数理解析研究所講究録 第2125巻 2019年 34-41

Dirac masses and isometric rigidity

György Pál Gehér Tamás Titkos Dániel Virosztek

This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

Introduction

The aim of this short note is to expound one particular issue that was discussed during the talk [10] given at the symposium "Researches on isometries as preserver problems and related topics" at Kyoto RIMS. That is, the role of Dirac masses by describing the isometry group of various metric spaces of probability measures. This article is of survey character, and it does not contain any essentially new results.

From an isometric point of view, in some cases, metric spaces of measures are similar to $C(K)$-type function spaces. Similarity means here that their isometries are driven by some nice transformations of the underlying space. Of course, it depends on the particular choice of the metric how nice these transformations should be. Sometimes, as we will see, being a homeomorphism is enough to generate an isometry. But sometimes we need more: the transformation must preserve the underlying distance as well. Statements claiming that isometries in questions are necessarily induced by homeomorphisms are called Banach-Stone-type results, while results asserting that the underlying transformation is necessarily an isometry are termed as isometric rigidity results.

As Dirac masses can be considered as building bricks of the set of all Borel measures, a natural question arises: *Is it enough to understand how an isometry acts on the set of Dirac masses? Does this action extend uniquely to all measures?* In what follows, we will thoroughly investigate this question.

1 Notions, notations

In this section we introduce all the notions and notations that are necessary to read the paper. Let $X \neq \emptyset$ be a set, and let $\rho : X^2 \to \mathbb{R}_+$ be a metric on X. In our considerations, the metric topology on X will always be complete and separable, so in order to simplify some notions, we assume that (X, ρ) is a Polish space. The symbols $\mathcal{P}(X)$ and $\mathcal{M}(X)$ stand for the sets of probability measures and nonnegative finite measures on the Borel σ-algebra of X, respectively. Given a measure μ, the support S_μ is the set of all points $x \in X$ for which every open neighbourhood of x has positive measure.

As usual, δ_x denotes the Dirac measure concentrated to $x \in X$. The set of all Dirac measures will be denoted by $\Delta(X)$.

If a metric space (Y, d) is given, a map $f : Y \to Y$ is called an isometric embedding if it preserves the distance, that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in Y$. Surjective isometric embeddings are termed as isometries.

For a measurable map $\psi : X \to X$, the push-forward $\psi_\# : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by $\big(\psi_\#(\mu)\big)(A) = \mu(\psi^{-1}[A])$, where $A \subseteq X$ is a Borel set, and $\psi^{-1}[A] = \{x \in X \,|\, \psi(x) \in A\}$. We call a metric space of measures isometrically rigid, if all their isometries are of the form $\psi_\#$ for some isometry $\psi : X \to X$. A map $f : \mathcal{P}(X) \to \mathcal{P}(X)$ is called shape preserving if for all $\mu \in \mathcal{P}(X)$ there exists a $\psi \in \mathrm{Isom}(X)$ (depending on μ) such that $f(\mu) = \psi_\#(\mu)$. An isometry is called exotic if it is not shape preserving.

The cumulative distribution function and its right-continuous generalized inverse are key notions of this short note. We recall these well known notions in the following two special cases: when $X = \mathbb{R}$ and when $X = [0,1]$. If $\left(X, \varrho\right) = \left(\mathbb{R}, |\cdot|\right)$, the cumulative distribution function of $\mu \in \mathcal{P}(\mathbb{R})$ is defined as

$$F_\mu(x) := \mu((-\infty, x]) \qquad (x \in \mathbb{R}).$$

Its right-continuous generalized inverse is defined as $F_\mu^{-1}(y) := \sup\{x \in \mathbb{R} : F_\mu(x) \leq y\}$ for $y \in (0,1)$. If $\left(X, \varrho\right) = \left([0,1], |\cdot|\right)$, we consider F_μ and F_μ^{-1} as $[0,1] \to [0,1]$ functions. In this case, F_μ^{-1} is defined by right-continuity at 0 and it takes the value 1 at 1.

2 Banach–Stone-type theorems and isometric rigidity

In this section we will provide some examples of Banach-Stone-type and isometric rigidity results from the last decade. We do not wish to give a complete overview of the recent progress in this flourishing field, we consider only those results which are closely related to our organizing principle. Namely, the role of Dirac masses.

We start by highlighting an idea of Molnár, which is some kind of core of the results listed in this section. Assume that we have a metric d on $\mathcal{P}(\mathbb{R})$, and consider a set $S \subset Y$. Define

$$u(S) := \{\nu \in \mathcal{P}(X) | d(\nu, \mu) = 1 \text{ for all } \mu \in S\},$$

and observe that, if ϕ is a distance preserving bijection on $\mathcal{P}(\mathbb{R})$ with respect to d, then the cardinality of $u(u(\{\mu\}))$ and $u(u(\{\phi(\mu)\}))$ are the same. Consequently, the following characterization (which is valid for the Kolmogorov–Smirnov, Kuiper, and Lévy metrics) guarantees that an isometry restricted to $\Delta(X)$ is a bijection of $\Delta(X)$:

$$\mu \in \Delta(X) \qquad \Longleftrightarrow \qquad u(u(\{\mu\})) = \{\mu\}.$$

After this important remark we proceed with two Banach–Stone type theorems. We recall that the Kolmogorov–Smirnov distance d_{KS} on $\mathcal{P}(\mathbb{R})$ is defined by

$$d_{KS}(\mu, \nu) := \|F_\mu - F_\nu\|_\infty = \sup_{x \in \mathbb{R}} |F_\mu(x) - F_\nu(x)|.$$

The following characterization was obtained by Dolinar and Molnár in [2].

Theorem 1. *Let $\phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be a Kolmogorov-Smirnov isometry, that is, a bijection on $\mathcal{P}(\mathbb{R})$ with the property that*

$$d_{KS}(\phi(\mu), \phi(\nu)) = d_{KS}(\mu, \nu) \qquad (\mu, \nu \in \mathcal{P}(\mathbb{R})).$$

Then either there exists a strictly increasing bijection $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$F_{\phi(\mu)}(t) = F_\mu(\psi(t)) \qquad (t \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R})), \tag{1}$$

or there exits a strictly decreasing bijection $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ such that

$$F_{\phi(\mu)}(t) = 1 - F_\mu(\tilde{\psi}(t)-) \qquad (t \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R})), \tag{2}$$

where $F_\eta(x-)$ denotes the left limit of the distribution function F_η at the point x. Moreover, any transformation of the form (1) *or* (2) *is a Kolmogorov-Smirnov isometry.*

A recent work concerning the closely related Kuiper metric provides a Banach-Stone-type result as well. Recall that the Kuiper distance of $\mu, \nu \in \mathcal{P}(\mathbb{R})$ is given by the formula

$$d_K(\mu, \nu) := \sup_{I \in \mathcal{I}} |\mu(I) - \nu(I)|, \quad \text{where} \quad \mathcal{I} = \{I \subset R \,|\, \#I > 1 \text{ and } I \text{ is connected}\}.$$

Now, the characterization of Kuiper isometries reads as follows. (For more details see [3].)

Theorem 2. *Let $\phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be a Kuiper isometry, that is, a bijection on $\mathcal{P}(\mathbb{R})$ with the property that*

$$d_K\left(\phi(\mu), \phi(\mu)\right) = d_K\left(\mu, \nu\right) \qquad (\mu, \nu \in \mathcal{P}(\mathbb{R}))\,.$$

Then there exists a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(\mu) = g_{\#}(\mu) \qquad (\mu \in \mathcal{P}(\mathbb{R}))\,.$$

Moreover, every transformation of this form is a Kuiper isometry on $\mathcal{P}(\mathbb{R})$.

Before continuing, let us make an observation. For any two real numbers $x \neq y$ we have $d_{KS}(\delta_x, \delta_y) = 1$ and $d_K(\delta_x, \delta_y) = 1$ regardless to the value of $|x - y|$. This means that although $\mathcal{P}(\mathbb{R})$ does contain a natural copy of $\mathbb{R}$, the embedding $x \mapsto \delta_x$ does not need to carry over any metric information from X. As it will turn out soon, the form of isometries changes radically, once we consider a metric on $\mathcal{P}(\mathbb{R})$ that takes care of distances attained in the underlying space.

The first rigidity result that we mention is about the Lévy distance. For $\mu, \nu \in \mathcal{P}(\mathbb{R})$ define

$$d_L(\mu, \nu) := \inf\left\{\varepsilon > 0 \,|\, F_\mu(t - \varepsilon) - \varepsilon \leq F_\nu(t) \leq F_\mu(t + \varepsilon) + \varepsilon \ (\forall t \in \mathbb{R})\right\}.$$

Obviously, $d_L(\delta_x, \delta_y) = \min\{1, |x - y|\}$, and thus it is not surprising at all that a Lévy isometry $\phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ must be related to an isometry of $\mathbb{R}$. In fact, Molnár proved that every Lévy isometry is implemented by a translation and a reflection [9].

Theorem 3. *Let $\phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be a Lévy isometry, that is, a bijection on $\mathcal{P}(\mathbb{R})$ with the property that*

$$d_L\left(\phi(\mu), \phi(\nu)\right) = d_L\left(\mu, \nu\right) \qquad (\mu, \nu \in \mathcal{P}(\mathbb{R}))\,.$$

Then there is a constant $c \in \mathbb{R}$ such that either

$$F_{\phi(\mu)}(t) = F_\mu\left(t + c\right) \qquad (t \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R})) \tag{3}$$

or

$$F_{\phi(\mu)}(t) = 1 - F_\mu\left((-t + c)-\right) \qquad (t \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R})) \tag{4}$$

holds. Moreover, any transformation of the form (3) *or* (4) *is a Lévy isometry on $\mathcal{P}(\mathbb{R})$.*

The second isometric rigidity result is about Borel probability measures living on real separable Banach spaces endowed with the Lévy-Prokhorov distance

$$d_{LP}\left(\mu, \nu\right) = \inf\left\{\varepsilon > 0 \,|\, \mu(A) \leq \nu\left(A^\varepsilon\right) + \varepsilon \text{ for all } A \in \mathcal{B}_X\right\},$$

where

$$A^\varepsilon = \bigcup_{x \in A} B_\varepsilon(x) \text{ and } B_\varepsilon(x) = \{y \in X \,|\, d(x, y) < \varepsilon\}\,.$$

Molnár's trick on characterizing Dirac masses as measures satisfying $u\left(u\left(\{\mu\}\right)\right) = \{\mu\}$ works here as well. Moreover, we have again that $d_{LP}(\delta_x, \delta_y) = \min\{1, |x - y|\}$, but it is not so obvious for first sight that an isometry acts like a distance preserving bijection on $\Delta(X)$. For the details see [4].

Theorem 4. *Let $(X, ||\cdot||)$ be a separable real Banach space and let $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ be a Lévy-Prokhorov isometry, that is, a bijection satisfying*

$$d_{LP}\left(\phi(\mu), \phi(\nu)\right) = d_{LP}\left(\mu, \nu\right) \qquad (\mu, \nu \in \mathcal{P}(X))$$

holds. Then there exists an affine isometry $\psi : X \to X$ which induces ϕ, that is, we have

$$\phi(\mu) = \psi_{\#}(\mu) \qquad (\mu \in \mathcal{P}(X))\,. \tag{5}$$

Moreover, any transformation of the form (5) *is a Lévy-Prokhorov isometry.*

After these Banach–Stone type and rigidity results one can have the feeling that

- an isometry maps $\Delta(X)$ onto $\Delta(X)$
- the action on $\Delta(X)$ determines the isometry uniquely

In what follows, we will see that none of these statements are true.

3 Quadratic Wasserstein spaces

First, we recall the notion of a Wasserstein space $\mathcal{W}_p(X)$. For a parameter value $p \geq 1$ let us denote the set of Borel measures with finite pth moment by

$$\mathcal{W}_p(X) := \left\{ \mu \in \mathcal{P}(X) \,\middle|\, \exists x_0 \in X : \int_X \rho(x, x_0)^p \, d\mu(x) < \infty \right\},$$

where (X, ρ) is a complete and separable metric space. A Borel probability measure π on X^2 is a coupling for $\mu, \nu \in \mathcal{P}_p(X)$ ($\pi \in \mathcal{C}(\mu, \nu)$, in symbols), if their marginals are μ and ν, i.e., for all Borel sets $A \subseteq X$ it satisfies

$$\pi(A \times X) = \mu(A) \qquad \text{and} \qquad \pi(X \times A) = \nu(A). \tag{6}$$

The set $\mathcal{W}_p(X)$ endowed with the metric

$$d_{W_p}(\mu, \nu) = \left(\inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{X^2} \rho(x, y)^p \, d\pi(x, y) \right)^{1/p} \tag{7}$$

is called shortly as the p-Wasserstein space (on X). One of the features of the metric d_{W_p} is that it takes care of large distances in X. In fact, the embedding of X into $\mathcal{W}_p(X)$ as the set of Dirac masses is distance preserving. For more details and historical comments we refer the reader to [12].

From our point of view, the most important results were obtained by Bertrand and Kloeckner for quadratic ($p = 2$) Wasserstein spaces. In [7], Kloeckner provided a detailed study of quadratic Wasserstein spaces built on finite dimensional Euclidean spaces. According to his results, considering the quadratic Wasserstein distance, none of the Wasserstein spaces built on Euclidean spaces are isometrically rigid. Moreover, if the underlying Euclidean space is of dimension 1, then even exotic isometries exist. For more details see Section 5 in [7].

Theorem 5. *The isometry group of the space $\mathcal{W}_2(\mathbb{R})$ is a semidirect product*

$$\mathrm{Isom}\, \mathbb{R} \ltimes \mathrm{Isom}\, \mathbb{R}. \tag{8}$$

In (8) the left factor is the image of $\#$ and the right factor consists of all isometries that fix pointwise the set of Dirac measures. Moreover, the right factor decomposes as $\mathrm{Isom}\, \mathbb{R} = C_2 \ltimes \mathbb{R}$, where the C_2 factor (the group of order 2) is generated by a non-trivial involution that preserve shapes and the $\mathbb{R}$ factor is a flow of exotic isometries.

According to this desrciption, there are many isometries with identical action on Dirac masses, so that it cannot be true that an isometry is determined by its action on Dirac masses.

Later, it turned out that negative curvature makes the structure of the isometries simpler [1] in the sense that the quadratic Wasserstein space built on a negatively curved geodesically complete Hadamard space is isometrically rigid.

4 Splitting masses

After showing in [7] that $\mathcal{W}_2(\mathbb{R})$ admits exotic isometries, Kloeckner posed the following two questions.

- Does there exist a Polish (or Hadamard) space $X \neq \mathbb{R}$ such that $\mathcal{W}_2(X)$ admits exotic isometries?
- Does there exist a Polish space X whose Wasserstein space $\mathcal{W}_2(X)$ possess an isometry that does not preserve the set of Dirac masses?

In this short section we will highlight that the choice of parameter value $p = 2$ is essential in these questions. In fact, we will prove by showing an example that the answer is affirmative for both questions if $p = 1$.

Set X to be the unit interval $[0,1]$. The special feature of $\mathcal{W}_p([0,1])$ is that the p-Wasserstein distance d_{W_p} can be calculated as

$$d_{W_p}(\mu,\nu) = \left(\int_0^1 |F_\mu^{-1}(t) - F_\nu^{-1}(t)|^p dt\right)^{\frac{1}{p}} \qquad (\mu,\nu \in \mathcal{W}_p([0,1])).$$

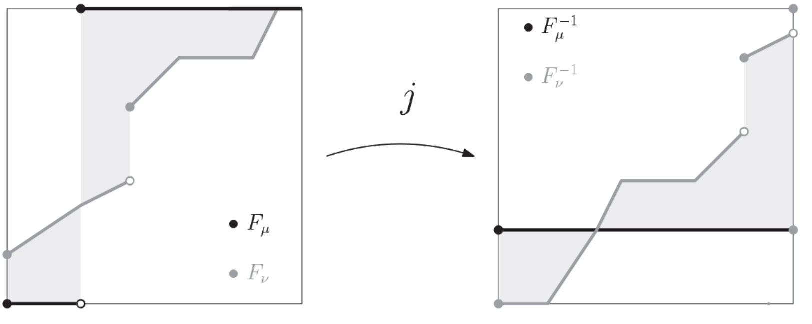
Furthermore, according to Vallender [11], in the special case of $p = 1$ and $X = [0,1]$, the Wasserstein distance can be calculated by means of the distribution functions as well

$$d_{W_1}(\mu,\nu) = \int_0^1 |F_\mu(t) - F_\nu(t)| dt \qquad (\mu,\nu \in \mathcal{W}_1([0,1])).$$

Recall that a cumulative distribution function of a $\mu \in \mathcal{P}([0,1])$ is monotone increasing, continuous from the right and takes the value 1 at the point 1. Conversely, any function $F : [0,1] \to [0,1]$ satisfying the above three conditions is the cumulative distribution function of some Borel probability measure on $[0,1]$. Consequently, for any measure $\mu \in \mathcal{P}([0,1])$, the function F_μ^{-1} is a cumulative distribution function of some measure $\nu \in \mathcal{P}([0,1])$, that is, $F_\nu = F_\mu^{-1}$. It is easy to see that the map $j : \mathcal{W}_1([0,1]) \to \mathcal{W}_1([0,1])$ defined by the equation

$$F_{j(\mu)} = F_\mu^{-1} \qquad (\mu \in \mathcal{W}_1([0,1]))$$

preserves the distance. As $j \circ j$ is the identity of $\mathcal{W}_1([0,1])$, we see also that j is a bijection, and thus an isometry.



Finally, observe that j does not send Dirac masses to Dirac masses. Indeed, (as it can be seen on the figure), $j(\delta_t) = t\delta_0 + (1-t)\delta_1$ for all $0 \le t \le 1$. More details about isometries and isometric embeddings of $\mathcal{W}_p([0,1])$ and $\mathcal{W}_p(\mathbb{R})$ spaces can be found in [6].

5 Some remarks on isometric embeddings

We close this short note by mentioning our recent result on the discrete case [5]. Our aim to do so is to show how difficult the description of distance preserving maps can be, when one drops bijectivity.

Let $X \neq \emptyset$ be a countable set, and let $\rho : X^2 \to \{0,1\}$ be the discrete metric, i.e., $\rho(x,y) := 1$ if $x \neq y$ and $\rho(x,x) := 0$ for all $x, y \in X$. To avoid trivialities, we assume that X has at least two elements.

Before showing an example and stating the theorem, we emphasize that we do not assume affinity or any other algebraic property when speaking about isometric embeddings.

Let us fix a parameter value $p \in (0, \infty)$, and let X be the set of natural numbers endowed with the discrete metric. Define $f : \mathcal{W}_p(X) \to \mathcal{W}_p(X)$ as

$$f\Big(\sum_{x \in S_\mu} c_x \cdot \delta_x \Big) = \sum_{x \in S_\mu} \big[\ln(1 + c_x) \cdot \delta_{2x} + \big(c_x - \ln(1 + c_x) \big) \cdot \delta_{2x+1} \big].$$

One can show by definition that this is a non-surjective isometric embedding. (Observe that the range of f does not contain Dirac masses.) What happens here is roughly speaking the following: f splits Dirac masses as

$$f(\delta_x) = \ln 2 \cdot \delta_{2x} + (1 - \ln 2) \cdot \delta_{2x+1},$$

and redistributes weights. On the one hand, if f $x \neq y$, then $S_{f(\delta_x)} \cap S_{f(\delta_y)} = \emptyset$, thus f induces a partition of X, in fact, the support of $f(\mu)$ is the disjoint union

$$S_{f(\mu)} = \bigcup_{x \in S_\mu} S_{f(\delta_x)}.$$

On the other hand, we see that if $\mu(\{x\}) = c_x$, then $f(\mu)\big(\{2x, 2x + 1\}\big) = c_x$, and if $\mu(\{x\}) \leq \nu(\{x\})$ then

$$f(\mu)|_{\{2x,2x+1\}} \leq f(\nu)|_{\{2x,2x+1\}}.$$

We will see that every non-surjective isometric embedding looks like this in a particular sense. The action of f on $\Delta(X)$ will induce a partition and a family of nonnegative finite measures satisfying some special properties. It can be seen easily that only the lack of surjectivity is responsible for such phenomena, because bijective isometries are basically just permutations of the underlying space.

Theorem 6. *Let $p \in (0, \infty)$ be fixed, and let $f : \mathcal{W}_p(X) \to \mathcal{W}_p(X)$ be an isometric embedding, i.e.,*

$$d_{W_p}(\mu, \nu) = d_{W_p}(f(\mu), f(\nu)) \qquad \text{for all} \quad \mu, \nu \in \mathcal{W}_p(X). \tag{9}$$

Then there exists a unique family Φ of measures indexed by the set $X \times (0, 1]$, that is

$$\Phi := \big(\varphi_{x,t}\big)_{x \in X, t \in (0,1]} \in \mathcal{M}(X)^{X \times (0,1]} \tag{10}$$

that satisfies the following properties

(*a*) *for all $x \neq y$: $S_{\varphi_{x,1}} \cap S_{\varphi_{y,1}} = \emptyset$*

(*b*) *for all $x \in X$ and $t \in (0, 1]$: $\varphi_{x,t}(X) = t$*

(*c*) *$0 < s < t \leq 1$ implies $\varphi_{x,s} \leq \varphi_{x,t}$ for all $x \in X$,*

and that generates f in the following sense

$$f(\mu) = \sum_{x \in S_\mu} \varphi_{x,\mu(\{x\})} \qquad \text{for all} \quad \mu \in \mathcal{W}_p(X). \tag{11}$$

Conversely, every $X \times (0, 1]$-indexed family of measures satisfying properties $(a) - (c)$ generates an isometric embedding via the formula (11).

Acknowledgements

This paper is part of a long term collaboration investigating the isometric structure of Wasserstein spaces. The authors would like to thank the warm hospitality and generosity of László Erdős and his group at Institute of Science and Technology Austria.

T. Titkos wants to thank Oriental Business and Innovation Center - OBIC for providing financial support to participate in the symposium at the Kyoto RIMS.

Gy. P. Gehér was supported by the Leverhulme Trust Early Career Fellowship (ECF-2018-125), and also by the Hungarian National Research, Development and Innovation Office (K115383). T. Titkos was supported by the Hungarian National Research, Development and Innovation Office - NKFIH (PD128374), by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by the ÚNKP-18-4-BGE-3 New National Excellence Program of the Ministry of Human Capacities. D. Virosztek was supported by the ISTFELLOW program of the Institute of Science and Technology Austria (project code IC1027FELL01) and partially supported by the Hungarian National Research, Development and Innovation Office NKFIH (grant no. K124152 and grant no. KH129601).

References

[1] J. Bertrand, and B. Kloeckner, *A geometric study of Wasserstein spaces: isometric rigidity in negative curvature,* Int. Math. Res. Notices **2016(5)** (2016), 1368-1386.

[2] G. Dolinar, and L. Molnár, *Isometries of the space of distribution functions with respect to the KolmogorovSmirnov metric,* J. Math. Anal. Appl. **348** (2008), 494498.

[3] Gy. P. Gehér, *Surjective Kuiper isometries,* Houston Journal of Mathematics **44(1)** (2018), 263-281.

[4] Gy. P. Gehér, and T. Titkos, *A characterisation of isometries with respect to the Lévy-Prokhorov metric,* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze (2018), in press.

[5] G. P. Gehér, T. Titkos, and D. Virosztek, *On isometric embeddings of Wasserstein spaces — the discrete case,* arXiv:1809.01101 (2018).

[6] G. P. Gehér, T. Titkos, and D. Virosztek, *On isometric embeddings of Wasserstein spaces – the real line*, Manuscript in preparation (2019).

[7] B. Kloeckner, *A geometric study of Wasserstein spaces: Euclidean spaces,* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **IX, 2** (2010), 297-323.

[8] L. Molnár, *Kolmogorov-Smirnov isometries and affine automorphisms of spaces of distribution functions,* Cent. Eur. J. Math. **9** (2011), 789-796.

[9] L. Molnár, *Lévy isometries of the space of probability distribution functions,* J. Math. Anal. Appl. **380** (2011), 847-852.

[10] T. Titkos, *On the isometry group of metric spaces of probability measures*, talk at the symposium: Researches on isometries as preserver problems and related topics, Kyoto RIMS.

[11] S. S. Vallender, *Calculation of the Wasserstein distance between probability distributions on the line,* Theory Probab. Appl. **18** (1973), 784786.

[12] C. Villani, *Optimal Transport, Old and New,* Springer, 2009.

[13] C. Villani, *Topics in Optimal Transportation,* Graduate Studies in Mathematics, vol. **58**, American Mathematical Society, Providence, RI, 2003.

[14] D. Virosztek, Maps on probability measures preserving certain distances — a survey and some new results. *Acta Sci. Math. (Szeged)* **84** (2018), 65-80.

György Pál Gehér
University of Reading, Department of Mathematics and Statistics
Whiteknights, P.O. Box 220,Reading RG6 6AX
United Kingdom
E-mail address: G.P.Geher@reading.ac.uk or gehergyuri@gmail.com

Tamás Titkos
Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences
H-1052 Budapest, Reáltanoda u. 13-15
and
BBS University of Applied Sciences
H-1054 Budapest, Alkotmány u. 9.
Hungary
E-mail address: titkos.tamas@renyi.mta.hu

Dániel Virosztek
Institute of Science and Technology Austria
Am Campus 1, 3400 Klosterneuburg
Austria
Email address: daniel.virosztek@ist.ac.at