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On Spaces of Lipschitz Maps with Values in a Uniform Algebra

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In this paper, linear always means complex linear, especially Banach algebra always means complex Banach algebra. Isometries with respect to the s-norm between vector valued Lipschitz spaces were studied by Hatori and Oi [2]. We prove a version of their results (Main Theorem A). There are literatures which study isometries with respect to the max-norm between vector valued Lipschitz spaces [1, 4]. In this paper, we exhibit the form of isometries with respect to the max-norm under an additional condition (Main Theorem B) (cf. [7].).

1 Definitions

In this section, we introduce some basic definitions.

Definition 1.1. Let X be a compact metric space and E a normed space. A map $f : X \longrightarrow E$ is called *a Lipschitz map* if

$$(L(f) :=) \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x) - f(y)\|_E}{d(x,y)} < \infty.$$

The number $L(f)$ is called *the Lipschitz constant of* f. We shall denote by $\mathrm{Lip}(X, E)$ the space of all Lipschitz maps from X into E. We write the space $\mathrm{Lip}(X, \mathbb{C})$ just by

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Lip (X) for simplification. There are several norms on the Lipschitz space Lip (X, E): *s-norm* $\|\cdot\|_s$ is defined by $\|\cdot\|_s = \|\cdot\|_\infty + L(\cdot)$, and *max-norm* $\|\cdot\|_{max}$ by $\|\cdot\|_{max} = \max\{\|\cdot\|_\infty, L(\cdot)\}$. If B is a Banach algebra, the space $(\mathrm{Lip}(X, B), \|\cdot\|_s)$ is a Banach algebra, and the space $(\mathrm{Lip}(X, B), \|\cdot\|_{max})$ is a Banach space (In general, submultiplicativity needs not hold.).

2 Main Theorem A

The next is the theorem of isometries with respect to the s-norm between Lipschitz spaces. In this section, we give an outline of the proof of this theorem.

Theorem 2.1 (Main Theorem A)**.** *For $j = 1, 2$, let X_j be a compact metric space, Y_j a compact Hausdorff space, and A_j a uniform algebra. If $U : (\mathrm{Lip}(X_1, A_1), \|\cdot\|_s) \longrightarrow (\mathrm{Lip}(X_2, A_2), \|\cdot\|_s)$ is a unital surjective linear isometry, then there exist*

- *a continuous map $\psi : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$ such that for every $y' \in \mathrm{Ch}(A_2)$ $\psi(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry,*

and

- *a homeomorphism $\tau : \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$*

such that $(U(F)(x'))(y') = (F(\psi(x', y')))(\tau(y'))$ for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, and $F \in \mathrm{Lip}(X_2, A_2)$.

Remark 2.2. A map U being unital means $U(1) = 1$. The space $\mathrm{Ch}(A)$ denotes a Choquet boundary of A. (If Y is a compact metric space and A is a subspace of $(C(Y), \|\cdot\|_\infty)$, $\mathrm{Ch}(A) = \{y \in Y \mid \tau_y \in \mathrm{ext}\{\varphi \in A^* \mid \|\varphi\| = \varphi(1) = 1\}\}$ where τ_y is the evaluation map at $y \in Y$.)

Outline of the proof of the Main Theorem A

First, we regard $\mathrm{Lip}(X_j, A_j)$ as a subspace of $C(X_j \times Y_j)$. We apply a theorem of Jarosz [3], then we find that U is an isometry also with respect to the supremum norm. Using partition of unity, we find that the uniform closure of $\mathrm{Lip}(X_j, A_j)$ coincides with $C(X_j, A_j)$. So we can extend U from $C(X_1, A_1)$ onto $C(X_2, A_2)$ which is a unital surjective linear isometry with respect to the supremum norm. We denote this map by $\tilde{U}^{\infty}$. We define maps

$$S : \left\{ \varphi' \in C\left(X_2, A_2\right)^* \mid \|\varphi'\| = \varphi'\left(1\right) = 1 \right\} \\ \longrightarrow \left\{ \varphi \in C\left(X_1, A_1\right)^* \mid \|\varphi\| = \varphi\left(1\right) = 1 \right\}$$

by $S\left(\varphi'\right) := \varphi' \circ \tilde{U}^{\infty}$ and

$$S' : \left\{ \varphi \in C\left(X_1, A_1\right)^* \mid \|\varphi\| = \varphi\left(1\right) = 1 \right\} \\ \longrightarrow \left\{ \varphi' \in C\left(X_2, A_2\right)^* \mid \|\varphi'\| = \varphi'\left(1\right) = 1 \right\}$$

by $S'\left(\varphi\right) := \varphi \circ \left(\tilde{U}^{\infty}\right)^{-1}$. Then, S and S' are well-defined, S' is an inverse map of S, and S is a w^*-homeomorphism.
For $j = 1, 2$, we define a set

$$K_j := \operatorname{ext}\left\{ \varphi \in C\left(X_j, A_j\right)^* \mid \|\varphi\| = \varphi\left(1\right) = 1 \right\}.$$

Then we find that $S\left(K_2\right) = K_1$ by some easy argument of extreme points. We note that the Choquet boundary of $C\left(X_j, A_j\right)$ coincides with $X_j \times \mathrm{Ch}\left(A_j\right)$. If we define a homeomorphism $\Phi_j : X_j \times \mathrm{Ch}\left(A_j\right) \longrightarrow K_j$ by $\Phi_j\left(x, y\right) = \varphi_{(x,y)}$ where $\varphi_{(x,y)}$ is the evaluation at (x, y) for $j = 1, 2$, then the map $\Phi_1^{-1} \circ S \circ \Phi_2$ is a homeomorphism between $X_2 \times \mathrm{Ch}\left(A_2\right)$ and $X_1 \times \mathrm{Ch}\left(A_1\right)$. So we can define continuous maps $\psi_1 : X_2 \times \mathrm{Ch}\left(A_2\right) \longrightarrow X_1$, $\psi_2 : X_2 \times \mathrm{Ch}\left(A_2\right) \longrightarrow \mathrm{Ch}\left(A_1\right)$ by $\left(\psi_1, \psi_2\right) = \Phi_1^{-1} \circ S \circ \Phi_2$. By the similar way, we consider the homeomorphism $\Phi_2^{-1} \circ S^{-1} \circ \Phi_1$ between $X_1 \times \mathrm{Ch}\left(A_1\right)$ and $X_2 \times \mathrm{Ch}\left(A_2\right)$, and define continuous maps $\psi_1' : X_1 \times \mathrm{Ch}\left(A_1\right) \longrightarrow X_2$, $\psi_2' : X_1 \times \mathrm{Ch}\left(A_1\right) \longrightarrow \mathrm{Ch}\left(A_2\right)$ by $\left(\psi_1', \psi_2'\right) = \Phi_2^{-1} \circ S^{-1} \circ \Phi_1$. Then for every $x' \in X_2$ and $y' \in \mathrm{Ch}\left(A_2\right)$, $\left(\left(U\left(F\right)\right)\left(x'\right)\right)\left(y'\right) = S\left(\varphi'_{(x',y')}\right)\left(F\right) = \left(F\left(\psi_1\left(x', y'\right)\right)\right)\left(\psi_2\left(x', y'\right)\right)$. We shall observe the maps ψ_1, ψ_2.

At the first, We show that the map ψ_2 needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2\left(x_1', y'\right) = \psi_2\left(x_2', y'\right)$ holds for any $x_1', x_2' \in X_2$ and $y' \in \mathrm{Ch}\left(A_2\right)$. Suppose that there are $x_1^{\circ} \neq x_2^{\circ} \in X_2$ and $y^{\circ} \in \mathrm{Ch}\left(A_2\right)$ such that $\psi_2\left(x_1^{\circ}, y^{\circ}\right) \neq \psi_2\left(x_2^{\circ}, y^{\circ}\right)$. Then there is $h \in A_1$ such that $h\left(\psi_2\left(x_1^{\circ}, y^{\circ}\right)\right) \neq h\left(\psi_2\left(x_2^{\circ}, y^{\circ}\right)\right)$ since A_1 is a uniform algebra. By the direct computation, we assert that $L\left(1 \otimes h\right) = 0$ and $L\left(U\left(1 \otimes h\right)\right) \neq 0$. On the other hand U preserves the Lipschitz constant because U preserves the s-norm and the supremum norm. This is a contradiction. Hence the map ψ_2 needs not depend on the first variable.

Then we define continuous maps $\tau : \mathrm{Ch}\left(A_2\right) \longrightarrow \mathrm{Ch}\left(A_1\right)$ by $\tau\left(y'\right) = \psi_2\left(x', y'\right)$ $\left(y' \in \mathrm{Ch}\left(A_2\right)\right)$ for some $x' \in X_2$, and $\tau' : \mathrm{Ch}\left(A_1\right) \longrightarrow \mathrm{Ch}\left(A_2\right)$ by $\tau'\left(y\right) = \psi_2'\left(x, y\right)$

$(y \in \mathrm{Ch}(A_1))$ for some $x \in X_1$. We can check that the map τ is a homeomorphism between $\mathrm{Ch}(A_2)$ and $\mathrm{Ch}(A_1)$. Moreover, τ' is an inverse map of τ. On the other hand, the maps $\psi_1(\cdot, y') : X_2 \longrightarrow X_1$ and $\psi_1' : X_1 \longrightarrow X_2$ are bijective for each $y' \in \mathrm{Ch}(A_2)$ and $y \in \mathrm{Ch}(A_1)$ respectively. Moreover, $\psi_1(\cdot, y') = \psi_1'(\cdot, \tau(y'))^{-1}$ and $\psi_1'\left(\cdot, \tau^{-1}(y)\right)^{-1}$ hold for each $y' \in \mathrm{Ch}(A_2)$ and $y \in \mathrm{Ch}(A_1)$ respectively.

These indicate that it is sufficient to show that $\psi_1(\cdot, y_0') : X_2 \longrightarrow X_1$ is a contractive map for each $y_0' \in \mathrm{Ch}(A_2)$ which proves the Main Theorem A. Take $y_0' \in \mathrm{Ch}(A_2)$ arbitrarily. We prove that $d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) \leq d(x_1', x_2')$ for every distinct $x_1', x_2' \in X_2$. We define a function $f_{\psi_1(x_2', y_0')} : X_1 \longrightarrow \mathbb{C}$ by $f_{\psi_1(x_2', y_0')}(x) = d(x, \psi_1(x_2', y_0'))$ for $x \in X_1$. Then $f_{\psi_1(x_2', y_0')}$ is in $\mathrm{Lip}(X_1)$ and $L\left(f_{\psi_1(x_2', y_0')}\right) = 1$. Therefore,

$$
\begin{aligned}
& d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) \\
&= d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) - d(\psi_1(x_2', y_0'), \psi_1(x_2', y_0')) \\
&= \left| f_{\psi_1(x_2', y_0')}(\psi_1(x_1', y_0')) - f_{\psi_1(x_2', y_0')}(\psi_1(x_2', y_0')) \right| \\
&= \left| \left(\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right)(\psi_1(x_1', y_0')) \right)(\tau(y_0')) - \left(\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right)(\psi_1(x_2', y_0')) \right)(\tau(y_0')) \right| \\
&= \left| \left(\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_1') \right)(y_0') - \left(\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_2') \right)(y_0') \right| \\
&\leq \left\| \left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_1') - \left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_2') \right\|_{\infty(Y_2)} \\
&\leq d(x_1', x_2')\, L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right).
\end{aligned}
$$

Since U preserves the Lipschitz constant, we have

$$
L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right) = L\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) = L\left(f_{\psi_1(x_2', y_0')} \right) = 1.
$$

Hence we have

$$
\begin{aligned}
d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) &\leq d(x_1', x_2')\, L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right) \\
&= d(x_1', x_2')
\end{aligned}
$$

and $\psi_1(\cdot, y_0')$ is a contractive map. We complete the outline of the proof of the Main Theorem A.

□

3 Main Theorem B

In this section, we consider isometries with respect to the max-norm between Lipschitz spaces. We exhibit the Main Theorem B and give an outline of the proof of this theorem.

Definition 3.1 (K pair)**.** Let X_1 and X_2 be compact metric spaces. We say that the ordered pair (X_1, X_2) of these two sets is *a K pair* if the following two conditions are satisfied.

- (K 1) For $j = 1, 2$, if we take any $x_1, x_2 \in X_j$, there are finitely many $x_1^\circ, ..., x_n^\circ \in X_j$ such that $d(x_1, x_1^\circ) < 1$, $d\left(x_i^\circ, x_{i+1}^\circ\right) < 1$ $(i = 1, ...n-1)$, $d(x_n^\circ, x_2) < 1$.
- (K 2) For any bijection $\psi : X_2 \longrightarrow X_1$ and positive ε, the following statement holds; if $x_1', x_2' \in X_2$ and $d(x_1', x_2') < \varepsilon$ implies that $d(\psi(x_1'), \psi(x_2')) = d(x_1', x_2')$, then ψ is an isometry.

Theorem 3.2 (Main Theorem B)**.** *For $j = 1, 2$, let X_j be a compact metric space, Y_j a compact Hausdorff space, and A_j a uniform algebra. We assume that (X_1, X_2) is a K pair. If $U : (\mathrm{Lip}(X_1, A_1), \|\cdot\|_{max}) \longrightarrow (\mathrm{Lip}(X_2, A_2), \|\cdot\|_{max})$ is a unital surjective linear isometry, then there exist*

- *a continuous map $\psi : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$ such that for every $y' \in \mathrm{Ch}(A_2)$ $\psi(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry,*

and

- *a homeomorphism $\tau : \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$*

such that $(U(F)(x'))(y') = (F(\psi(x', y')))(\tau(y'))$ for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, and $F \in \mathrm{Lip}(X_2, A_2)$.

Outline of the proof of the Main Theorem B

We can prove by the same way as the outline of the proof of Theorem 2.1 that there are continuous maps $\psi_1 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$, $\psi_2 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$ such that for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, $((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))$ holds.

At the first, we show that the map ψ_2 needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2(x'_1, y') = \psi_2(x'_2, y')$ holds for any $x'_1, x'_2 \in X_2$ and $y' \in \mathrm{Ch}(A_2)$. By the condition (K 1), it suffices to show that the equality $\psi_2(x'_1, y') = \psi_2(x'_2, y')$ holds for every $x'_1, x'_2 \in X_2$ with $d(x'_1, x'_2) < 1$ and $y' \in \mathrm{Ch}(A_2)$. If not, there exist $x^\circ_1, x^\circ_2 \in X_2$ with $d(x^\circ_1, x^\circ_2) < 1$ and $y^\circ \in \mathrm{Ch}(A_2)$ such that $\psi_2(x^\circ_1, y^\circ) \neq \psi_2(x^\circ_2, y^\circ)$. Let $\varepsilon_0 = \frac{1-d(x^\circ_1, x^\circ_2)}{2}$. We take an open neighborhood $V \subset Y_1$ of $\psi_2(x^\circ_1, y^\circ)$ which doesn't contain $\psi_2(x^\circ_2, y^\circ)$. Then there is a peaking function $h \in A_1$ such that $h(\psi_2(x^\circ_1, y^\circ)) = 1$, and $|h(y)| < \varepsilon_0$ for every $y \in Y_1 \setminus V$. Especially $|h(\psi_2(x^\circ_2, y^\circ))| < \varepsilon_0$. It is clear that $\|1 \otimes h\|_{max} = 1$. On the other hand,

$$\begin{aligned} L(U(1 \otimes h)) &\geq \frac{|U(1 \otimes h)(x^\circ_1, y^\circ) - U(1 \otimes h)(x^\circ_2, y^\circ)|}{d(x^\circ_1, x^\circ_2)} \\ &= \frac{|h(\psi_2(x^\circ_1, y^\circ)) - h(\psi_2(x^\circ_2, y^\circ))|}{d(x^\circ_1, x^\circ_2)} \\ &> \frac{1 - 2\varepsilon_0}{d(x^\circ_1, x^\circ_2)} = 1 \end{aligned}$$

holds, hence we get $\|U(1 \otimes h)\|_{max} > 1$. This contradicts to the fact that U preserves the max-norm. Thus ψ_2 needs not depend on the first variable.

By the Theorem of Jarosz [3], U is also an isometry with respect to the supremum norm. We can extend U from the uniform closure of $\mathrm{Lip}(X_1, A_1)$, which is $C(X_1, A_1)$, onto the uniform closure of $\mathrm{Lip}(X_2, A_2)$, which is $C(X_2, A_2)$, that is a unital surjective linear isometry with respect to the supremum norm. Since $C(X_j, A_j)$ is a uniform algebra, a theorem of Nagasawa [5] yields that U is multiplicative. For each $y' \in \mathrm{Ch}(A_2)$, we define a map $U_{y'} : \mathrm{Lip}(X_1) \longrightarrow \mathrm{Lip}(X_2)$ by

$$U_{y'}(f) = ((U(f \otimes 1))(\cdot))(y')$$

for $f \in \mathrm{Lip}(X_1)$. $U_{y'}$ is a unital homomorphism. So by [6, Theorem 5-1] , there is a Lipschitz map $\phi_{y'} : X_2 \longrightarrow X_1$ such that $U_{y'}(f) = f \circ \phi_{y'}$ for every $f \in \mathrm{Lip}(X_1)$. It is easy to check the equality $\phi_{y'} = \psi_1(\cdot, y')$. Hence $\psi_1(\cdot, y')$ is a Lipschitz map.

Next we prove that $\psi_1(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry for each $y' \in \mathrm{Ch}(A_2)$. Let $\varepsilon_0 = \frac{1}{\max\{1, L(\psi_1(\cdot, y'))\}}$. By (K 2) and the descriptions in the outline of the proof of Theorem 2.1, it suffices to show that for every $x'_1, x'_2 \in X_2$ with $d(x'_1, x'_2) < \varepsilon_0$, the equality $d(x'_1, x'_2) \geq d(\psi_1(x'_1, y'), \psi_1(x'_2, y'))$ holds. We define $f_{\psi_1(x'_2, y')} \in \mathrm{Lip}(X_1)$ by $f_{\psi_1(x'_2, y')}(x) = \min\{d(x, \psi_1(x'_2, y')), 1\}$ for

$x \in X_1$, then we have $L\left(f_{\psi_1(x_2',y')}\right) \leq 1$, $\left\|f_{\psi_1(x_2',y')}\right\|_\infty \leq 1$. By the definition of ε_0, we get $d\left(\psi_1\left(x_1', y'\right), \psi_1\left(x_2', y'\right)\right) \leq 1$. Hence $f_{\psi_1(x_2',y')}\left(\psi_1\left(x_1', y'\right)\right) = d\left(\psi_1\left(x_1', y'\right), \psi_1\left(x_2', y'\right)\right)$, and

$$\begin{aligned}
& d\left(\psi_1\left(x_1', y'\right), \psi_1\left(x_2', y'\right)\right) \\
&= \left|f_{\psi_1(x_2',y')}\left(\psi_1\left(x_1', y'\right)\right) - f_{\psi_1(x_2',y')}\left(\psi_1\left(x_2', y'\right)\right)\right| \\
&= \left|\left(U\left(f_{\psi_1(x_2',y')} \otimes 1\right)\left(x_1'\right)\right)\left(y'\right) - \left(U\left(f_{\psi_1(x_2',y')} \otimes 1\right)\left(x_2'\right)\right)\left(y'\right)\right| \\
&\leq \left\|U\left(f_{\psi_1(x_2',y')} \otimes 1\right)\left(x_1'\right) - U\left(f_{\psi_1(x_2',y')} \otimes 1\right)\left(x_2'\right)\right\|_{\infty(Y_2)} \\
&\leq L\left(U\left(f_{\psi_1(x_2',y')} \otimes 1\right)\right) d\left(x_1', x_2'\right) \leq d\left(x_1', x_2'\right).
\end{aligned}$$

Thus we get the desired inequality. Now we complete the outline of the proof of the Main Theorem B.

□

In the next section, we observe some examples of K pairs, and Main Theorem B without the condition, K pair.

4 K pairs

In the Main Theorem B, we assume that (X_1, X_2) is a K pair. We give some examples of K pairs.

Example 4.1.

1. If $a < b$, the pair of closed intervals $([a, b], [a, b])$ is a K pair.
2. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with the usual distance, then $\left(\overline{\mathbb{D}}, \overline{\mathbb{D}}\right)$ is a K pair.
3. Let $K = (\{0\} \times [-1, 1]) \cup \left(\left\{\left(t, \frac{1}{2}t\right) \mid 0 \leq t \leq 2\right\}\right) \cup \left(\left\{\left(t, -\frac{1}{2}t\right) \mid 0 \leq t \leq 2\right\}\right) \subset \mathbb{R}^2$ with the usual distance, then (K, K) is a K pair.

It is not difficult to check that these three pairs above are K pairs.

Example 4.2. Let $X_1 = X_2 = Y_1 = Y_2 = \{a, b\}$ where the distance of a and b is 2, then (X_1, X_2) is not a K pair because it doesn't satisfy (K 1). We define a map $\phi : X_2 \times Y_2 \longrightarrow X_1 \times Y_1$ by

$$\phi\left((a, a)\right) = (a, a), \ \phi\left((a, b)\right) = (b, a)$$

$$\phi((b,a)) = (a,b), \ \phi((b,b)) = (b,b)$$

and maps $\psi_1 : X_2 \times Y_2 \longrightarrow X_1$, $\psi_2 : X_2 \times Y_2 \longrightarrow Y_1$ by $\phi = (\psi_1, \psi_2)$. Let U : $\mathrm{Lip}(X_1, C(Y_1)) \longrightarrow \mathrm{Lip}(X_2, C(Y_2))$ be

$$((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))$$

for $x' \in X_2$, $y' \in Y_2$, and $F \in \mathrm{Lip}(X_1, C(Y_1))$. Then U is a unital surjective linear isometry with respect to the max-norm. Actually for every $F \in \mathrm{Lip}(X_j, C(Y_j))$,

$$L(F) = \frac{\|F(a) - F(b)\|_\infty}{2} \leq \|F\|_\infty .$$

Hence the max-norm coincides with the supremum norm in this case. The map U is clearly an isometry with respect to the supremum norm. But U cannot be represented as the form in Theorem 3.2.

Example 4.3. Let $H = (\{0\} \times [-1,1]) \cup ([0,3] \times \{0\}) \cup (\{3\} \times [-1,1]) \subset \mathbb{R}^2$ with the usual distance. Then (H, H) is not a K pair. To prove this, we define a bijection $\psi : H \longrightarrow H$ by

$$\psi((x,y)) = \begin{cases} (x,y) & ((x,y) \in \{0\} \times [-1,1]) \\ (x,y) & ((x,y) \in [0,3] \times \{0\}) \\ (x,-y) & ((x,y) \in \{3\} \times [-1,1]) . \end{cases}$$

Then $d((x_1,y_1),(x_2,y_2)) < 2$ implies that $d((x_1,y_1),(x_2,y_2)) = d(\psi((x_1,y_1)), \psi((x_2,y_2)))$ but ψ is not an isometry. Let Y be any compact Hausdorff space. We define a map $U : \mathrm{Lip}(H, C(Y)) \longrightarrow \mathrm{Lip}(H, C(Y))$ by

$$((U(F))(x'))(y') = (F(\psi(x')))(y')$$

for $x' \in H$, $y' \in Y$, and $F \in \mathrm{Lip}(H, C(Y))$. This U is not represented by the form in Theorem 3.2, but U is a unital surjective linear isometry with respect to the max-norm.

References

[1] Botelho, Fernanda; Fleming, Richard J.; Jamison, James E. *Extreme points and isometries on vector-valued Lipschitz spaces.* J. Math. Anal. Appl. **381** (2011), no. 2, 821-832.

[2] Hatori, Osamu; Oi, Shiho, *Hermitian operators on Banach algebras of vector-valued Lipschitz maps.* J. Math. Anal. Appl. **452** (2017), no. 1, 378-387.

[3] Jarosz, Krzysztof, *Isometries in semisimple, commutative Banach algebras.* Proc. Amer. Math. Soc. **94** (1985), no. 1, 65-71.

[4] Jiménez-Vargas, A.; Villegas-Vallecillos, Moisés, *Linear isometries between spaces of vector-valued Lipschitz functions.* Proc. Amer. Math. Soc. **137** (2009), no. 4, 1381-1388.

[5] Nagasawa, Masao, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions.* Kōdai Math. Sem. Rep. **11** (1959), 182-188. 46.00

[6] Sherbert, Donald R. *Banach algebras of Lipschitz functions.* Pacific J. Math. **13** (1963), 1387-1399.

[7] Tanabe, Kyohei, *Isometries on Spaces of Lipschitz Maps with Values in a Uniform Algebra*, master's thesis in Niigata University (2019)