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Blow-up profile for a nonlinear heat equation with the Neumann boundary condition

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This paper is concerned with the nonlinear diffusion equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, ν is the unit outward normal vector on $\partial\Omega$, $p>1$ is a constant and $u_0 \in L^\infty(\Omega)$ is a nonnegative function with $\|u_0\|_\infty \neq 0$. For the solution $u(x,t)$ of the nonlinear diffusion equation, the *blow-up time* T is defined by

$$T = \sup\{\tau > 0 \mid u(x,t) \text{ is bounded in } \bar{\Omega} \times (0,\tau)\}.$$

Then, $0 < T < +\infty$ and $\overline{\lim}_{t\to T}\|u(x,t)\|_{C(\bar{\Omega})} = +\infty$ hold. The *blow-up set* of the solution $u(x,t)$ is defined as the set

$$\{x \in \bar{\Omega} \mid \text{there is a sequence } (x_n,t_n) \text{ in } \bar{\Omega} \times (0,T) \text{ such that}$$

$$(x_n,t_n) \to (x,T) \text{ and } u(x_n,t_n) \to +\infty \text{ as } n \to \infty\}.$$

This set is a nonempty closed set in $\bar{\Omega}$. From standard parabolic estimates, we can obtain the *blow-up profile*, which is a continuous function defined by

$$u_*(x) = \lim_{t\to T} u(x,t)$$

outside the blow-up set.

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The blow-up problem has been studied by many authors since the pioneering work due to Fujita [13]. There are a number of results for the nature of the blow-up set. For the Cauchy problem with $(N-2)p < N+2$, Velázquez [34] showed that the $(N-1)$-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of $\mathbf{R}^N$ whenever the solution is not the constant blow-up one $(p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$. For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with $(N-2)p < N+2$, Merle and Zaag [25] showed that for any finite set $D \subset \Omega$, there exists u_0 such that the blow-up set is D (See also [1] and [3]). For the Cauchy problem with $N=1$, Herrero and Velázquez [17] showed that for any point $\bar{x}$ in the blow-up set of a solution $\bar{u}$ and $\varepsilon > 0$, there exists u_0 with $\|u_0 - \bar{u}_0\|_C \leq \varepsilon$ such that the blow-up set of u consists of a single point x with $|x - \bar{x}| \leq \varepsilon$. For the Cauchy-Dirichlet problem in an ellipsoid centred at the origin with $(N-2)p < N$, Filippas and Merle [10] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with $(N-2)p < N+2$, the second author [27] showed the following. For any nonnegative function $\phi \in C(\bar{\Omega})$ and $\delta > 0$, if $\varepsilon > 0$ is small, then any point x in the blow-up set satisfies $\phi(x) \geq \max_y \phi(y) - \delta$ for $u_0 = \varepsilon^{-1}\phi$. For the Cauchy-Neumann problem, the first author [18] showed the following. Suppose that $\Omega = (0,\pi) \times \Omega_0$ is a cylindrical domain with a bounded smooth domain Ω_0 in $\mathbf{R}^{N-1}$ and that a nonnegative function $\phi \in L^\infty(\Omega)$ satisfies $\int_\Omega \phi(x_1, x_2, \cdots, x_N) \cos x_1 dx > 0$. If $\varepsilon > 0$ is small, then the blow-up set is contained in the base plane $\{0\} \times \bar{\Omega}_0$ for $u_0 = \varepsilon\phi$. Recently, for the Cauchy-Neumann problem with $(N-2)p < N+2$, the first and second authors [20] obtained the following. Let P be the orthogonal projection in $L^2(\Omega)$ onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function $\phi \in L^\infty(\Omega)$ and neighborhood W of $\{x \in \bar{\Omega} \,|\, (P\phi)(x) = \max_{y \in \bar{\Omega}} (P\phi)(y)\} \cup \partial\Omega$, if $\varepsilon > 0$ is small, then the blow-up set is contained in W for $u_0 = \varepsilon\phi$. See, e.g., the references in this paper for related results or other studies on blow-up formation in $u_t = \Delta u + u^p$.

In this papar, we study the blow-up profile.

For large initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$, we have the following.

Theorem 1 ([35]) *Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$ is contained in the set $S := \{x \in \bar{\Omega} \,|\, \phi(x) \geq \max_{y\in\bar{\Omega}} \phi(y) - \delta\}$ and the blow-up profile u_*^ε satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(\phi(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}}\phi(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Theorems 2 and 3 are instability results for constant blow-up solutions.

Theorem 2 ([36]) *Let $f \in C(\bar{\Omega})$ be a positive function, and let δ and T_0 be positive constants. Then, there exist C and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ satisfying $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$\left\| u_0^\varepsilon(x) - (p-1)^{-\frac{1}{p-1}} T_0^{-\frac{1}{p-1}} \right\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$$

such that the blow-up time of the solution u^ε with initial data $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ is larger than T_0 and the inequality

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \leq \delta$$

holds.

Theorem 3 ([36]) *Let $f \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$, and let δ and c be positive constants. Then, there exist C and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ with $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and $\|u_0^\varepsilon - c\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$ such that the blow-up set of the solution u^ε with the initial data u_0^ε is contained in the set $S := \{x \in \bar{\Omega} \,|\, f(x) \geq \max_{y\in\bar{\Omega}} f(y) - \delta\}$ and the blow-up profile u_*^ε satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(f(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}} f(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Let λ_i be the i-th eigenvalue of $-\Delta\varphi = \lambda\varphi$ with the Neumann boundary condition $\frac{\partial\varphi}{\partial\nu} = 0$, where $0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. We denote the orthogonal projection in $L^2(\Omega)$ onto the eigenspace X_i corresponding to the i-th eigenvalue by P_i. Here, we remark that $P_1\phi = \frac{1}{|\Omega|}\int_\Omega \phi\, dx$ is a constant.

For small initial data $u_0^\varepsilon = \varepsilon\phi$, the first and second authors already showed Propositions 4 and 5 below.

Proposition 4 ([20]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist a constant $\varepsilon_0 > 0$ and a family $\{(t^\varepsilon, \delta^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]} \subset \mathbf{R}^2$ such that the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time T^ε satisfy $\lim_{\varepsilon\to+0} t^\varepsilon = 1$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}T^\varepsilon = (p-1)^{-1}(P_1\phi)^{-(p-1)}$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}e^{\lambda_2 T^\varepsilon}\delta^\varepsilon = (p-1)^{-1}(P_1\phi)^{-p}$ and*

$$\lim_{\varepsilon\to+0}\left\| \frac{t^\varepsilon}{\delta^\varepsilon}\left(1-(p-1)^{\frac{1}{p-1}}t^{\varepsilon\frac{1}{p-1}}u^\varepsilon(x,T^\varepsilon-1)\right)\right.$$

$$\left. -e^{\lambda_2}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right)\right\|_{L^\infty(\Omega)} = 0.$$

Proposition 5 ([19]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist C and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0]$, the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time T^ε satisfy $u^\varepsilon(x,t) \le C(T^\varepsilon - t)^{-\frac{1}{p-1}}$ for all $(x,t) \in \bar\Omega \times [T^\varepsilon - 1, T^\varepsilon)$.*

We obtain the following as a corollary of the propositions above.

Theorem 6 ([21]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0]$, the blow-up set of the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ is contained in the set $S := \{x \in \bar\Omega \mid (P_2\phi)(x) \ge \max_{y\in\bar\Omega}(P_2\phi)(y) - \delta\}$. Further, the blow-up time T^ε and the blow-up profile u_*^ε satisfy the inequality*

$$\left|\varepsilon^{p-1}T^\varepsilon - (p-1)^{-1}(P_1\phi)^{-(p-1)}\right| + \left\|\varepsilon^{-1}e^{-\frac{\lambda_2 T^\varepsilon}{p-1}}u_*^\varepsilon(x)\right.$$

$$\left. -(p-1)^{-\frac{1}{p-1}}(P_1\phi)^{\frac{p}{p-1}}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right)^{-\frac{1}{p-1}}\right\|_{C(\bar\Omega\setminus S)} \le \delta.$$

REFERENCES

[1] D. Amadori, Unstable blow-up patterns, Diff. and Integral Eq., 8 (1995), 1977-1996.

[2] P. Baras and L. Cohen, Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation, J. Funct. Anal., 71 (1987), 142-174.

[3] J. Bebernes, A. Bressan and V. A. Galaktionov, On symmetric and nonsymmetric blowup for a weakly quasilinear heat equation, NoDEA, 3 (1996), 269-286.

[4] H. Bellout and A. Friedman, Blow-up estimates for a nonlinear hyperbolic heat equation, SIAM J. Math. Anal., 20 (1989), 354-366.

[5] J. Bricmont and A. Kupiainen, Universality in blow-up for nonlinear heat equations, Nonlinearity, 7 (1994), 539-575.

[6] L. A. Caffarrelli and A. Friedman, Blow-up solutions of nonlinear heat equations, J. Math. Anal. Appl., 129 (1988), 409-419.

[7] X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, J. Differential Equations, 78 (1989), 160-190.

[8] Y.-G. Chen, Blow-up solutions of a semilinear parabolic equation with the Neumann and Robin boundary conditions, J. Fac. Sci. Univ. Tokyo IA, 37 (1990), 537-574.

[9] S. Filippas and W. Liu, On the blowup of multidimensional semilinear heat equations, Ann. Inst. Henri Poincaré Analyse non linéaire, 10 (1993), 313-344.

[10] S. Filippas and F. Merle, Compactness and single-point blowup of positive solutions on bounded domains, Proc. Roy. Soc. Edinburgh A, 127 (1997), 47-65.

[11] A. Friedman and A. A. Lacey, The blow-up time for solutions of nonlinear heat equations with small diffusion, SIAM J. Math. Anal., 18 (1987), 711-721.

[12] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J., 34 (1985), 425-447.

[13] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo I, 13 (1966), 109-124.

[14] V. A. Galaktionov and S. A. Posashkov, Application of new comparison theorem in the investigation of unbounded solutions of nonlinear parabolic equations, Differential Equations, 22 (1986), 809-815.

[15] V. A. Galaktionov and J. L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math., 50 (1997), 1-67.

[16] Y. Giga and R. V. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math., 42 (1989), 845-884.

[17] M. A. Herrero and J. J. L. Velázquez, Generic behaviour of one-dimensional blow up patterns, Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV, 19 (1992), 381-450.

[18] K. Ishige, Blow-up time and blow-up set of the solutions for semilinear heat equations with large diffusion, Adv. Diff. Eqns., 7 (2002), 1003-1024.

[19] K. Ishige and N. Mizoguchi, Blow-up behavior for semilinear heat equations with boundary conditions, *preprint.*

[20] K. Ishige and N. Mizoguchi, Location of blow-up set for a semilinear parabolic equation with large diffusion, *preprint.*

[21] K. Ishige, N. Mizoguchi and H. Yagisita, Blow-up profile of a solution for a nonlinear heat equation with large diffusion, *in preparation.*

[22] C. F. Kammerer, F. Merle and H. Zaag, Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view, Math. Ann., 317 (2000), 347-387.

[23] A. A. Lacey and D. Tzanetis, Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition, IMA J. Appl. Math., 41 (1988), 207-215.

[24] K. Masuda, Analytic solutions of some nonlinear diffusion equations, Math. Z., 187 (1984), 61-73.

[25] F. Merle and H. Zaag, Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J., 86 (1997), 143-195.

[26] F. Merle and H. Zaag, A Liouville theorem for vector-valued nonlinear heat equations and applications, Math. Ann., 316 (2000), 103-137.

[27] N. Mizoguchi, Location of blowup points of solutions for a semilinear parabolic equation, *preprint.*

[28] N. Mizoguchi, H. Ninomiya and E. Yanagida, Critical exponent for the bipolar blowup in a semilinear parabolic equation, J. Math. Anal. Appl., 218 (1998), 495-518.

[29] N. Mizoguchi and E. Yanagida, Life span of solutions for a semilinear parabolic problem with small diffusion, J. Math. Anal. Appl., 261 (2001), 350-368.

[30] C. E. Mueller and F. B. Weissler, Single point blow-up for a general semilinear heat equation, Indiana Univ. Math. J., 34 (1985), 881-913.

[31] C.-C. Poon, Blow-up behavior for semilinear heat equations in non-convex domains, Diff. and Integral Eq., 13 (2000), 1111-1138.

[32] S. Sakaguchi and T. Suzuki, Interior imperfect ignition cannot occur on a set of positive measure, Arch. Rational Mech. Anal., 142 (1998), 143-153.

[33] J. J. L. Velázquez, Higher dimensional blow up for semilinear parabolic equations, Comm. in PDE, 17 (1992), 1567-1596.

[34] J. J. L. Velázquez, Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, Indiana Univ. Math. J., 42 (1993), 445-476.

[35] H. Yagisita, Blow-up profile of a solution for a nonlinear heat equation with small diffusion, *to appear*, J. Math. Soc. Japan.

[36] H. Yagisita, Variable instability of a constant blow-up solution in a nonlinear heat equation, *to appear*, J. Math. Soc. Japan.

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