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Blow-up profile for a nonlinear heat equation with the Neumann boundary condition

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This paper is concerned with the nonlinear diffusion equation

\[
\begin{cases}
    u_t = \Delta u + u^p & \text{in } \Omega \times (0,T), \\
    \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0,T), \\
    u(x,0) = u_0(x) & x \in \bar{\Omega},
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( \nu \) is the unit outward normal vector on \( \partial \Omega \), \( p > 1 \) is a constant and \( u_0 \in L^\infty(\Omega) \) is a nonnegative function with \( ||u_0||_\infty \neq 0 \). For the solution \( u(x,t) \) of the nonlinear diffusion equation, the blow-up time \( T \) is defined by

\[ T = \sup\{ \tau > 0 \mid u(x,t) \text{ is bounded in } \bar{\Omega} \times (0,\tau) \}. \]

Then, \( 0 < T < +\infty \) and \( \lim_{t \to T} ||u(x,t)||_{C(\bar{\Omega})} = +\infty \) hold. The blow-up set of the solution \( u(x,t) \) is defined as the set

\[ \{ x \in \bar{\Omega} \mid \text{there is a sequence } (x_n, t_n) \text{ in } \bar{\Omega} \times (0,T) \text{ such that } \]

\[ (x_n, t_n) \to (x, T) \text{ and } u(x_n, t_n) \to +\infty \text{ as } n \to \infty \}. \]

This set is a nonempty closed set in \( \bar{\Omega} \). From standard parabolic estimates, we can obtain the blow-up profile, which is a continuous function defined by

\[ u_*(x) = \lim_{t \to T} u(x,t) \]

outside the blow-up set.

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The blow-up problem has been studied by many authors since the pioneering work due to Fujita [13]. There are a number of results for the nature of the blow-up set. For the Cauchy problem with \((N-2)p < N + 2\), Velázquez [34] showed that the \((N-1)\)-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of \(\mathbb{R}^N\) whenever the solution is not the constant blow-up one \((p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}\). For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with \((N-2)p < N + 2\), Merle and Zaag [25] showed that for any finite set \(D \subset \Omega\), there exists \(u_0\) such that the blow-up set is \(D\) (See also [1] and [3]). For the Cauchy problem with \(N = 1\), Herrero and Velázquez [17] showed that for any point \(\bar{x}\) in the blow-up set of a solution \(\bar{u}\) and \(\varepsilon > 0\), there exists \(u_0\) with \(\|u_0 - \bar{u}_0\|_C \leq \varepsilon\) such that the blow-up set of \(u\) consists of a single point \(x\) with \(|x - \bar{x}| \leq \varepsilon\).

For the Cauchy-Dirichlet problem in an ellipsoid centred at the origin with \((N-2)p < N\), Filippas and Merle [10] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with \((N-2)p < N + 2\), the second author [27] showed the following. For any nonnegative function \(\phi \in C(\overline{\Omega})\) and \(\delta > 0\), if \(\varepsilon > 0\) is small, then any point \(x\) in the blow-up set satisfies \(\phi(x) \geq \max_y \phi(y) - \delta\) for \(u_0 = \varepsilon^{-1}\phi\). For the Cauchy-Neumann problem, the first author [18] showed the following. Suppose that \(\Omega = (0, \pi) \times \Omega_0\) is a cylindrical domain with a bounded smooth domain \(\Omega_0\) in \(\mathbb{R}^{N-1}\) and that a nonnegative function \(\phi \in L^\infty(\Omega)\) satisfies \(\int_\Omega \phi(x_1, x_2, \cdots, x_N) \cos x_1 dx > 0\). If \(\varepsilon > 0\) is small, then the blow-up set is contained in the base plane \(\{0\} \times \tilde{\Omega}_0\) for \(u_0 = \varepsilon\phi\). Recently, for the Cauchy-Neumann problem with \((N-2)p < N + 2\), the first and second authors [20] obtained the following. Let \(P\) be the orthogonal projection in \(L^2(\Omega)\) onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function \(\phi \in L^\infty(\Omega)\) and neighborhood \(W\) of \(\{x \in \tilde{\Omega} \mid (P\phi)(x) = \max_{y \in \Omega}(P\phi)(y)\} \cup \partial \Omega\), if \(\varepsilon > 0\) is small, then the blow-up set is contained in \(W\) for \(u_0 = \varepsilon\phi\). See, e.g., the references in this paper for related results or other studies on blow-up formation in \(u_t = \Delta u + u^p\).
In this paper, we study the blow-up profile.

For large initial data $u_0^\epsilon = \epsilon^{-1}\phi$, we have the following.

**Theorem 1** ([35]) Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial \Omega$, and let $\delta > 0$ be a constant. Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, the blow-up set of the solution $u^\epsilon$ with the initial data $u_0^\epsilon = \epsilon^{-1}\phi$ is contained in the set $S := \{x \in \bar{\Omega} | \phi(x) \geq \max_{y \in \Omega} \phi(y) - \delta\}$ and the blow-up profile $u_*^\epsilon$ satisfies the inequality

$$\left\| \epsilon u_*^\epsilon(x) - \left( \phi(x)^{-(p-1)} - (\max_{y \in \Omega} \phi(y))^{-(p-1)} \right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega} \setminus S)} \leq \delta.$$ 

Theorems 2 and 3 are instability results for constant blow-up solutions.

**Theorem 2** ([36]) Let $f \in C(\bar{\Omega})$ be a positive function, and let $\delta$ and $T_0$ be positive constants. Then, there exist $C$ and $\epsilon_0 > 0$ satisfying the following: For any $\epsilon \in (0, \epsilon_0]$, there exists $u_0^\epsilon \in C^2(\bar{\Omega})$ satisfying $\frac{\partial u_0^\epsilon}{\partial \nu} = 0$ on $\partial \Omega$ and

$$\left\| u_0^\epsilon(x) - (p-1)^{-\frac{1}{p-1}} T_0^{-\frac{1}{p-1}} \right\|_{C^2(\bar{\Omega})} \leq C \epsilon^{p-1}$$

such that the blow-up time of the solution $u^\epsilon$ with initial data $u^\epsilon(x,0) = u_0^\epsilon(x)$ is larger than $T_0$ and the inequality

$$\left\| \epsilon u^\epsilon(x,T_0) - f(x) \right\|_{C(\bar{\Omega})} \leq \delta$$

holds.

**Theorem 3** ([36]) Let $f \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial f}{\partial \nu} = 0$ on $\partial \Omega$, and let $\delta$ and $c$ be positive constants. Then, there exist $C$ and $\epsilon_0 > 0$ satisfying the following: For any $\epsilon \in (0, \epsilon_0]$, there exists $u_0^\epsilon \in C^2(\bar{\Omega})$ with $\frac{\partial u_0^\epsilon}{\partial \nu} = 0$ on $\partial \Omega$ and $\| u_0^\epsilon - c \|_{C^2(\bar{\Omega})} \leq C \epsilon^{p-1}$ such that the blow-up set of the solution $u^\epsilon$ with the initial data $u_0^\epsilon$ is contained in the set $S := \{x \in \bar{\Omega} | f(x) \geq \max_{y \in \Omega} f(y) - \delta\}$ and the blow-up profile $u_*^\epsilon$ satisfies the inequality

$$\left\| \epsilon u_*^\epsilon(x) - \left( f(x)^{-(p-1)} - (\max_{y \in \Omega} f(y))^{-(p-1)} \right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega} \setminus S)} \leq \delta.$$
Let \( \lambda_i \) be the \( i \)-th eigenvalue of \(-\Delta \varphi = \lambda \varphi \) with the Neumann boundary condition \( \frac{\partial \varphi}{\partial \nu} = 0 \), where \( 0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \). We denote the orthogonal projection in \( L^2(\Omega) \) onto the eigenspace \( X_i \) corresponding to the \( i \)-th eigenvalue by \( P_i \). Here, we remark that \( P_1 \phi = \frac{1}{|\Omega|} \int_{\Omega} \phi \, dx \) is a constant.

For small initial data \( u_0^\varepsilon = \varepsilon \phi \), the first and second authors already showed Propositions 4 and 5 below.

**Proposition 4** ([20]) Let \( \phi \in L^\infty(\Omega) \) be a nonnegative function with \( \|\phi\|_\infty \neq 0 \). Then, there exist a constant \( \varepsilon_0 > 0 \) and a family \( \{(t^\varepsilon, \delta^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]} \subset \mathbb{R}^2 \) such that the solution \( u^\varepsilon \) with the initial data \( u_0^\varepsilon = \varepsilon \phi \) and its blow-up time \( T^\varepsilon \) satisfy
\[
\lim_{\varepsilon \to +0} t^\varepsilon = 1, \quad \lim_{\varepsilon \to +0} \varepsilon^{p-1} T' = (p-1)^{-1}(P_1 \phi)^{-(p-1)}, \quad \lim_{\varepsilon \to +0} \varepsilon^{p-1} e^{\lambda_2 T^\varepsilon} \delta^\varepsilon = (p-1)^{-1}(P_1 \phi)^{-p}
\]
and
\[
\lim_{\varepsilon \to +0} \left\| \frac{t^\varepsilon}{\delta^\varepsilon} \left( 1 - (p-1)\frac{1}{p-1} t^\varepsilon \frac{1}{p-1} u^\varepsilon(x, T^\varepsilon - 1) \right) \right\|_{L^\infty(\Omega)} = 0.
\]

**Proposition 5** ([19]) Let \( \phi \in L^\infty(\Omega) \) be a nonnegative function with \( \|\phi\|_\infty \neq 0 \). Then, there exist \( C \) and \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), the solution \( u^\varepsilon \) with the initial data \( u_0^\varepsilon = \varepsilon \phi \) and its blow-up time \( T^\varepsilon \) satisfy
\[
u^\varepsilon(x, t) \leq C(T^\varepsilon - t)^{-\frac{1}{p-1}} \text{ for all } (x, t) \in \overline{\Omega} \times [T^\varepsilon - 1, T^\varepsilon).
\]

We obtain the following as a corollary of the propositions above.

**Theorem 6** ([21]) Let \( \phi \in L^\infty(\Omega) \) be a nonnegative function with \( \|\phi\|_\infty \neq 0 \), and let \( \delta > 0 \) be a constant. Then, there exist \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), the blow-up set of the solution \( u^\varepsilon \) with the initial data \( u_0^\varepsilon = \varepsilon \phi \) is contained in the set \( S := \{ x \in \overline{\Omega} | (P_2 \phi)(x) \geq \max_{y \in \overline{\Omega}} (P_2 \phi)(y) - \delta \} \). Further, the blow-up time \( T^\varepsilon \) and the blow-up profile \( u_*^\varepsilon \) satisfy the inequality
\[
\left| \varepsilon^{p-1} T^\varepsilon - (p-1)^{-1}(P_1 \phi)^{-(p-1)} \right| + \left\| e^{-\frac{\lambda_2 T^\varepsilon}{p-1}} u_*^\varepsilon(x) - \left( (\max_{y \in \Omega} (P_2 \phi)(y)) - (P_2 \phi)(x) \right)^{-\frac{1}{p-1}} \right\|_{C(\overline{\Omega} \setminus S)} \leq \delta.
\]
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