w_1 = f_*(u_1)\), since \((u_1, w_1) \in \Gamma_*\). We can find a positive constant \(M\) such that

\[
\gamma v + \delta f_*(v) \geq M \text{ for } u_1 \leq v \leq \bar{u}. \tag{4.5}\]

Now, consider the Cauchy problem

\[
\hat{u}'(t) = \frac{\gamma \hat{u}(t) + \delta f_*(\hat{u}(t))}{cf'_*(\hat{u}(t)) + d}, \quad t_1 \leq t < \hat{t}_1^* \tag{4.6}
\]

\[
\hat{u}(t_1) = u_1 \tag{4.7}
\]

where \(\hat{t}_1^*\) is the supremum of positive number \(t_1^* (> t_1)\) such that problem (4.6)-(4.7) has a solution on \([t_1, t_1^*]\). In fact, since the function \(v \mapsto \frac{\gamma v + \delta f_*(v)}{cf'_*(v) + d}\) is Lipschitz continuous in a neighborhood of \(v = u_1\), by the general theory of ODEs the problem (4.6)-(4.7) has a (unique) local (in time) solution \(\hat{u}(t)\). It is easy to see from (4.5) that \(\hat{u}(\cdot)\) is monotonically increasing and reaches the value \(\bar{u}\) in a finite time \(t_2 \in (t_1, t_1^*)\). Now,
putting \( \hat{w}(t) = f_*(\hat{u}(t)) \) on \([t_1, t_2]\), we have that \( \{\hat{u}, \hat{w}\} \) satisfies our system (1.1) and (1.2) on \([t_1, t_2]\). In fact, it follows from (4.6) that
\[
cf'_* (\hat{u}(t))\hat{u}'(t) + d\hat{u}'(t) = \gamma \hat{u}(t) + \delta f_*(\hat{u}(t)),
\]
which implies \( c\hat{w}'(t) + d\hat{u}'(t) = \gamma \hat{u}(t) + \delta \hat{w}(t) \) on \([t_1, t_2]\). Thus (1.2) is satisfied. Equation (1.1) is checked as follows. By assumption (A1) and (4.2), calculating \( \alpha \hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}' \), we obtain
\[
\alpha \hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}' = \alpha \hat{u} + \beta f_*(\hat{u}) - \frac{\gamma \hat{u} + \delta f_*(\hat{u})}{cf'_*(\hat{u}) + d} (af'_*(\hat{u}) + b)
\]
\[
= \frac{(\alpha \hat{u} + \beta f_*(\hat{u}))(cf'_*(\hat{u}) + d) - (\gamma \hat{u} + \delta f_*(\hat{u}))(af'_*(\hat{u}) + b)}{cf'_*(\hat{u}) + d}
\]
\[
= \frac{((c\alpha - a\gamma)\hat{u} + (c\beta - a\delta)f_*(\hat{u}))(f'_*(\hat{u}) - (b\gamma - d\alpha)\hat{u} - (b\delta - d\beta)f_*(\hat{u})}{cf'_*(\hat{u}) + d}
\]
\[
= \frac{(c\beta - a\delta)f_*(u)f'_*(u) - (b\gamma - d\alpha)u}{cf'_*(u) + d}
\]
\[
\leq 0
\]
on \([t_1, t_2]\). By the definition of subdifferentials (see (1.4)) we have \( \partial I_\hat{u}(\hat{u}) = (-\infty, 0] \) for \( \hat{w} = f_*(\hat{u}) \). Therefore
\[
\alpha \hat{u} + \beta \hat{w} - a\hat{w}' - b\hat{u}' \in \partial I_\hat{u}(\hat{w}) \text{ on } [t_1, t_2].
\]
Thus, by the uniqueness, \( \{\hat{u}, \hat{w}\} \) must be the solution \( \{u, w\} \) of (1.1)-(1.2) on \([t_1, t_2]\). Next we show (4.4). Since (4.2) and (4.3) hold on \([t_1, t_2]\), we obtain
\[
\frac{d}{dt} B(u, w) = \frac{u'}{B(u, w)} \{(c\beta - a\delta) f'_*(u) f_*(u) - (b\gamma - d\alpha)u\}
\]
\[
\leq 0 \text{ on } [t_1, t_2].
\]
Next we prove (b). Let us recall that \( \bar{u} < 0, f_*(\bar{u}) < 0 \) by Lemma 4.2 (b). We obtain automatically
\[
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f'_*(v)} \leq f'_*(v) \text{ for } u_1 \leq v \leq \bar{u}.
\]
(4.8)
Therefore, in the same way as in (a), \( \{u, w\} \) satisfies (4.3) for a moment after the time \( t_1 \) and the orbit \( (u(t), w(t)) \) moves along the curve \( \Gamma_* \), starting from \( (u(t_1), w(t_1)) \). We now
show that \((u(t), w(t))\) converges to \((\overline{u}, \overline{w})\) if \(t \to +\infty\). Let \(T\) be the supremum of all \(s(\geq t_1)\) such that
\[
u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{cf_*(u(t)) + d}, \quad w(t) = f_*(u(t)) \quad \forall t \in [t_1, s].
\]
Then, just as in the case of (a), we see that \(T > t_1\). Since \(u\) is non-decreasing on \([t_1, T]\), \(\lim_{t \to T} u(t)\) exists. We want to see that \(\lim_{t \to T} u(t) = \overline{u}\). We show it by contradiction. Now, assume that \(\lim_{t \to T} u(t) < \overline{u}\). Then we consider the following statements:

(i) \(T = +\infty, \quad u_\infty := \lim_{t \to +\infty} u(t)\) and \(w_\infty := \lim_{t \to +\infty} w(t)\) give a pair of stationary solutions

or

(ii) \(T < +\infty\) and \(\frac{(d\alpha - b\gamma)u(t)}{(a\delta - c\beta)f_*(u(t))} > f'_*(u(t))\) for some \(t > T\).

But these cases do not occur in our situations considered now. In fact, the case (i) yields that \(u(t_1) \leq u_\infty < \overline{u}\) and \(\gamma u_\infty + \delta f_*(u_\infty) = 0\), which contradicts our assumption. Also, the case (ii) yields a contradiction to (4.8).

Assertion (c) is similarly proved to (b).

Finally we prove (d). By the same argument as above, we have
\[
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f'_*(v) \quad \text{for all} \quad v \leq u_1,
\]
and find a negative constant \(\tilde{M}\) such that
\[
\gamma v + \delta f_*(v) \leq \tilde{M} \quad \text{for all} \quad v \leq u_1.
\]
Hence \(\{u, w\}\) satisfies (4.3) for all \(t \geq t_1\) and \(u(\cdot)\) is monotonically decreasing on \([t_1, \infty)\). By assumption (A2), \((u(t), w(t))\) diverges to \((-\infty, f_\infty)\) as \(t \to +\infty\).

**Lemma 4.4** Assume that \((u(t_1), w(t_1)), \ t_1 \geq 0,\) is on \(\Gamma^*\) and \(w(t_1) > 0\). Then:

(a) if \(\gamma u(t_1) + \delta w(t_1) < 0\) and if there exists \(\overline{u} < u(t_1)\) such that \(\gamma v + \delta f^*(v) < 0\) for \(\overline{u} \leq v \leq u(t_1)\) and moreover if the following condition hold that
\[
\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f^*(v)} \leq f''(v) \quad \text{for} \quad \overline{u} \leq v \leq u(t_1),
\]
then \( \{u, w\} \) satisfies

\[
\begin{align*}
u'(t) &= \frac{\gamma u + \delta f^*(u)}{c f^*(u) + d}, \\
w'(t) &= f^*/(u)u'(t)
\end{align*}
\]

on a compact interval \([t_1, t_2]\), where \( t_2 \) is the earliest time at which \( u(t_2) = \bar{u} \), and the orbit \((u(t), w(t))\) moves along \( \Gamma_* \) from \((u(t_1), w(t_1))\) to \((\bar{u}, f^*(\bar{u}))\) for \( t_1 \leq t \leq t_2 \). Moreover

\[
\frac{d}{dt}B(u, w) \leq 0 \text{ on } [t_1, t_2].
\]

(b) if \( \gamma u(t_1) + \delta w(t_1) < 0 \) and if there exists a stationary point \((\bar{u}, \bar{w}) \in \Gamma^*(l)\) with \( \bar{u} < u(t_1) \) such that

\[
\gamma v + \delta f^*(v) < 0 \text{ for } \bar{u} < v \leq u(t_1),
\]

then \( \{u, w\} \) satisfies (4.4) on \([t_1, +\infty)\), and the orbit \((u(t), w(t))\) moves downward along the curve \( \Gamma^* \) and converges to \((\bar{u}, \bar{w})\) as \( t \to +\infty \).

(c) if \( \gamma u(t_1) + \delta w(t_1) > 0 \) and if there exists a stationary point \((u, w) \in \Gamma^*(l)\) with \( u > u(t_1) \) such that

\[
\gamma v + \delta f^*(v) > 0 \text{ for } u(t_1) \leq v < u,
\]

then \( \{u, w\} \) satisfies (4.4) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) moves upward along the curve \( \Gamma^* \) and converges to \((u, w)\) as \( t \to +\infty \).

(d) if \( \gamma v + \delta f^*(v) > 0 \) holds for all \( v \geq u(t_1) \), then \( \{u, w\} \) satisfies (4.4) for \([t_1, +\infty)\). Hence the orbit \((u(t), w(t))\) diverges to \((\infty, f^\infty)\) as \( t \to +\infty \).

5 Large time behaviour of orbits

In this section, we prove Theorem 2.2 in the case (1) in section 3. Any other cases can be treated by a simple modification of them. We investigate the behaviour of the solution \( \{u, w\} \) when the initial data \((u_0, w_0)\) belong to each of \( S_0, S_1, S_2 \) and \( S_3 \).

In the case of \((u_0, w_0) \in S_0\)

When \((u_0, w_0) \in S_0\), we obtain \( B(u_0, w_0) \leq r_0 \). Therefore, by Lemma 4.1(a), we see that the orbit \((u(t), w(t))\) draws anticlockwise ellipse \( B(u, w) = B(u_0, w_0) \) for all \( t \geq 0 \), and is periodic in time.

In the case of \((u_0, w_0) \in S_1\)

First, we consider the case of \((u_0, w_0) \in S_1^-\) and \( w_0 \leq 0 \). Clearly \( B(u_0, w_0) > r_0 \).
By Lemma 4.1 (b), the orbit \((u(t), w(t))\) draws an anticlockwise ellipse on \(B(u, w) = B(u_0, w_0)\), until it reaches \(\Gamma_*\), satisfying
\[
\begin{align*}
u'(t) &= \frac{c\beta - a\delta}{ad - bc} w(t), \quad 0 \leq t \leq t_1, \\
w'(t) &= \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad 0 \leq t \leq t_1,
\end{align*}
\]
where \(t_1\) is the earliest time such that \((u(t_1), w(t_1)) \in \Gamma_*\). We have \(w(t_1) = f_*(u(t_1))\), \(B(u(t_1), w(t_1)) < r_{1*}\) and
\[
\gamma v + \delta f_*(v) > 0 \quad \text{for} \quad u(t_1) \leq v \leq u_*
\]
Next, take the number \(u_2\) so that \(u_2 = \sup \{\tilde{u} \mid u(t_1) \leq \tilde{u} \leq u_*, \quad \frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f(u)} \leq f_*(u) \quad \text{for} \quad u \in [u(t_1), \tilde{u}]\} \).
Then we have the following three possibilities: (i) \(u(t_1) < u_2 < u_*\), (ii) \(u_2 = u(t_1)\), (iii) \(u_2 = u_*\).

In the case of (i), by Lemma 4.3 (a)
\[
\begin{align*}
u'(t) &= \frac{\gamma u(t) + \delta f_*(u(t))}{c f_*(u(t)) + d} , \quad w(t) = f_*(u(t)), \quad t \in [t_1, t_2]
\end{align*}
\]
where \(t_2\) is the earliest time such that \(u(t_2) = u_2\). We denote by \(C_2\) the ellipse \(B(u, w) = B(u_2, f_*(u_2)) =: r_2\). By assumption (A3) and the definition of \(u_2\), we see that an arc \([(u, w) \mid u_2 \leq u \leq \tilde{u}_3], \quad B(u, w) = r_2\) on \(C_2\) is contained in \(\mathcal{S}\). Now, denote by \(u_3\) the largest one of such numbers \(\tilde{u}_3\), we have \(u_3 > u_2\). Moreover, by Lemma 4.1 (b), \(\{u, w\}\) is given by
\[
\begin{align*}
u'(t) &= \frac{c\beta - a\delta}{ad - bc} w(t) , \quad w'(t) = \frac{d\alpha - b\gamma}{ad - bc} u(t), \quad t \in [t_2, t_3],
\end{align*}
\]
where \(t_3\) is the earliest time such that \(u(t_3) = u_3\). Our assumption (A3) guarantees that the orbit \((u(t), w(t))\) reaches \((u_*, f_*(u_*))\) at \(t = t_* < \infty\) by repeating finitely many times such behaviours as above. Here, after the time \(t_*\), the orbit \((u(t), w(t))\) draws the anticlockwise ellipse \(B(u, w) = r_0\) periodically in time (see Lemma 4.1 (a)).

In the case of (ii), it is the case that \(t_1 = t_2\) with the same notation as above, and the behaviour of \((u(t), w(t))\) is similar to the case of (i) after the time \(t_2\).

In the case of (iii), it is the case that \(t_2 = t_*\), and the behaviour of \((u(t), w(t))\) is the
anticlockwise ellipse $B(u, w) = r_0$ after the time $t_*$.

Next, consider the case of $(u_0, w_0) \in S_1^-$ with $w_0 > 0$. In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_1$. If $(u(s_1), w(s_1)) \in \Gamma_*$, then the behaviour of $(u(t), w(t))$ is exactly the same as in the previous case after time $s_1$. On the other hand, if $(u(s_1), w(s_1)) \in \Gamma^*$, then the orbit $(u(t), w(t))$ moves downward for a time interval $[s_1, s_2]$ with $s_1 \leq s_2$ along the curve $\Gamma^*$ by Lemma 4.4 (a) (in this step assumption (A3) regarding the function $f^*(\cdot)$ is used), where $s_2$ is the largest time of $s_2$ such that $(u(t), w(t)) \in \Gamma^*$ for $\forall t \in [s_1, s_2]$. It is easy to see that $w(s_2) > 0$ and $s_2 < +\infty$. After time $s_2$, the orbit $(u(t), w(t))$ draws an anticlockwise arc on $B(u, w) = B(u(s_2), w(s_2))$ until it reaches $\Gamma_*$ or $\Gamma^*$ at time $s_3$. Repeating such procedures finitely many times, the orbit $(u(t), w(t))$ arrives at $\Gamma_*$ at time $t = t_1$ in the last step. After time $t_1$, the behaviour of $(u(t), w(t))$ was already seen in the case of $(u_0, w_0) \in S_1^-$ with $w_0 \leq 0$.

Finally, we consider the case of $(u_0, w_0) \in S_1^+$. We have the following three cases:

(i) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) \geq r_0^*$,

(ii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r_0^*$ and $w_0 \geq 0$,

(iii) $(u_0, w_0) \in S_1^+$ with $B(u_0, w_0) < r_0^*$ and $w_0 < 0$.

First, we consider the case (i). In this case, the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = r \in [r_0^*, r_1^*]$ and a part of $\Gamma^*$ alternately and reaches the point $(u^*, f^*(u^*))$ at a finite time $t = t^*$. Since $(u^*, f^*(u^*)) \in S_1^-$, the behaviour of $(u(t), w(t))$ after the time $t^*$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 > 0$.

In the second case (ii), the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse $B(u, w) = B(u_0, w_0)$ and reaches a point $(u_0, w_0) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as in the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the third case (iii), the orbit $(u(t), w(t))$ possibly draws an anticlockwise arc on the ellipse $B(u, w) = r \in (r_0^*, r_1^*)$ and a part of $\Gamma_*$ alternately and reaches a point $(u_1, w_1) \in \Gamma_*$ with $u_1 < u_*$ and $w_1 < 0$ at a finite time $t = t_1$. After the time $t_1$, the behaviour of $(u(t), w(t))$ is the same as the case $(u_0, w_0) \in S_1^-$ with $w_0 < 0$.

In the case of $(u_0, w_0) \in S_2$

We give a proof only in the case of $(u_0, w_0) \in S_2^-$, since the proof of the case of $(u_0, w_0) \in S_2^+$ is quite similar. In a way similar to that in the case of $(u_0, w_0) \in S_1$, we see that the orbit $(u(t), w(t))$, drawing an anticlockwise arc on the ellipse $B(u, w) = r \in [r_1^*, R_1^*]$, arrives at a point $(u_1, w_1) \in \Gamma_*$ at a certain finite time $t = t_1$. If $(u(t_1), w(t_1)) = (u_1, w_1)) \in \Gamma_*(l)$, then $(u(t_1), w(t_1))$ is a stationary solution of (1.1)-(1.3) by Lemma 4.2 (b). If $(u(t_1), w(t_1)) \notin \Gamma_*(l)$, then we have the following two cases:
(i) $\gamma u(t_1) + \delta w(t_1) > 0,$

(ii) $\gamma u(t_1) + \delta w(t_1) < 0.$

Suppose now that (i) holds. Then there is a closed interval $[u, \bar{u}] \subset (-\infty, 0)$ on the $u$-axis such that $u < u(t_1) < \bar{u}$, $\gamma v + \delta f_*(v) > 0$ for all $v \in (u, \bar{u})$ and $\gamma \bar{u} + \delta f_*(\bar{u}) = \gamma u + \delta f_*(u) = 0$. Therefore, the orbit $(u(t), w(t))$ converges to $(\bar{u}, f_*(\bar{u})) \in \Gamma_*(l)$ as $t \to +\infty$ by Lemma 4.3 (b). On the other hand, when (ii) holds, the orbit $(u(t), w(t))$ converges to a stationary point as $t \to +\infty$, too.

In the case of $(u_0, w_0) \in S_3$

It is enough to consider only the case $(u_0, w_0) \in S_3^-$. In the same way as in the case of $(u_0, w_0) \in S_2$, the orbit $(u(t), w(t))$ reaches $\Gamma_*$ in a finite time $t_1$. Also, we obtain $B(u(t_1), w(t_1)) < R_1$, and $\gamma v + \delta f_*(v) < 0$ for $v < u_1$. Therefore, by Lemma 4.3 (d), we see that $(u(t), w(t))$ diverges to $(-\infty, f_\infty)$ as $t \to +\infty$. Similarly, in the case $(u_0, w_0) \in S_3^+$, we see that $(u(t), w(t))$ diverges to $(\infty, f^\infty)$ as $t \to +\infty$.

Remark 5.1 We have many cases about the stability around stationary points in $S_2$. If, for instance, we restrict our geometry of the curves $\Gamma_*, \Gamma^*$ and $l$ to the one as illustrated by the picture (Fig. 1), then stationary points are classified into the following three categories: Let $(u_\infty, w_\infty)$ be any stationary point in $S_2$. Then one of the following cases happens.

![Diagram](image)

Fig. 1

(1) $(u_\infty, w_\infty)$ is stable. Namely, there is a neighborhood $U_1$ of $(u_\infty, w_\infty)$ in $R^2$ such that the orbit $(\tilde{u}(t), \tilde{w}(t))$ stays in $U_1 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ as $t \to +\infty$, whenever $(\tilde{u}_0, \tilde{w}_0) = (\tilde{u}(0), \tilde{w}(0)) \in U_1 \cap S$.

(2) $(u_\infty, w_\infty)$ is semistable. Namely, there is a neighborhood $U_2$ of $(u_\infty, w_\infty)$ in $R^2$ such that the following properties (i) and (ii) are satisfied:

(i) For any initial point $(\tilde{u}_0, \tilde{w}_0) \in U_2 \cap S \cap K_\infty$, the orbit $(\tilde{u}(t), \tilde{w}(t))$ stays in $U_2 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ as $t \to +\infty$, whenever $(\tilde{u}_0, \tilde{w}_0) = (\tilde{u}(0), \tilde{w}(0)) \in U_2 \cap S$.

(ii) For any initial point $(\tilde{u}_0, \tilde{w}_0) \in U_2 \cap S \cap K_\infty$, the orbit $(\tilde{u}(t), \tilde{w}(t))$ gets out of $U_2$ after a certain time $t_1$. 

where $\mathcal{K}_\infty := \{(u, w) | B(u, w) \geq B(u_\infty, w_\infty)\}$.

(3) $(u_\infty, w_\infty)$ is unstable. Namely, there is a neighborhood $U_3$ of $(u_\infty, w_\infty)$ in $R^2$ such that the following properties (iii) and (iv) are satisfies:

(iii) For any initial point $(\hat{u}_0, \hat{w}_0) \in U_3 \cap S \cap C_\infty$, the orbit $(\hat{u}, \hat{w})$ stays in $U_3 \cap S$ for all $t \geq 0$ and converges to $(u_\infty, w_\infty)$ in a finite time $t_1$.

(iv) For any initial point $(\check{u}_0, \check{w}_0) \in U_3 \cap S \cap C_\infty^c$, the orbit $(\check{u}(t), \check{w}(t))$ gets out of $U_3$ after a certain time $t_1$.

where $C_\infty := \{(u, w) | B(u, w) = B(u_\infty, w_\infty)\}$.

6 Some numerical simulations

In this section, we give some numerical experiments to verify Theorem 2.2. In order to catch the behaviour of solutions, we simply take the coefficients $a, b, c, d$ and functions $F, h$ satisfying (A1) with $\Delta > 0$ such that the orbits of solutions are anticlockwise circles without subdifferential term $\partial I_u^\lambda(w)$. Now we fix the coefficients $a, b, c, d$ and functions $F, h$ as follows:

$a = 1, b = -1, c = 1, d = 1, F(u, w) = u + w, h(u, w) = u - w$.

In this case, our system is of the following form:

\[ w' - u' + \partial I_u^\lambda(w) \ni u + w, \quad 0 < t < T, \]

\[ w' + u' = u - w, \quad 0 < t < T, \]

\[ u(0) = u_0, \quad w(0) = w_0. \]

Now let $\lambda$ and $\Delta t$ be small positive numbers, and $n$ be a large natural number. Then the difference scheme for our numerical simulation is of the form

\[ \frac{w^{k+1} - w^k}{\Delta t} - \frac{u^{k+1} - u^k}{\Delta t} + \partial I_u^\lambda(w^{k+1}) = u^k + w^k, \]

\[ \frac{w^{k+1} - w^k}{\Delta t} + \frac{u^{k+1} - u^k}{\Delta t} = u^k - w^k, \quad k = 0, 1, 2, \ldots, \]

\[ u^0 = u_0, \quad w^0 = w_0, \]

where

\[ \partial I_u^\lambda(w^{k+1}) = \frac{[w^{k+1} - f^*(u^k)]^+}{\lambda} - \frac{[f_*(u^k) - w^{k+1}]^+}{\lambda}. \]

The graphs of $I_u^\lambda$ and $\partial I_u^\lambda$ are illustrated in Figures 2 and 3, respectively.
In our actual computation

\[ \Delta t = \frac{1}{1000}, \quad \lambda = \frac{1}{1000}, \]

and we examine the following items:

- We define the subset \( S_i (i = 0, 1, 2, 3) \) by the geometries of the given functions \( f_*(u) \) and \( f^*(u) \) and the line \( \gamma u + \delta w = 0 \).

- By numerical simulations, we verify that the behaviour of solutions satisfies the statements of Theorem 2.2 when the initial data belong to each subset \( S_i (i = 0, 1, 2, 3) \).

Experiment 1:
We take the functions \( f_*(u) \), \( f^*(u) \) as follows:

\[
 f_*(u) = \begin{cases} 
 -1 & \text{if } u \leq 0.4, \\
 5u^2 - 4u - 0.2 & \text{if } 0.4 < u \leq 0.6, \\
 2u - 2 & \text{if } 0.6 < u \leq 1.4, \\
 -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
 1 & \text{if } 1.6 < u, 
\end{cases} \quad f^*(u) = \begin{cases} 
 -1 & \text{if } u \leq -1.6, \\
 5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
 2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
 -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
 1 & \text{if } -0.4 < u. 
\end{cases}
\]

\( f_*(u) \) and \( f^*(u) \) are symmetric with respect to origin. In this case, by our choice of \( f_*(u) \), \( f^*(u) \) and the line \( u + w = 0 \), we obtain that

\[ r_0 := r_{0*} = r_{0}^* = \frac{2\sqrt{5}}{5}, \quad r_{1*} = r_{1}^* = R_{1*} = R_{1}^* = \sqrt{2} \]

and stationary points are \((1, 1), (0, 0)\), and \((-1, -1)\).

Therefore, subsets \( S_i (i = 0, 1, 2, 3) \) are defined by (3.1)-(3.9) and are illustrated by Figure 4. Now we take the initial data which belong to each subset \( S_i \) and numerical experiments are shown as follows:

<table>
<thead>
<tr>
<th>subset</th>
<th>data</th>
<th>( u_0 )</th>
<th>( w_0 )</th>
<th>( (u_0, w_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>Fig. 5</td>
<td>-0.2</td>
<td>-0.6</td>
<td>( (u_0, w_0) \in S_0 )</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>Fig. 6</td>
<td>-0.7</td>
<td>-0.8</td>
<td>( (u_0, w_0) \in S_1 )</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>Fig. 7</td>
<td>1.3</td>
<td>0.8</td>
<td>( (u_0, w_0) \in S_2 )</td>
</tr>
</tbody>
</table>
When the initial data belong to $S_0$, the orbit draws anticlockwise circle from the initial point $(u_0, w_0)$ (Fig. 5). In the case when $(u_0, w_0) \in S_1$, the orbit draws an anticlockwise arc and a part of $\Gamma_*$ alternately and reaches a periodic circle $B(u, w) = r_0$ in a finite time (Fig. 6). On the other hand, in the case when $(u_0, w_0) \in S_3^-$ or $S_3^+$, the orbit diverges to $(-\infty, -1)$ or $(+\infty, 1)$ as $t \to +\infty$ (Fig. 7).

**Experiment 2:**
We take the functions $f_*(u), f^*(u)$ as follows:

$$f_*(u) = \begin{cases} -1 & \text{if } u \leq -1, \\ 3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\ -u^2 - 0.25 & \text{if } -0.75 < u \leq 0, \\ -0.25 & \text{if } 0 < u \leq 0.75, \\ 4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\ 2u - 2 & \text{if } 1 < u \leq 1.4, \\ -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\ 1 & \text{if } 1.6 < u. \end{cases}$$

$$f^*(u) = \begin{cases} -1 & \text{if } u \leq -2, \\ u^2 + 4u + 3 & \text{if } -2 < u \leq -1.5, \\ u + 0.75 & \text{if } -1.5 < u \leq 0, \\ -u^2 + u + 0.75 & \text{if } 0 < u \leq 0.5, \\ 1 & \text{if } 0.5 < u. \end{cases}$$
In this case, we obtain that
\[ r_0 := r_{0*} = \frac{1}{4}, \quad r_0^* = \frac{3\sqrt{2}}{8}, \quad r_1* = \frac{\sqrt{2}}{2}, \quad R_1* = \sqrt{2}, \quad R_1^{*} = \sqrt{2} \]
and

stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

Since \(r_{0*} < r_0^* < r_1* < R_1*\), \(S_i(i = 0, 1, 2, 3)\) are defined by (3.1)-(3.9) and are illustrated by Figure 8. The initial data and the subsets \(S_i\) in which the initial data are given in this experiments are as follows

<table>
<thead>
<tr>
<th>data</th>
<th>(u_0)</th>
<th>(w_0)</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 9</td>
<td>-1.1</td>
<td>0.4</td>
<td>((u_0, w_0) \in S_1)</td>
</tr>
<tr>
<td>Fig. 10</td>
<td>-1.4</td>
<td>-0.4</td>
<td>((u_0, w_0) \in S_1^{-})</td>
</tr>
</tbody>
</table>

We also see that the behaviour of orbits of solutions for each initial data \((u_0, w_0) \in S_i(i = 0, 1, 2, 3)\) guarantee Theorem 2.2 (Fig. 9-11). Especially by Fig. 10 and 11, we
can recognize that the point \((-0.5, -0.5)\) is a semi stable stationary solution.

**Experiment 3:**
We take the functions \(f_*(u), f^*(u)\) as follows

\[
\begin{align*}
    f_*(u) &= \begin{cases} 
-1 & \text{if } u \leq -1, \\
    3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\
    -u^2 - 0.25 & \text{if } -0.75 < u \leq 0, \\
    -0.25 & \text{if } 0 < u \leq 0.75, \\
    4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
    2u - 2 & \text{if } 1 < u \leq 1.4, \\
    -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
    1 & \text{if } 1.6 < u.
\end{cases}
\]
\[f^*(u) = \begin{cases} 
-1 & \text{if } u \leq -1.6, \\
    5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
    2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
    -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
    1 & \text{if } -0.4 < u.
\end{cases}
\]

In this case, we obtain that

\[
r_0 := r_{0*} = \frac{1}{4}, \quad r_1 = \frac{2\sqrt{5}}{5}, \quad r_1^* = \frac{\sqrt{2}}{2}, \quad R_1 = \sqrt{2},
\]

and stationary solutions are \((1, 1), (0, 0), (-0.5, -0.5)\) and \((-1, -1)\).

This implies that \(r_{0*} < r_{1*} < r_1^* < R_1\). Therefore, \(S_i(i = 0, 1, 2, 3)\) are defined by (3.10)-(3.17) (Fig. 12). Given initial data, our experiments are the following (Fig. 13-15):
Note that the function \( f_*(u) \) is the same as in experiment 2 but \( f^*(u) \) is not. We see that the orbit starting from \((0.8, 0.8)\) draws an anticlockwise arc and a part of \( \Gamma^* \) alternately and reaches \( \Gamma_* \) in a finite time, and then it goes to the semi stable stationary point \((-0.5, -0.5)\) as \( t \to +\infty \).

**Experiment 4:**
We take the functions \( f_*(u), f^*(u) \) as follows

\[
\begin{align*}
\mathcal{L} f_*(u) &= \begin{cases}
-1 & \text{if } u \leq -1.25, \\
3u^2 + 7u + 3.6875 & \text{if } -1.25 < u \leq -1, \\
-2u^2 + u + 0.3125 & \text{if } -1 < u \leq -0.25, \\
-0.25 & \text{if } -0.25 < u \leq 0.75, \\
4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\
2u - 2 & \text{if } 1 < u \leq 1.4, \\
-5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\
1 & \text{if } 1.6 < u.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L} f^*(u) &= \begin{cases}
1 & \text{if } u \leq -1.6, \\
5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\
2u + 2 & \text{if } -1.4 < u \leq -0.6, \\
-5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\
1 & \text{if } -0.4 < u.
\end{cases}
\end{align*}
\]

\( f^*(u) \) is the same function as in experiment 3 and \( f_*(u) \) slightly changes from the one in experiment 3. In this case, we obtain that

\[
\begin{align*}
&\mathcal{L} r_0 := r_{0*} = \frac{1}{16}, \quad r^*_0 = \frac{4}{5}, \quad r_1* = R_{1*} = \frac{1}{8}, \\
\text{and}
\end{align*}
\]

stationary solutions are \((1, 1), (0, 0)\) and \((-0.5, -0.5)\).

Since \( r_{0*} < r_{1*} = R_{1*} < r^*_0 \), subset \( \mathcal{S}_i (i = 0, 1, 2, 3) \) are defined by \((3.18)-(3.24)\) (see Fig. 16). We take the initial data as follows:

<table>
<thead>
<tr>
<th>data</th>
<th>( u_0 )</th>
<th>( w_0 )</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 17</td>
<td>0.22</td>
<td>0.22</td>
<td>((u_0, w_0) \in \mathcal{S}_1)</td>
</tr>
<tr>
<td>Fig. 18</td>
<td>0.8</td>
<td>0.8</td>
<td>((u_0, w_0) \in \mathcal{S}_1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>data</th>
<th>( u_0 )</th>
<th>( w_0 )</th>
<th>subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 19</td>
<td>1.3</td>
<td>0.8</td>
<td>((u_0, w_0) \in \mathcal{S}_3^*)</td>
</tr>
</tbody>
</table>

Then our experiments results are shown by Fig. 17-19.
These numerical experiments show that the subsets $S_i$ ($i = 0, 1, 2, 3$) are completely different from those in experiment 3. When the initial datum is $(0.8, 0.8)$, the orbit draws an anticlockwise arc and a part of $\Gamma^*$ alternately and reaches $\Gamma_*$ in a finite time, and moving along the curve $w = f_*(u)$ downward and diverges to $(-\infty, -1)$ as $t \to +\infty$.

References


