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Multiplicative linear functional on the Zygmund F-algebra

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1 Zygmund F-algebra

Let consider the function $\varphi(t) = t\log(e+t)$ for $t \in [0,\infty)$. The *Zygmund F-algebra* $N\log N$ consists of analytic functions f on the unit disc $\mathbb{D}$ for which

$$\sup_{0\le r<1}\int_{\mathbb{T}}\varphi(\log^+|f(r\zeta)|)d\sigma(\zeta)<\infty,$$

where $\log^+ x = \max\{0,\log x\}$ for $x \ge 0$. It is easily verified that the above condition is equivalent to the condition:

$$\sup_{0\le r<1}\int_{\mathbb{T}}\varphi(\log(1+|f(r\zeta)|))d\sigma(\zeta)<\infty.$$

This class was considered by A. Zygmund [5] first. O.M. Eminyan [1] studied linear space properties of this class. Since the function $\varphi(\log(1+x))$ satisfies

$$\varphi(\log(1+x)) \le x \quad \text{for } x \ge 0,$$

we see that the inclusion $H^1 \subset N\log N$ holds. More precisely it is known that it holds the following relation:

$$\bigcup_{p>0} H^p \subset N\log N \subset N^* \subset N.$$

This implies that the boundary function f^* exists for any $f \in N\log N$. By using this boundary value of f, we can define the quasi-norm $\|f\|$ on $N\log N$ by

$$\|f\| = \int_{\mathbb{T}} \varphi(\log(1 + |f^*(\zeta)|))d\sigma(\zeta).$$

Since this quasi-norm satisfies the triangle inequality, $d(f, g) := \|f - g\|$ defines a translation invariant metric on $N\log N$. So $N\log N$ is an F-space in the sense of Banach with respect to this metric d. Moreover Eminyan [1] proved that $N\log N$ forms F-algebra with respect to d. The author and et al. [2, 4] have considered isometries of $N\log N$.

2 Results

In a general theory on Banach algebra, it is well known that every nontrivial multiplicative linear functional is continuous and that every maximal ideal is the kernel of a multiplicative linear functional. In [3], Roberts and Stoll proved that for the class N^* it is still true that every nontrivial multiplicative linear functional is continuous. However they showed that a maximal ideal in N^* is not necessarily the kernel of a multiplicative linear functional. Since the space $N\log N$ is also topological algebra, we will consider the same problems for $N\log N$.

First we will observe elementary examples. Fix $a \in \mathbb{D}$ and put $\phi_a(f) = f(a)$ for $f \in N\log N$. By applying the Poisson integral of $\varphi(\log(1 + |f^*|))$, we see that ϕ_a is a continuous multiplicative linear functional on $N\log N$. Furthermore, for each $a \in \mathbb{D}$ we define

$$\mathcal{M}_a = \{f \in N\log N \,:\, f(a) = 0\},$$

that is $\mathcal{M}_a = \mathrm{Ker}(\phi_a)$. Since ϕ_a is a surjective multiplicative linear functional on $N\log N$, $\mathcal{M}_a$ is a maximal ideal of $N\log N$. The continuity of ϕ_a implies $\mathcal{M}_a$ is closed in $N\log N$. Hence we see that $\mathcal{M}_a$ is a closed maximal ideal in $N\log N$.

The following result claim that every nontrivial multiplicative linear functional on $N\log N$ is represented by a point evaluation at some point of $\mathbb{D}$. Since $N\log N$ is a subspace in N^*, each function $f \in N\log N \setminus \{0\}$ has a canonical factorization form as follows:

$$f(z) = B(z)S(z)F(z),$$

where B is the Blaschke product, S is the singular inner function and F is the outer function. This result implies that $\mathcal{M}_a = (\pi - a)N\log N$ for some point $a \in \mathbb{D}$. Thus we have the following result.

Theorem 1. *Suppose that ϕ is a nontrivial multiplicative linear functional on $N\log N$. Then there exists $a \in \mathbb{D}$ such that $\phi(f) = f(a)$ for $f \in N\log N$ and ϕ is continuous on $N\log N$.*

As a corollary, we also can characterize a nontrivial algebra homomorphism of $N\log N$.

Corollary 2. *If $\Gamma : N\log N \to N\log N$ is a nontrivial algebra homomorphism, then there is a analytic self-map Φ of $\mathbb{D}$ such that $\Gamma(f) = f \circ \Phi$ for $f \in N\log N$.*

Remark. Every composition operator induced by an analytic self-map of $\mathbb{D}$ is continuous on $N\log N$.

As in the case N^*, we also obtain some information on the structure of a maximal ideal in $N\log N$. Let ν be a positive singular measure and put S a singular inner function with respect to ν, namely

$$S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta)\right).$$

Since $S^{-1} \notin N\log N$, $S \cdot N\log N$ is proper ideal. By Zorn's lemma, we see that $S \cdot N\log N$ is contained in a maximal ideal $\mathcal{M}$ in $N\log N$. Thus we have $S \in \mathcal{M}$. If $\mathcal{M}$ is the kernel of some multiplicative linear functional on $N\log N$, then Theorem 1 shows that $\mathcal{M} = \mathcal{M}_a$ for some point $a \in \mathbb{D}$. This implies that $S \notin \mathcal{M}$. We reach a contradiction. Hence we have the following result.

Proposition 3. *A maximal ideal need not be the kernel of a multiplicative linear functional on $N\log N$.*

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