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Kyoto University
2-local isometries on spaces of differentiable functions

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Abstract

Let $C^{(2)}([0,1])$ be the Banach space of 2-times continuously differentiable functions on the closed unit interval $[0,1]$ equipped with the norm $\|f\|_\sigma = |f(0)| + |f'(0)| + \|f''\|_\infty$, where $\|g\|_\infty = \sup\{|g(t)| : t \in [0,1]\}$ for $g$. If $T : (C^{(2)}([0,1]), \|\cdot\|_\sigma) \to (C^{(2)}([0,1]), \|\cdot\|_\sigma)$ is a 2-local isometry, then $T$ is a surjective complex-linear isometry.

1 Introduction

Let $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ be normed linear spaces over the complex number $\mathbb{C}$. A mapping $T : M \to N$ is called an isometry if $\|T(f) - T(g)\|_N = \|f - g\|_M$ for all $f, g \in M$. The linear isometries on various function spaces have been studied by many mathematicians (see [2]). The source of this subject is the classical Banach-Stone theorem, which characterizes the surjective complex-linear isometry on $C(X)$, the Banach space of all complex-valued continuous functions on a compact Hausdorff space $X$ with the supremum norm $\|\cdot\|_\infty$.

Theorem 1.1 (Banach-Stone). A mapping $T$ is a surjective complex-linear isometry on $C(X)$ if and only if there exist a unimodular continuous function $w : X \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and a homeomorphism $\varphi : X \to X$ such that $T(f) = w(f \circ \varphi)$ for all $f \in C(X)$.

In this paper, we treat with the space of continuously differentiable functions. Let $C^{(n)}([0,1])$ be the Banach space of all $n$-times continuously differentiable functions on the closed unit interval $[0,1]$ with a norm. For example, $C^{(n)}([0,1])$ with one of
the following norms is a Banach space;

\[\|f\|_C = \sup_{t \in [0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!},\]

\[\|f\|_\Sigma = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!},\]

\[\|f\|_\sigma = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty,\]

\[\|f\|_m = \max\{|f(0)|, |f'(0)|, \ldots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\},\]

for \( f \in C^{(n)}([0,1]) \). Among them, \((C^{(n)}([0,1]), \|\cdot\|_C)\) and \((C^{(n)}([0,1]), \|\cdot\|_\Sigma)\) are unital semisimple commutative Banach algebras. In 1965, Cambern [1] characterized surjective complex-linear isometries on \((C^{(1)}([0,1]), \|\cdot\|_C)\). In 1981, Pathak [10] extended this result to \((C^{(n)}([0,1]), \|\cdot\|_C)\). On the other hand, Rao and Roy [11] gave the characterization of surjective complex-linear isometries on \((C^{(1)}([0,1]), \|\cdot\|_\sigma)\) in 1971. Those results say that every surjective complex-linear isometry has the canonical form; \(T(f) = w(f \circ \varphi)\). However, the author [6, 7] proved that surjective complex-linear isometries on \((C^{(n)}([0,1]), \|\cdot\|_\sigma)\) or \((C^{(n)}([0,1]), \|\cdot\|_m)\) have a different form.

In [9], Molnár introduced the notion of 2-local isometry as follows. For a Banach space \( \mathcal{B} \), a mapping \( T : \mathcal{B} \to \mathcal{B} \) is called a 2-local isometry if for each \( f, g \in \mathcal{B} \) there exists a surjective complex-linear isometry \( T_{f,g} : \mathcal{B} \to \mathcal{B} \) such that \( T(f) = T_{f,g}(f) \) and \( T(g) = T_{f,g}(g) \). Note that no surjectivity or linearity of \( T \) is assumed. Molnár studied 2-local isometries on \( B(H) \), the Banach algebra of all bounded linear operators on an infinite dimensional separable Hilbert space \( H \). Let \( C_0(X) \) be the Banach algebra of all complex-valued continuous functions on a locally compact Hausdorff space \( X \) which vanish at infinity equipped with the supremum norm \( \|\cdot\|_\infty \). For a first countable \( \sigma \)-compact Hausdorff space \( X \), Győry [3] showed that every 2-local isometry on \( C_0(X) \) is a surjective complex-linear isometry. Hosseini [4] studied generalized 2-local isometries on \( (C^{(n)}([0,1]), \|\cdot\|_m) \). The authors, in [5, 8], considered 2-local isometries on the spaces \( (C^{(n)}([0,1]), \|\cdot\|_C) \), \( (C^{(1)}([0,1]), \|\cdot\|_\Sigma) \) and \( (C^{(1)}([0,1]), \|\cdot\|_\sigma) \).

2 Results

The following theorem is the main result of this paper.
Theorem 2.1. Every 2-local isometry on \((C^{(2)}([0,1]), \| \cdot \|_{\sigma})\) is a surjective complex-linear isometry.

The following characterization of surjective complex-linear isometries on \((C^{(2)}([0,1]), \| \cdot \|_{\sigma})\) is important to the proof of the theorem. For any \(f \in C([0,1])\), define \(Sf \in C^{(1)}([0,1])\) by \((Sf)(t) = \int_{0}^{t} f(s) \, ds\) \((\forall t \in [0,1])\).

Lemma 2.2 ([7]). A mapping \(T\) is a surjective complex-linear isometry on \((C^{(2)}([0,1]), \| \cdot \|_{\sigma})\) if and only if there exist unimodular constants \(\lambda, \mu \in \mathbb{T}\), a unimodular continuous function \(w : [0,1] \to \mathbb{T}\) and a homeomorphism \(\varphi : [0,1] \to [0,1]\) such that one of the following holds:

(i) \(T(f)(t) = \lambda f(0) + \mu f'(0)t + (S^{2}(w(f'' \circ \varphi)))(t)\) \((\forall f \in C^{(2)}([0,1]), \forall t \in [0,1])\).

(ii) \(T(f)(t) = \lambda f'(0) + \mu f(0)t + (S^{2}(w(f'' \circ \varphi)))(t)\) \((\forall f \in C^{(2)}([0,1]), \forall t \in [0,1])\).

From now on, we write simply \(C^{(2)}\) for the Banach space \((C^{(2)}([0,1]), \| \cdot \|_{\sigma})\). Let \(T\) be a 2-local isometry on \(C^{(2)}\). We define the map \(U : C([0,1]) \to C([0,1])\) by \(U(f) = (T(S^{2}f))''\) for all \(f \in C([0,1])\).

Lemma 2.3. There exist a unimodular continuous function \(w : [0,1] \to \mathbb{T}\) and a homeomorphism \(\varphi : [0,1] \to [0,1]\) such that \((T(f))'' = w(f'' \circ \varphi)\) for all \(f \in C^{(2)}\).

Proof. Let \(f, g \in C([0,1])\). Since \(T\) is a 2-local isometry on \(C^{(2)}\), there exists a surjective complex-linear isometry \(T_{S^{2}f,S^{2}g}\) on \(C^{(2)}\) such that \(T(S^{2}f) = T_{S^{2}f,S^{2}g}(S^{2}f)\) and \(T(S^{2}g) = T_{S^{2}f,S^{2}g}(S^{2}g)\). By Lemma 2.2, there exist a unimodular continuous function \(w_{f,g} : [0,1] \to \mathbb{T}\) and a homeomorphism \(\varphi_{f,g} : [0,1] \to [0,1]\) such that \((T_{S^{2}f,S^{2}g}(h))'' = w_{f,g}(h'' \circ \varphi_{f,g})\) for all \(h \in C^{(2)}\). Define \(U_{f,g}(h) = w_{f,g}(h \circ \varphi_{f,g})\) for all \(h \in C([0,1])\). By the Banach-Stone theorem, we see that \(U_{f,g}\) is a surjective complex-linear isometry on \(C([0,1])\). We have

\[
U(f) = (T(S^{2}f))'' = (T_{S^{2}f,S^{2}g}(S^{2}f))'' = w_{f,g}(f \circ \varphi_{f,g}) = U_{f,g}(f).
\]

Similarly, \(U(g) = U_{f,g}(g)\). Hence \(U\) is a 2-local isometry on \(C([0,1])\). By [3, Theorem 2], \(U\) is a surjective complex-linear isometry on \(C([0,1])\). Hence the Banach-Stone theorem implies that there exist a unimodular continuous function \(w : [0,1] \to \mathbb{T}\) and a homeomorphism \(\varphi : [0,1] \to [0,1]\) such that

\[
U(f) = w(f \circ \varphi) \quad (2.1)
\]
for all $f \in C([0,1])$.

Let $f \in C^{(2)}$. Put $g = S^2(f'')$. Since $T$ is a 2-local isometry on $C^{(2)}$, there exists a surjective complex-linear isometry $T_{f,g}$ on $C^{(2)}$ such that $T(f) = T_{f,g}(f)$ and $T(g) = T_{f,g}(g)$. By Lemma 2.2, there exist a unimodular continuous function $w_{f,g} : [0,1] \to \mathbb{T}$ and a homeomorphism $\varphi_{f,g} : [0,1] \to [0,1]$ such that $(T_{f,g}(h))'' = w_{f,g}(h'' \circ \varphi_{f,g})$ for all $h \in C^{(2)}$. Then we have

$$(T(f))'' = (T_{f,g}(f))'' = w_{f,g}(f'' \circ \varphi_{f,g}) = (T_{f,g}(g))'' = (T(g))'',$

since $g'' = (S^2(f''))'' = f''$. Substituting $f = f''$ into (2.1), we have

$$(T(f))'' = (T(g))'' = (T(S^2(f'')))' = U(f'') = w(f'' \circ \varphi).$$

Hence the lemma completes the proof. \qed

We define the functions $1$ and $\text{id}$ by $1(t) = 1$ ($\forall t \in [0,1]$) and $\text{id}(t) = t$ ($\forall t \in [0,1]$), respectively.

**Lemma 2.4.** There exist unimodular constants $\lambda, \mu \in \mathbb{T}$ such that one of the following holds:

(i) $T(1) = \lambda 1$ and $T(\text{id}) = \mu \text{id}$.

(ii) $T(1) = \mu \text{id}$ and $T(\text{id}) = \lambda 1$.

**Proof.** Since $T$ is a 2-local isometry, there exists a surjective complex-linear isometry $T_{1,\text{id}}$ on $C^{(2)}$ such that $T(1) = T_{1,\text{id}}(1)$ and $T(\text{id}) = T_{1,\text{id}}(\text{id})$. By Lemma 2.2, there exist unimodular constants $\lambda, \mu \in \mathbb{T}$, a unimodular continuous function $w_{1,\text{id}}$ and a homeomorphism $\varphi_{1,\text{id}}$ such that one of the following holds:

(i) $T_{1,\text{id}}(f)(t) = \lambda f(0) + \mu f'(0)t + (S^2(w_{1,\text{id}}(f'' \circ \varphi_{1,\text{id}})))(t)$ ($\forall f \in C^{(2)}, \forall t \in [0,1]$).

(ii) $T_{1,\text{id}}(f)(t) = \lambda f'(0) + \mu f(0)t + (S^2(w_{1,\text{id}}(f'' \circ \varphi_{1,\text{id}})))(t)$ ($\forall f \in C^{(2)}, \forall t \in [0,1]$).

If (i) holds, then we have $T(1)(t) = T_{1,\text{id}}(1)(t) = \lambda$ and $T(\text{id})(t) = T_{1,\text{id}}(\text{id})(t) = \mu t$.

If (ii) holds, then we have $T(1)(t) = T_{1,\text{id}}(1)(t) = \mu t$ and $T(\text{id})(t) = T_{1,\text{id}}(\text{id})(t) = \lambda$.

Hence the lemma is proven. \qed

**Lemma 2.5.** One of the following holds:

(a) $T(f)(0) = T(1)(0)f(0)$ ($\forall f \in C^{(2)}$) and $(Tf)'(0) = (T(\text{id}))(0)f'(0)$ ($\forall f \in C^{(2)}$).

(b) $T(f)(0) = T(\text{id})(0)f'(0)$ ($\forall f \in C^{(2)}$) and $(Tf)'(0) = (T(1))(0)f(0)$ ($\forall f \in C^{(2)}$).

**Proof.** Let $f \in C^{(2)}$. Since $T$ is a 2-local isometry, there exist surjective complex-linear isometries $T_{1,f}$ and $T_{\text{id},f}$ such that $T(f) = T_{1,f}(f) = T_{\text{id},f}(f)$, $T(1) = T_{1,\text{id}}(1) = T_{\text{id},\text{id}}(\text{id})$. Then we have

$$(T(f))'(0) = T_{1,f}(f)'(0) = T_{\text{id},f}(f)'(0) = (T_{1,\text{id}}(1))(0)f'(0) = (T_{\text{id},\text{id}}(\text{id}))(0)f(0).$$

Hence the lemma completes the proof. \qed
$T_{1,f}(1)$ and $T(id) = T_{id,f}(id)$. By Lemma 2.2, there exist unimodular constants $\lambda_{1,f}, \mu_{1,f}, \lambda_{id,f}, \mu_{id,f} \in \mathbb{T}$ such that one of the following (i) and (ii) and one of the following (I) and (II) hold:

(i) $T_{1,f}(g)(0) = \lambda_{1,f}g(0)$, $(T_{1,f}(g))'(0) = \mu_{1,f}g'(0)$ for all $g \in C^{(2)}$.

(ii) $T_{1,f}(g)(0) = \lambda_{1,f}g'(0)$, $(T_{1,f}(g))'(0) = \mu_{1,f}g(0)$ for all $g \in C^{(2)}$.

(I) $T_{id,f}(g)(0) = \lambda_{id,f}g(0)$, $(T_{id,f}(g))'(0) = \mu_{id,f}g'(0)$ for all $g \in C^{(2)}$.

(II) $T_{id,f}(g)(0) = \lambda_{id,f}g'(0)$, $(T_{id,f}(g))'(0) = \mu_{id,f}g(0)$ for all $g \in C^{(2)}$.

If (i) and (I) hold, we have $T(f)(0) = T_{1,f}(f)(0) = \lambda_{1,f}f(0)$ and $(T(f))'(0) = (T_{id,f}(f))'(0) = \mu_{id,f}f'(0)$. Also, we have $T(1)(0) = T_{1,f}(1)(0) = \lambda_{1,f}$ and $T(id)(0) = T_{id,f}(id)(0) = \lambda_{id,f}$. Hence we obtain (a).

If (i) and (II) hold, we have $T(f)(0) = T_{id,f}(f)(0) = \lambda_{1,f}f'(0)$ and $(T(f))'(0) = (T_{id,f}(f))'(0) = \mu_{id,f}f(0)$. We also have $T(id)(0) = T_{id,f}(id)(0) = \lambda_{id,f}$. This contradicts Lemma 2.4.

If (ii) and (I) hold, $T(1)(0) = T_{1,f}(1)(0) = 0$ and $T(id)(0) = T_{id,f}(id)(0) = 0$. This contradicts Lemma 2.4.

If (ii) and (II) hold, we have $T(f)(0) = T_{id,f}(f)(0) = \lambda_{id,f}f'(0)$ and $(T(f))'(0) = (T_{id,f}(f))'(0) = \mu_{1,f}f(0)$. We also have $T(id)(0) = T_{id,f}(id)(0) = \lambda_{id,f}$ and $T(1)(0) = T_{1,f}(1)(0) = \lambda_{1,f}$. Hence we obtain (b).

**Proof of Theorem 2.1.** Let $T$ be a 2-local isometry on $C^{(2)}$. We note that if Lemma 2.4(i) holds, then Lemma 2.5(a) holds. Suppose that Lemma 2.5(b) holds. Then $T(f)(0) = 0$ for all $f \in C^{(2)}$, which is a contradiction. Similarly, we see that if Lemma 2.4(ii) holds, then Lemma 2.5(b) holds.

By Lemmas 2.3, 2.4 and 2.5, we have

$$T(f)(t) = T(f)(0) + (T(f))'(0)t + (S^2(T(f))')(t)$$

$$= T(id)(0)f(0) + (T(id))'(0)f'(0)t + (S^2(w(f'' \circ \varphi)))(t)$$

$$= \lambda f(0) + \mu f'(0)t + (S^2(w(f'' \circ \varphi)))(t)$$

or

$$T(f)(t) = T(f)(0) + (T(f))'(0)t + (S^2(T(f))')(t)$$

$$= T(id)(0)f'(0) + (T(id))'(0)f(0)t + (S^2(w(f'' \circ \varphi)))(t)$$

$$= \lambda f'(0) + \mu f(0)t + (S^2(w(f'' \circ \varphi)))(t).$$

Hence Lemma 2.2 implies that $T$ is a surjective complex-linear isometry on $C^{(2)}$. □
References


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