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A property of the undominated core for TU games

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Abstract

For a coalitional game with transferable utility, the undominated core is a set of imputations which are not dominated by any other imputations. This set is characterized by reduced game property, individual rationality and a kind of monotonicity.

1 Introduction

In this note we treat solutions for coalitional games with transferable utility. The solutions are the core and the undominated core which were considered in Gillies[3]. We characterize the undominated core, that is, the set of all undominated imputations. The characterization is by axioms, one of which is the reduced game property. In Tadenuma[10], the reduced game by Moulin[7] is used for characterizing the core. We use a variation of the reduced game by Moulin[7]. Llerena/Rafels[6] characterizes the undominated core by another reduced game. The results by Rafels/Tijs[9] and Chang[2] connects the undominated core with the core, and these are effective in our study. For other earlier contributions in this area, see the Reference of [6] and see [8]. For other contributions related to this area, see [1],[4] and [5].

2 Definition of a game

Let \( \mathbb{N} \) be the set of natural numbers and let it be the set of players. A cooperative game with transferable utility (abbreviated as a game) is an ordered pair \((N, v)\), where \( N = \{1, \ldots, n\} \subset \mathbb{N} \) is a finite set of \( n \) players and \( v \), called the characteristic function, is a real-valued function on the power set of \( N \), satisfying \( v(\emptyset) = 0 \). A coalition is a subset of \( N \). We denote by \( \Gamma \) the set of all games. For a finite set \( Z \), \(|Z|\) denotes the cardinality of \( Z \). For a coalition \( S \), \( \mathbb{R}^{S} \) is the \(|S|\)-dimensional product space \( \mathbb{R}^{|S|} \) with coordinates indexed by players in \( S \). The \( i \)th component of \( x \in \mathbb{R}^{S} \) is denoted by \( x_{i} \). For \( S \subseteq N \) and \( x \in \mathbb{R}^{N} \), \( x_{S} \) means the restriction of \( x \) to \( S \). We call \( x \in \mathbb{R}^{N} \) a (payoff) vector. For \( S \subseteq N \) and \( x \in \mathbb{R}^{N} \), we define \( x(S) = \sum_{i \in S} x_{i} \) (if \( S \neq \emptyset \)) and \( = 0 \) (if \( S = \emptyset \)). A pre-imputation for a
game \((N, v) \in \Gamma\) is a vector \(x \in \mathbb{R}^{N}\) that satisfies

\[
x(N) = v(N).
\] (1)

The set of all pre-imputations for a game \((N, v) \in \Gamma\) is denoted by \(X(N, v)\). An *imputation* for a game \((N, v) \in \Gamma\) is a vector \(x \in X(N, v)\) that satisfies

\[
x_i \geq v(\{i\}), \quad \forall i \in N.
\] (2)

\(I(N, v)\) is the set of all imputations for a game \((N, v) \in \Gamma\). A *feasible* vector for a game \((N, v) \in \Gamma\) is a vector \(x \in \mathbb{R}^{N}\) that satisfies

\[
x(N) \leq v(N).
\] (3)

The set of all feasible vectors for a game \((N, v)\) is denoted by \(X^*(N, v)\). Let \(\sigma\) be a mapping that associates with every game \((N, v) \in \Gamma'\) a set \(\sigma(N, v) \subseteq X^*(N, v)\) where \(\Gamma'\) is a subset of \(\Gamma\). \(\sigma\) is called a *solution* on \(\Gamma'\).

**Definition 2.1** A solution \(\sigma\) on \(\Gamma'\) satisfies the *Pareto optimality* (PO) if for every game \((N, v) \in \Gamma'\), \(\sigma(N, v) \subseteq X(N, v)\).

**Definition 2.2** A solution \(\sigma\) on \(\Gamma'\) satisfies the *individual rationality* (IR) if for every game \((N, v) \in \Gamma'\), any \(x \in \sigma(N, v)\), \(x_i \geq v(\{i\})\) for all \(i \in N\).

For a game \((N, v) \in \Gamma\), define a game \((N, v^-)\) by

\[
v^-(S) = \min\{v(S), v(N) - \sum_{i \in N \setminus S} v(\{i\})\}, \quad \forall S \subseteq N.
\] (4)

**Definition 2.3** A solution \(\sigma\) on \(\Gamma'\) satisfies the property I (PR-I) if for games \((N, v), (N, w) \in \Gamma'\) such that \(v^-(S) \geq w^-(S)\) for all \(S \subset N\), and \(v^-(N) = w^-(N)\), \(\sigma(N, v) \subseteq \sigma(N, w)\).

For a game \((N, v) \in \Gamma\), \(x \in X^*(N, v)\) and \(S \subseteq N\), a *reduced* game is a game \((S, v^x_S) \in \Gamma\). Here \(S\) is the player set and \(v^x_S\) is the characteristic function which is defined by \(v\), \(x\) and \(S\).

**Definition 2.4** A solution \(\sigma\) on \(\Gamma'\) satisfies the *reduced game property* (RGP) if for a game \((N, v) \in \Gamma'\), any \(x \in \sigma(N, v)\) and any \(S \subset N, S \neq \emptyset\), \((S, v^x_S) \in \Gamma'\) and \(x_S \in \sigma(S, v^x_S)\).

**Definition 2.5** A solution \(\sigma\) on \(\Gamma'\) satisfies the property II (PR-II) if for a game \((N, v) \in \Gamma'\), \(v(S) = \sum_{i \in S} v(\{i\})\) for all \(S \subseteq N\), then \(x \in \sigma(N, v)\), where \(x_i = v(\{i\})\) for all \(i \in N\).

\[\text{In [6], this game is expressed as } (N, v').\]
3 Core for TU games

In this section the undominated core on $\Gamma$ is characterized by axioms where the reduced game is defined as follows.

**Definition 3.1** For $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$ and $S \subseteq N$, we define a reduced game $(S, v_S^x) \in \Gamma$ by

$$v_S^x(T) = \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v\{i\}\} - x(N \setminus S)$$

$$= v^-(T \cup (N \setminus S)) - x(N \setminus S), \quad \forall T \subseteq S, T \neq \emptyset,$$

$$v_S^x(\emptyset) = 0. \quad (5)$$

**Remark 3.2** This reduced game is a variation of the reduced game by Moulin [7]. The latter is used for characterizing the core (See [10]).

**Definition 3.3** For a game $(N, v) \in \Gamma$ and for $x, y \in X(N, v)$, $x$ dominates $y$ via $S \subset N$ if

$$x_i > y_i, \forall i \in S,$$

$$x(S) \leq v(S). \quad (6)$$

**Definition 3.4** The undominated core of a game $(N, v) \in \Gamma$, denoted by $DC(N, v)$, is defined by

$$DC(N, v) = \{x \in I(N, v) : x \text{ is not dominated by any } y \in I(N, v)\}. \quad (7)$$

The core of a game $(N, v) \in \Gamma$, denoted by $C(N, v)$, is defined by

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S), \forall S \subseteq N, S \neq \emptyset\}. \quad (8)$$

The core and the undominated core were considered in Gillies [3]. The following is the main theorem of this paper.

**Theorem 3.5** The undominated core is the only solution on $\Gamma$ which satisfies RGP, IR, PR-I, and PR-II.

To prove this theorem, we need 6 lemmas.

**Lemma 3.6** The undominated core on $\Gamma$ satisfies RGP.

**Proof:** It suffices to see when the unmoderated core is nonempty. For $(N, v) \in \Gamma$, suppose $DC(N, v) \neq \emptyset$ and let $x \in DC(N, v)$. Hence $x \in I(N, v)$. For $S \subset N, S \neq \emptyset$, consider $(S, v_S^x)$. By definition,

$$x(S) = v(N) - x(N \setminus S) = v_S^x(S). \quad (9)$$
Claim 3.6 A. \( x_i \geq v_S^x(\{i\}) \) for all \( i \in S \).

Proof of Claim 3.6 A: If \(|S| = 1\), that is, \( S = \{i\} \) then \( v_S^x(\{i\}) = x_i \) because \( x \in I(N, v) \). Let \(|S| \geq 2\). Assume \( x_i < v_S^x(\{i\}) \) for \( i \in S \). Then

\[ x(N \setminus S) + x_i < x(N \setminus S) + v_S^x(\{i\}) = \min\{v(\{i\} \cup (N \setminus S)), v(N) - \sum_{j \in S \setminus \{i\}} v(\{j\})\} \leq v(N) - \sum_{j \in S \setminus \{i\}} v(\{j\}). \]  

From this,

\[ x(N \setminus S) + x_i + \sum_{j \in S \setminus \{i\}} v(\{j\}) < v(N) = x(N). \tag{11} \]

That is,

\[ \sum_{j \in S \setminus \{i\}} v(\{j\}) < x(S \setminus \{i\}). \tag{12} \]

This implies that there exists \( j^* \in S \setminus \{i\} \) such that

\[ x_{j^*} > v(\{j^*\}). \tag{13} \]

Define \( z \in \mathbb{R}^N \) by

\[ z_j = \begin{cases} 
  x_j + \varepsilon, & \text{if } j \in \{i\} \cup (N \setminus S); \\
  x_j - \delta, & \text{if } j = j^*; \\
  x_j, & \text{otherwise},
\end{cases} \tag{14} \]

where \( \delta \) and \( \varepsilon \) are determined so that

\[ 0 < \varepsilon \min\{(\{i\} \cup (N \setminus S)), v^{-}(\{i\} \cup (N \setminus S)) - x(N \setminus S), v^{-\prime}(\{i\} \cup (N \setminus S)) - x(N \setminus S)\}. \tag{15} \]

Then \( z \in I(N, v) \) and \( z \) dominates \( x \) via \( \{i\} \cup (N \setminus S) \) in \( (N, v) \). This contradicts \( x \in DC(N, v) \). This completes the proof of Claim 3.6 A. \( \square \)

From Claim 3.6 A and (9), we see \( (S, v_S^x) \in \Gamma_I \) and \( x_S \in I(S, v_S^x) \). We shall show \( x_S \in DC(S, v_S^x) \).

Assume that \( y \in I(S, v_S^x) \) dominates \( x_S \) via \( T \subset S \) in \( (S, v_S^x) \). That is,

\[
  \begin{align*}
    y(S) &= v_S^x(S) = x(S), \\
    y_i &\geq v_S^x(\{i\}) = v^{-}(\{i\} \cup (N \setminus S)) - x(N \setminus S), \forall i \in S, \\
    y_i &> x_i, \forall i \in T, \\
    y(T) &\leq v_S^x(T) = v^{-}(T \cup (N \setminus S)) - x(N \setminus S). 
  \end{align*} \tag{16}
\]
We let $Q \equiv \{ i \in S \setminus T : x_i > v(\{i\}) \}$ and $P \equiv \{ i \in S \setminus T : x_i = v(\{i\}) \}$. By (16),

$$x(T) + x(N \setminus S) < y(T) + x(N \setminus S)$$

$$\leq v^{-}(T \cup (N \setminus S)) \equiv \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\}$$

$$\leq v(N) - \sum_{i \in S \setminus T} v(\{i\})\).$$

(17)

This implies

$$\sum_{i \in S \setminus T} v(\{i\}) < v(N) - x(T) - x(N \setminus S) = x(S \setminus T).$$

(18)

Hence there exists $i \in S \setminus T$ such that $x_i > v(\{i\})$. That is, $Q \neq \emptyset$. Define $z \in \mathbb{R}^N$ as follows.

$$z_i = \begin{cases} 
  x_i + \varepsilon_i, & \text{if } i \in N \setminus S; \\
  y_i - \delta_i, & \text{if } i \in T; \\
  v(\{i\}), & \text{if } i \in P; \\
  x_i - \eta_i, & \text{if } i \in Q,
\end{cases}$$

(19)

where

$$0 < \delta_i < y_i - x_i, \forall i \in T,$$

$$\varepsilon_i > 0, \forall i \in N \setminus S,$$

$$0 < \eta_i \leq x_i - v(\{i\}), \forall i \in Q$$

(20)

$$y(T) - x(T) - \delta(T) + \varepsilon(N \setminus S) = \eta(Q),$$

$$\varepsilon(N \setminus S) \leq \delta(T).$$

Indeed, we can find $\delta_i, \varepsilon_i$ and $\eta_i$ which satisfy (20) as follows. Since $x(Q) - \sum_{i \in Q} v(\{i\}) > 0$, choose $k \geq 2$ so that

$$0 < \frac{y(T) - x(T)}{k} \leq x(Q) - \sum_{i \in Q} v(\{i\}).$$

(21)

Second, choose $\eta_i > 0, \forall i \in Q$ so that

$$\eta(Q) = \frac{y(T) - x(T)}{k} > 0 \text{ and } \eta_i \leq x_i - v(\{i\}), \forall i \in Q.$$ (22)

Choose $\delta_i > 0, i \in T$ so that $y_i - x_i - \delta_i < \frac{\eta(Q)}{k}$ for all $i \in T$. This implies $y(T) - x(T) - \delta(T) < \eta(Q)$.

Finally, determine $\varepsilon_i > 0, i \in N \setminus S$ so that the equality in (20) is satisfied. Then

$$\varepsilon(N \setminus S) - \delta(T) = \eta(Q) - [y(T) - x(T)] = \left(\frac{1}{k} - 1\right)[y(T) - x(T)] \leq 0.$$ (23)
So (20) is feasible with respect to $\delta_i, \varepsilon_i$ and $\eta_i$. From (19) and (20)

$$z(N) = x(N \setminus S) + \varepsilon(N \setminus S) + y(T) - \delta(T) + \sum_{i \in P} v(\{i\}) + x(Q) - \eta(Q)$$

$$= x(N) = v(N).$$

$$z(T \cup (N \setminus S)) = y(T) + x(N \setminus S) - \delta(T) + \varepsilon(N \setminus S)$$

$$\leq v^x_S(T) + x(N \setminus S) - \delta(T) + \varepsilon(N \setminus S)$$

$$= \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\} - \delta(T) + \varepsilon(N \setminus S)$$

$$\leq \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\}$$

$$\leq v(T \cup (N \setminus S)).$$

From (19) and (20), we see $z_i \geq v(\{i\})$ for all $i \in N$. From this and (24), $z \in I(N, v)$. Consequently, $z$ dominates $x$ via $T \cup (N \setminus S)$ in $(N, v)$, which contradicts $x \in DC(N, v)$. This completes the proof of Lemma 3.6. □

**Lemma 3.7** The undominated core on $\Gamma$ satisfies IR, PO , PR-I and PR-II.

**Proof:** By definition, the undominated core satisfies IR and PO. It is known (Rafels/Tijs(1997)) that for any game $(N, v)$ such that $I(N, v) \neq \emptyset$, $DC(N, v) = C(N, v^{-})$. By the definition of the core, $C(N, v^{-}) \subseteq C(N, w^{-})$ for any $(N, v), (N, w)$ such that $v^{-}(S) \geq w^{-}(S)$ for all $S \subset N$, and $v^{-}(N) = w^{-}(N)$. Since $I(N, v) \neq \emptyset$, we have $I(N, v) \neq \emptyset$, which implies $DC(N, v) = C(N_{v^{-}})$. Hence $DC(N, v) \subseteq DC(N, w)$ and the unmoderated core satisfies PR-I. It satisfies PR-II since any imputation can not dominate itself. □

**Lemma 3.8** If a solution $\sigma$ on $\Gamma$ satisfies RGP and IR, then it satisfies PO.

**Proof:** For $(N, v) \in \Gamma$, let $x \in \sigma(N, v)$. By RGP and IR,

$$x_i \geq \min\{v(\{j\} \cup (N \setminus \{i\}), v(N) - \sum_{j \in \{i\} \setminus \{i\}} v(\{j\}) - x(N \setminus \{i\})$$

$$= v(N) - x(N \setminus \{i\}).$$

From this, $x(N) \geq v(N)$. Since $\sigma(N, v) \subseteq X^*(N, v)$, $x(N) \leq v(N)$. Hence we have $x(N) = v(N)$. □

**Lemma 3.9** If a solution $\sigma$ on $\Gamma$ satisfies RGP, IR, PR-I and PR-II, then $DC(N, v) \subseteq \sigma(N, v)$ for all $(N, v) \in \Gamma$.
Proof: Suppose that a solution $\sigma$ satisfies RGP, IR and PR-I. For $(N, v) \in \Gamma$, if $DC(N, v) = \emptyset$, then it trivially holds. Suppose $DC(N, v) \neq \emptyset$. Let $x \in DC(N, v) \subseteq I(N, v)$. Since $DC(N, v) = C(N, v^{-})$, $x \in C(N, v^{-})$. Hence, $x(S) \geq v^{-}(S)$ for all $S \subseteq N$. Define a game $(N, v_{x}) \in \Gamma$ by $v_{x}(S) = x(S)$ for all $S \subseteq N$. Since $x(S) = v_{x}(S)$ for all $S \subseteq N$ and $(v_{x})^{-} = v_{x}$, we have $(v_{x})^{-}(S) \geq v^{-}(S)$ for all $S \subseteq N$ and $(v_{x})^{-}(N) = v^{-}(N) = v(N)$. By PR-I, $\sigma(N, v_{x}) \subseteq \sigma(N, v)$. By the assumption and by Lemma 3.8, $\sigma$ satisfies IR and PO. That is, $\sigma(N, v_{x}) \subseteq I(N, v_{x})$. By PR-II, $x \in \sigma(N, v_{x})$. Hence, $x \in \sigma(N, v)$. $\square$

Lemma 3.10 Suppose that $\sigma$ on $\Gamma$ satisfies RGP and IR. If $v(S) = v^{-}(S)$ for all $S \subseteq N$ then $\sigma(N, v) \subseteq C(N, v)$.

Proof: Let $x \in \sigma(N, v)$. By RGP, $x_{S} \in \sigma(S, v_{S}^{x})$ for all $S \subseteq N$. By IR, $x_{i} \geq v_{S}^{x} (\{i\})$ for all $i \in S$. Since $v(S) = v^{-}(S)$ for all $S \subseteq N$, we have $v(S) \leq v(N) - \sum_{j \in N \setminus S} v(\{j\})$ for all $S \subseteq N$. This implies $v_{S}^{x}(\{i\}) = v(\{i\} \cup (N \setminus S)) - x(N \setminus S)$ for all $i \in S$. Hence, $x(N \setminus S) + x_{i} \geq v(\{i\} \cup (N \setminus S))$ for all $i \in S$. This implies $x(T) \geq v(T)$ for all $T \subseteq N$ since

$$\{i \cup (N \setminus S) : i \in S, S \subseteq N\} = \{T \subseteq N\}. \quad (26)$$

Hence we have $x \in C(N, v)$. $\square$

Lemma 3.11 If a solution $\sigma$ on $\Gamma$ satisfies RGP, IR and PR-I, then $\sigma(N, v) \subseteq DC(N, v)$ for all $(N, v) \in \Gamma$.

Proof: Assume $I(N, v) \neq \emptyset$. Since $(v^{-})^{-}(S) = v^{-}(S)$ for all $S \subseteq N$, by PR-I and Lemma 3.10 we have $\sigma(N, v) = \sigma(N, v^{-})$ and $\sigma(N, v^{-}) \subseteq C(N, v^{-})$. Then $C(N, v^{-}) = DC(N, v)$. Hence $\sigma(N, v) \subseteq DC(N, v)$. Next assume $I(N, v) = \emptyset$. By Lemma 3.8 and IR, $\sigma(N, v) \subseteq I(N, v) = \emptyset$. Hence $\sigma(N, v) = \emptyset \subset DC(N, v)$. $\square$

From Lemmas 3.6 and 3.7, the undominated core satisfies all properties in the statement of the theorem. From Lemma 3.9 and 3.11, a solution on $\Gamma$ must coincide with the undominated core if it satisfies all properties in the statement of the theorem. This completes the proof of the theorem. $\square$

The next examples show that the properties in Theorem 3.5 are independent.

Example 3.12 Let $\sigma^{1}(N, v) = I(N, v)$ for all $(N, v) \in \Gamma$. By definition, $\sigma^{1}$ satisfies IR, PR-I and PR-II. Let $N = \{1, 2, 3\}$ and $v(N) = 3, v(13) = v(23) = 2, v(12) = 1$ and $v(i) = 0$ for $i = 1, 2, 3$. Then $x = (1, 2, 0) \in I(N, v)$. Let $S = \{1, 2\}$. We see $x_{\{1, 2\}} \notin I(\{1, 2\}, v_{\{1, 2\}}^{x}) = \sigma^{2}(\{1, 2\}, v_{\{1, 2\}}^{x})$ because $v_{\{1, 2\}}^{x} (\{1\}) = 2 > x_{1} = 1$. Hence it does not satisfy RGP.

Example 3.13 Let $\sigma^{2}(N, v) = \emptyset$ for all $(N, v) \in \Gamma$. Then $\sigma^{2}$ satisfies IR, PR-I and RGP. But it does not satisfy PR-II.
Example 3.14 Let $\sigma^3(N, v) = C(N, v)$ for all $(N, v) \in \Gamma$. By definition, $\sigma^3$ satisfies IR and PR-II. Let’s see it satisfies RGP. Let $x \in C(N, v)$. Then by definition, $v_{S}^{x}(S) = x(S)$ for all $S \subseteq N$. Hence $x_{S} \in C(N, v_{S}^{x})$. Next, let’s see it does not satisfy PR-I. For $N = \{1, 2, 3\}$, let $v(i) = w(i) = 0$ for $i = 1, 2, 3$ and $v(N) = w(N) = 5$. Let $v(12) = w(12) = 2$ and $v(13) = w(13) = 3$. Let $v(23) = 5$ and $w(23) = 6$. Then $C(N, v) = \{(0, 2, 3)\}$ and $C(N, w) = \emptyset$, while $v^{-}(S) = w^{-}(S)$ for all $S \subseteq N$.

Example 3.15 Let $\sigma^4(N, v) = \{x \in X^{*}(N, v) : x_{i} \leq v(N) - v^{-}(N \setminus \{i\}), \forall i \in N\}$ for all $(N, v) \in \Gamma$. For sufficiently large $\varepsilon > 0$, $y_{i} \equiv v(N) - v^{-}(N \setminus \{i\}) - \varepsilon < v(\{i\})$ for some $i \in N$ as well as $y(N) \leq v(N)$, but $y \in \sigma^4(N, v)$. So $\sigma^4(N, v)$ does not satisfy IR. Suppose $v^{-}(S) \geq w^{-}(S)$ for all $S \subseteq N$ and $v^{-}(N) \geq w^{-}(N)$. Then $v(N) = v^{-}(S) \subseteq N$ and $v^{-}(N) \geq w^{-}(N)$. Then $v(N) = w(N)$ and $v^{-}(N) \subseteq N$ for all $i \in N$. Hence it holds $\sigma^4(N, v) \subseteq \sigma^4(N, w)$. Hence $\sigma^4$ satisfies PR-I. Next suppose $v(S) = \sum_{i \in S}v(\{i\})$ for all $S \subseteq N$. Then $\sigma^4(N, v) = \{x \in X^{*}(N, v) : x_{i} \leq v(\{i\}), \forall i \in N\}$, which implies $x \in \sigma^4(N, v)$ where $x_{i} = v(\{i\})$ for all $i \in N$. Hence $\sigma^4$ satisfies PR-II. Next suppose $x \in \sigma^4(N, v)$. Let $S \subseteq N$. Since $x(N) \leq v(N)$, it holds $x(S) \leq v(N) - x(N \setminus S) = v_{S}^{x}(S)$.

\[
v_{S}^{x}(S) - (v_{S}^{x})^{-}(S \setminus \{i\}) = v_{S}^{x}(S) - \min\{v_{S}^{x}(S \setminus \{i\}), v_{S}^{x}(S) - v_{S}^{x}(\{i\})\}
\]
\[
= \max\{v_{S}^{x}(S) - v_{S}^{x}(S \setminus \{i\}), v_{S}^{x}(\{i\})\}
\] (27)

Here
\[
v_{S}^{x}(S) - v_{S}^{x}(S \setminus \{i\}) = v(N) - \min\{v((S \setminus \{i\}) \cup (N \setminus S)), v(N) - v(\{i\})\}
\]
\[
= \max\{v(N) - v((S \setminus \{i\}) \cup (N \setminus S)), v(\{i\})\}
\] (28)

So
\[
v_{S}^{x}(S) - (v_{S}^{x})^{-}(S \setminus \{i\}) = \max\{v(N) - v(N \setminus \{i\}), v(\{i\})\}
\]
\[
\geq \max\{v(N) - v(N \setminus \{i\}), v(\{i\})\}
\]
\[
= v(N) - v^{-}(N \setminus \{i\})
\] (29)

Hence $x_{S} \in \sigma^4(S, v_{S}^{x})$. So $\sigma^4$ satisfies RGP.

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